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EXPONENTIAL CONVEXITY, EULER-RADAU EXPANSIONS AND STOLARSKY MEANS

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Abstract. We use Euler and Radau two-point formulas in order to generalize Cauchy means defined in [5] that are closely related to Stolarsky means. The gain of this approach is twofold. First, we are able to construct exponentially convex functions that are an essential ingredient of our new means since this fact leads to proof of monotonicity of constructed Cauchy means. Second, constructed exponentially convex functions are added as non-trivial to sparse examples of exponentially convex functions since invention of exponential convexity back to 1929.

1. Introduction

Stolarsky means are defined in a well-known paper [8]:

$$E_{p,q}(x,y)=\begin{cases}\left(\frac{q(y^p-x^p)}{p(y^q-x^q)}\right)^{1/(p-q)}, & pq(p-q)\neq 0;\\ \left(\frac{y^q-x^q}{r(\ln y-\ln x)}\right)^{1/q}, & p=0, q\neq 0;\\ e^{-\frac{1}{q}}\left(\frac{x^{x^q}}{y^{y^q}}\right)^{1/(x^q-y^q)}, & p=q\neq 0;\\ \sqrt{xy}, & p=q=0.\end{cases} \tag{1.1}$$

where x and y are positive real numbers $x\neq y$, p and q are any real numbers but 0.
Stolarsky proved that the function $E_{p,q}(x,y)$ is increasing in both parameters p and q i.e. for $p\leq u$ and $q\leq v$, we have

$$E_{p,q}(x,y)\leq E_{u,v}(x,y). \tag{1.2}$$

In the recent paper [5] further generalizations are made using mean-value theorems for both sides of Hadamard's inequality from [6]. Two new means of Stolarsky type defined in [5] are

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$$(1.3) \qquad E^1_{p,q}(x,y) = \left(\frac{(q-1)(q-2)}{(p-1)(p-2)} \frac{\frac{x^p-y^p}{p(x-y)} - \left(\frac{x+y}{2}\right)^{p-1}}{\frac{x^q-y^q}{q(x-y)} - \left(\frac{x+y}{2}\right)^{q-1}} \right)^{\frac{1}{p-q}}$$

and

$$(1.4) \qquad E^2_{p,q}(x,y) = \left(\frac{(q-1)(q-2)}{(p-1)(p-2)} \frac{\frac{x^{p-1}+y^{p-1}}{2} - \frac{x^p-y^p}{p(x-y)}}{\frac{x^{q-1}+y^{q-1}}{2} - \frac{x^q-y^q}{q(x-y)}} \right)^{\frac{1}{p-q}},$$

$x, y > 0,\ x \neq y,\ p, q \neq 0, 1, 2,\ p \neq q$ (appropriate continuous extensions can be found in [5]).
The same monotonicity properties are valid in both cases i.e. for $i = 1, 2$ we have

$$E^i_{p,q}(x,y) \leq E^i_{u,v}(x,y),$$

for $p \leq u$ and $q \leq v$.

In the sequel we show that two means defined in (1.3) and (1.4) can be integrated into lager class of means using Euler and Radau two-point formulas from [4] and [6], but first we have to introduce a notion of exponential convexity.

2. Exponentially convex functions

Exponentially convex functions are invented by Bernstein in [2] as a subclass of convex functions in a given open interval. These functions have many nice properties, for example, they are analytical on their domain. Although we will need only few of these properties we point here that very good reference on general results about exponential convexity is [1].

In the later text I stands for an open interval in $\mathbb{R}$.

Definition 1. *A function $\psi : I \to \mathbb{R}$ is exponentially convex if it is continuous and*

$$\sum_{i,j=1}^{n} \xi_i \xi_j \psi\left(\frac{x_i+x_j}{2}\right) \geq 0$$

for all $n \in \mathbb{N}$ and all choices $\xi_i \in \mathbb{R},\ x_i \in I,\ i = 1, \ldots, n$.

From Definition 1 we get the following corollary.

Corollary 2.1. *If ψ is exponentially convex on I then the matrix*

$$\left[\psi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{n}$$

is positive semi-definite. Particularly

$$\det\left[\psi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{n} \geq 0,$$

for every $n \in \mathbb{N}, x_i \in I,\ i = 1, \ldots, n$.

Corollary 2.2. *If $\psi : I \to (0,\infty)$ is exponentially convex function, then ψ is a log-convex function:*

$$\psi\left(\lambda x + (1-\lambda)y\right) \leq \psi(x)^{\lambda}\psi(y)^{1-\lambda},$$

for all $x, y \in I$ and any $\lambda \in [0,1]$.

The following lemma will play great role in next sections.

Lemma 2.3. *Let f be log-convex function and let $x_1 \leq y_1,\ x_2 \leq y_2,\ x_1 \neq x_2,\ y_1 \neq y_2$. Then the following inequality is valid:*

$$\left(\frac{f(x_2)}{f(x_1)}\right)^{1/(x_2-x_1)} \leq \left(\frac{f(y_2)}{f(y_1)}\right)^{1/(y_2-y_1)}. \tag{2.1}$$

Proof. This follows from [7], Remark 1.2. □

3. Euler two-point formulae for Stolarsky-type means

In the paper [6] can be found Euler two-point formulae:

Theorem 3.1. *Let $f : [0,1] \to \mathbb{R}$ be such that f' is a continuous function of bounded variation on $[0,1]$. Then for each $s \in [0,1/2]$*

$$\int_0^1 f(t)\mathrm{d}t = \frac{1}{2}\left[f(s) + f(1-s)\right] + \frac{1}{4}\int_0^1 F_2^s(t)\,\mathrm{d}f'(t), \tag{3.1}$$

where

$$F_2^s(t) = \begin{cases} 2t^2, & 0 \leq t \leq s \\ 2t^2 - 2t + 2s, & s < t \leq 1-s \\ 2t^2 - 4t + 2, & 1-s < t \leq 1 \end{cases} \tag{3.2}$$

Proof. See, [6] p. 558. □

Theorem 3.2. *Let $\phi \in C^2[x,y]$. Then for each $t \in \{x\} \cup [\frac{x+y}{2}, \frac{x+3y}{4}]$ there exist some $\xi \in [x,y]$, such that*

$$\frac{\phi(t) + \phi(x+y-t)}{2} - \frac{1}{y-x}\int_x^y \phi(u)du = \phi''(\xi)R(x,y;t) \tag{3.3}$$

where

$$R(x,y;t)=\frac{6t^2-6t(x+y)+x^2+y^2+4xy}{12}. \tag{3.4}$$

Proof. We apply Theorem 3.1 for $f:[0,1]\to\mathbb{R}$, $f(u):=\phi((1-u)x+yu)$. Then for all $s\in[0,\frac{1}{2}]$

$$\int_x^y \phi(u)\mathrm{d}u = (y-x)\int_0^1 \phi((1-u)x+uy)\mathrm{d}u \tag{3.5}$$

$$= (y-x)\frac{\phi((1-s)x+sy)+\phi(sx+(1-s)y)}{2} - \tag{3.6}$$

$$- \frac{(y-x)^3}{4}\int_0^1 \phi''((1-u)x+uy)F_2^s(t)dt$$

It is obvious that for each $s\in[\frac{1}{4},\frac{1}{2}]$, $F_2^s(t)\geq 0$, for all $t\in[0,1]$ and that for $s=0$, $F_2^s(t)\leq 0$, for all $t\in[0,1]$. Then by the mean-value theorem for integrals, for every $s\in\{0\}\cup[\frac{1}{4},\frac{1}{2}]$ there exists $\xi_0\in[0,1]$ such that

$$\int_0^1 \phi''((1-u)x+uy)F_2^s(t)dt=\phi''((1-\xi_0)x+\xi_0 y)\int_0^1 F_2^s(t)dt.$$

After we put substitution $t=(1-s)x+sy$ in (3.6), for $s\in\{0\}\cup[\frac{1}{4},\frac{1}{2}]$, our proof is done.

□

Remark 3.3. *We observe that if $x<y$ then $R(x,y;t)>0$ for $t=x$ and $R(x,y;t)<0$ for $t\in[\frac{x+y}{2},\frac{x+3y}{4}]$, where $R(x,y;t)$ is defined with (3.4).*

Corollary 3.4. *Let $\phi\in C^2[x,y]$ be a convex function. Then*

$$\frac{1}{y-x}\int_x^y \phi(u)du \geq \frac{\phi(t)+\phi(x+y-t)}{2}$$

for each $t\in[\frac{x+y}{2},\frac{x+3y}{4}]$. For $t=x$ the above inequality is reversed.

Remark 3.5. *Observe that for $t=x$ we get the first half of Hadamard inequality.*

Corollary 3.6. *Let $\phi_1,\phi_2\in C^2[x,y]$. Then for some $\xi\in[x,y]$,*

$$\frac{\frac{1}{y-x}\int_x^y \phi_1(u)du-\frac{\phi_1(t)+\phi_1(x+y-t)}{2}}{\frac{1}{y-x}\int_x^y \phi_2(u)du-\frac{\phi_2(t)+\phi_2(x+y-t)}{2}}=\frac{\phi_1''(\xi)}{\phi_2''(\xi)}, \tag{3.7}$$

for each $t\in\{x\}\cup[\frac{x+y}{2},\frac{x+3y}{4}]$, assuming both denominators different than zero.

Proof. Let us define the linear functional

$$L(f)=\frac{1}{y-x}\int_x^y f(u)\mathrm{d}u-\frac{f(t)+f(x+y-t)}{2}.$$

Next, we define $\lambda(t)=\phi_1(t)L(\phi_2)-\phi_2(t)L(\phi_1)$. By Theorem 3.2 there exists $\xi\in[x,y]$ such that

$$L(\lambda)=\lambda''(\xi)R(x,y;t).$$

On the other hand, $L(\lambda)=0$ so it follows by Remark 3.3 $\lambda''(\xi)=\phi_1''(\xi)L(\phi_2)-\phi_2''(\xi)L(\phi_1)=0$ and (3.7) is proved.

□

Remark 3.7. *If ϕ_1''/ϕ_2'' has inverse, from* (3.7) *we have*

$$\xi=\left(\frac{\phi_1''}{\phi_2''}\right)^{-1}\left(\frac{\frac{1}{y-x}\int_x^y\phi_1(u)du-\frac{\phi_1(t)+\phi_1(x+y-t)}{2}}{\frac{1}{y-x}\int_x^y\phi_2(u)du-\frac{\phi_2(t)+\phi_2(x+y-t)}{2}}\right) \tag{3.8}$$

If we take $\phi_1(u)=u^{p-1},\ \phi_2(u)=u^{q-1}$, in Remark 3.7 we can define new means. Suppose $0<x<y<\infty$ and $p\neq q,\ p,q\neq 0,1,2$ are given. Let us define the following expressions

$$E^t_{p,q}(x,y)=\left(\frac{(q-1)(q-2)}{(p-1)(p-2)}\frac{\frac{x^p-y^p}{p(x-y)}-\frac{t^{p-1}+(x+y-t)^{p-1}}{2}}{\frac{x^q-y^q}{q(x-y)}-\frac{t^{q-1}+(x+y-t)^{q-1}}{2}}\right)^{\frac{1}{p-q}}, \tag{3.9}$$

where $t\in\{x\}\cup[\frac{x+y}{2},\frac{x+3y}{4}]$.
In order to deduce continuous extensions of (3.9) for parameters p and q, for $t\in\{x\}\cup[\frac{x+y}{2},\frac{x+3y}{4}]$, we consider the following function

(3.10)
$$\psi(r)=\begin{cases}\frac{1}{(r-1)(r-2)}\left(\frac{x^r-y^r}{r(x-y)}-\frac{t^{r-1}+(x+y-t)^{r-1}}{2}\right), & r\neq 0,1,2;\\ \frac{\frac{x+y}{4t(x+y-t)}-\ln x-\ln y}{x-y}, & r=0;\\ \frac{x(\ln x-1)+y(\ln y-1)}{x-y}-\frac{\ln t+\ln(x+y-t)}{2}, & r=1;\\ \frac{t\ln t+(x+y-t)\ln(x+y-t)}{2}+\frac{x+y}{4}-\frac{x^2\ln x-y^2\ln y}{2(x-y)}, & r=2.\end{cases}$$

Theorem 3.8. *Let $0<x<y<\infty$.*

(i) *For every $t\in[\frac{x+y}{2},\frac{x+3y}{4}]$ function ψ defined with* (3.10) *is exponentially convex on $\mathbb{R}$.*

(ii) *For every* $t \in [\frac{x+y}{2}, \frac{x+3y}{4}]$ *and for all* $t_k \in \mathbb{R}$, $k = 1, 2, ..., n$, *matrix* $\left[\psi\left(\frac{t_i+t_j}{2}\right)\right]_{i,j=1}^{n}$ *is positive semi-definite matrix. Particularly*

$$\det\left[\psi\left(\frac{t_i+t_j}{2}\right)\right]_{i,j=1}^{n} \geq 0.$$

(iii) *For every* $t \in \{x\} \cup [\frac{x+y}{2}, \frac{x+3y}{4}]$ *and* $p \neq q$

$$E_{p,q}^{t}(x,y) = \left(\frac{\psi(p)}{\psi(q)}\right)^{\frac{1}{p-q}}.$$

Proof. (i) It is easy to see from (3.10) that the function ψ is continuous. For arbitrary $n \in \mathbb{N}$ and $u_i,\ t_i \in \mathbb{R}$ $(i = 1, \ldots, n)$ let us define the function

$$\Phi(x) = \sum_{i,j=1}^{n} u_i u_j \varphi_{\frac{t_i+t_j}{2}-1}(x),$$

where $\{\varphi_r :\ r \in \mathbb{R}\}$ is the family of convex functions defined on $(0, \infty)$ with

$$\varphi_r(x) = \begin{cases} \frac{x^r}{r(r-1)}, & r \neq 0, 1; \\ -\ln x, & r = 0; \\ x \ln x, & r = 1. \end{cases} \tag{3.11}$$

Since $\Phi''(x) = \sum\limits_{i,j=1}^{m} u_i u_j x^{\frac{t_i+t_j}{2}-3} = \left(\sum\limits_{i=1}^{m} u_i x^{\frac{t_i}{2}-\frac{3}{2}}\right)^2 \geq 0$, the function Φ is convex on $[x, y]$. Hence, using Corollary 4.5 for function Φ and using the fact that

$$\psi(r) = \frac{1}{y-x}\int_{x}^{y} \varphi_{r-1}(u)du - \frac{\varphi_{r-1}(t) + \varphi_{r-1}(x+y-t)}{2},$$

we get $\sum\limits_{i,j=1}^{m} u_i u_j \psi\left(\frac{t_i+t_j}{2}\right) \geq 0$, concluding exponential convexity of function ψ.
(ii) part of theorem follows from (i)-part and exponential convexity.
(iii) part of theorem is obvious.

□

Now, all continuous extensions of means $E_{p,q}^{t}(x,y)$ are obvious but the cases $p = q$:

$$E_{p,p}^{t}(x,y) = \exp\left(\frac{3-2p}{(p-1)(p-2)} + \frac{\frac{\ln t+\ln(x+y-t)}{2} - \frac{\ln^2 x-\ln^2 y}{2(x-y)}}{\frac{t^{p-1}+(x+y-t)^{p-1}}{2} - \frac{x^p-y^p}{p(x-y)}}\right)$$

$$E^t_{0,0}(x,y) =$$
$$\exp\left(\frac{\frac{3(x+y)}{4t(x+y-t)} - \frac{3(\ln x-\ln y)}{2(x-y)} + \frac{(x+y-t)\ln t+t\ln(x+y-t)}{2t(x+y-t)} - \frac{\ln^2 x-\ln^2 y}{2(x-y)}}{\frac{x+y}{2t(x+y-t)} - \frac{\ln x-\ln y}{x-y}}\right)$$

$$E^t_{1,1}(x,y) = \exp\left(\frac{\frac{x\ln^2 x-y\ln^2 y}{2(x-y)} - \frac{\ln^2 t+\ln^2(x+y-t)}{4} - \frac{\ln t+\ln(x+y-t)}{2}}{\frac{x(\ln x-1)-y(\ln y-1)}{x-y} - \frac{\ln t+\ln(x+y-t)}{2}}\right)$$

$$E^t_{2,2}(x,y) =$$
$$\exp\left(\frac{\frac{t\ln^2 t-(x+y-t)\ln^2(x+y-t)}{4} - \frac{3(x^2\ln x-y^2\ln y)}{4(x-y)} - \frac{3}{8}(x+y)}{\frac{t\ln t+(x+y-t)\ln(x+y-t)}{2} + \frac{x+y}{4} - \frac{x^2\ln x-y^2\ln y}{2(x-y)}}\right) \cdot$$
$$\cdot \exp\left(-\frac{\frac{t\ln t-(x+y-t)\ln(x+y-t)}{2} + \frac{x^2\ln^2 x-y^2\ln^2 y}{4(x-y)}}{\frac{t\ln t+(x+y-t)\ln(x+y-t)}{2} + \frac{x+y}{4} - \frac{x^2\ln x-y^2\ln y}{2(x-y)}}\right).$$

Remark 3.9. *If we put $t = \frac{x+y}{2}$ and $t = x$ in above continuous extensions we get continuous extensions of $E^1_{p,q}(x,y)$ and $E^2_{p,q}(x,y)$ defined with* (1.3) *and* (1.4) *respectively.*

Now we prove monotonicity of the new means.

Theorem 3.10. *Let $p \le r,\ q \le s$. Then*

$$E^t_{p,q}(x,y) \le E^t_{r,s}(x,y) \tag{3.12}$$

for all $0 < x < y < \infty$ and every $t \in \{x\} \cup [\frac{x+y}{2}, \frac{x+3y}{4}]$.

Proof. Assume first that $p \ne q,\ r \ne s$. Since by (i)-part of Theorem 3.8 the function ψ defined with (3.10) is exponentially convex, by Corollary 2.2 ψ is log-convex function and then by Lemma 2.3 we have

$$\left(\frac{\psi(p)}{\psi(q)}\right)^{\frac{1}{p-q}} \le \left(\frac{\psi(r)}{\psi(s)}\right)^{\frac{1}{r-s}}. \tag{3.13}$$

Using (iii)-part of Theorem 3.8 and then using above continuous extensions of means $E^t_{p,q}(x,y)$ we conclude our proof. □

4. Radau-type quadratures for Stolarsky means

We proceed with similar generalizations using Radau-type quadratures from [4] (see also [3]).

Theorem 4.1. *Let $f : [-1,1] \to \mathbb{R}$ be such that f'' is continuous on $[-1,1]$ and let $s \in (-1,0] \cup \{1\}$. Then there exists $\xi \in [-1,1]$ such that*

$$\int_{-1}^{1} f(t)dt - \frac{2s}{1+s} f(-1) - \frac{2}{1+s} f(s) = \frac{1}{3}(1-s)f''(\xi) \tag{4.1}$$

and

$$\int_{-1}^{1} f(t)dt - \frac{2}{1+s} f(-s) - \frac{2s}{1+s} f(1) = \frac{1}{3}(1-s)f''(-\xi) \tag{4.2}$$

Theorem 4.2. *Let $\phi \in C^2[x,y]$. Then for each $t \in (x, \frac{x+y}{2}] \cup \{y\}$ there exist some $\xi \in [x,y]$, such that*

$$\tfrac{2}{y-x} \int_{x}^{y} \phi(t)dt - \tfrac{2t-x-y}{t-x}\phi(x) - \tfrac{y-x}{t-x}\phi(t) = \phi''(\xi)R(x,y;t) \tag{4.3}$$

and

$$\tfrac{2}{y-x} \int_{x}^{y} \phi(t)dt - \tfrac{y-x}{t-x}\phi(x+y-t) - \tfrac{2t-x-y}{t-x}\phi(y) = \phi''(x+y-\xi)R(x,y;t) \tag{4.4}$$

where

$$R(x,y;t) = \frac{(4y+2x-6t)(y-x)}{12}.$$

Proof. We apply Theorem 4.1 for $f : [-1,1] \to \mathbb{R}$, $f(u) := \phi(\frac{1-u}{2}x + \frac{1+u}{2}y)$. Then for all $s \in (-1,0] \cup \{1\}$

$$\int_{x}^{y} \phi(u)\mathrm{d}u = \tfrac{y-x}{2} \int_{-1}^{1} \phi(\tfrac{1-u}{2}x + \tfrac{1+u}{2}y)\mathrm{d}u \tag{4.5}$$

$$= \tfrac{y-x}{2}\Big[\tfrac{2s}{1+s}\phi(x) + \tfrac{2}{1+s}\phi(\tfrac{1-s}{2}x + \tfrac{1+s}{2}y) + \tfrac{(1-3s)(y-x)^2}{12}\phi''(\tfrac{1-\xi_0}{2}x + \tfrac{1+\xi_0}{2}y)\Big], \tag{4.6}$$

for some $\xi_0 \in [-1,1]$. After we put substitution $t = \frac{1-s}{2}x + \frac{1+s}{2}y$ and apply (4.1) and (4.2) we get (4.3) and (4.4).

□

Corollary 4.3. *Let $\phi \in C^2[x,y]$. Then for each $t \in (x, \frac{x+y}{2}] \cup \{y\}$ there exist some $\xi \in [x,y]$, such that*

$$\frac{2}{y-x}\int_x^y \phi(t)dt - \frac{2t-x-y}{t-x}\phi(x) - \frac{y-x}{t-x}\phi(t) = \phi''(\xi)R_1(x,y;t) \tag{4.7}$$

and for each $v \in [\frac{x+y}{2}, y) \cup \{x\}$ there exist some $\eta \in [x,y]$, such that

$$\frac{2}{y-x}\int_x^y \phi(t)dt - \frac{y-x}{y-v}\phi(v) - \frac{y+x-2v}{y-v}\phi(y) = \phi''(\eta)R_2(x,y;v) \tag{4.8}$$

where

$$R_1(x,y;t) = \frac{(4y+2x-6t)(y-x)}{12} \text{ and } R_2(x,y;v) = \frac{(6v-4x-2y)(y-x)}{12}.$$

Remark 4.4. *Assume $x < y$. Observe then $R_1(x,y;t) > 0$ for $t \in (x, \frac{x+y}{2}]$; and $R_1(x,y;t) < 0$ for $t = y$. Also $R_2(x,y;v) > 0$ for $v \in [\frac{x+y}{2}, y)$; and $R_2(x,y;v) < 0$ for $v = x$.*

Corollary 4.5. *Let $\phi \in C^2[x,y]$ be a convex function.*

(i) *For every $t \in (x, \frac{x+y}{2}]$*

$$\frac{2}{y-x}\int_x^y \phi(t)dt \geq \frac{2t-x-y}{t-x}\phi(x) + \frac{y-x}{t-x}\phi(t).$$

For $t = y$ the above inequality is reversed.

(ii) *For every $v \in [\frac{x+y}{2}, y)$*

$$\frac{2}{y-x}\int_x^y \phi(t)dt \geq \frac{y-x}{y-v}\phi(v) + \frac{y+x-2v}{y-v}\phi(y).$$

For $v = x$ the above inequality is reversed.

Similar to Corollary (3.6) we can also prove the following corollary.

Corollary 4.6. *Let $\phi_1, \phi_2 \in C^2[x,y]$. Then for each $t \in (x, \frac{x+y}{2}] \cup \{y\}$ there exist some $\xi \in [x,y]$, such that*

$$\frac{\frac{2}{y-x}\int_x^y \phi_1(u)du - \frac{2t-x-y}{t-x}\phi_1(x) - \frac{y-x}{t-x}\phi_1(t)}{\frac{2}{y-x}\int_x^y \phi_2(u)du - \frac{2t-x-y}{t-x}\phi_2(x) - \frac{y-x}{t-x}\phi_2(t)} = \frac{\phi_1''(\xi)}{\phi_2''(\xi)} \tag{4.9}$$

and for each $v \in [\frac{x+y}{2}, y) \cup \{x\}$ there exist some $\eta \in [x,y]$, such that

$$\frac{\frac{2}{y-x}\int_x^y \phi_1(u)du - \frac{y-x}{y-v}\phi_1(v) - \frac{y+x-2v}{y-v}\phi_1(y)}{\frac{2}{y-x}\int_x^y \phi_2(u)du - \frac{y-x}{y-v}\phi_2(v) - \frac{y+x-2v}{y-v}\phi_2(y)} = \frac{\phi_1''(\eta)}{\phi_2''(\eta)}, \tag{4.10}$$

assuming that denominators in (4.9) and (4.10) are not equal zero.

Remark 4.7. *If ϕ_1''/ϕ_2'' has inverse, from* (4.9) *we have*

$$\xi = \left(\frac{\phi_1''}{\phi_2''}\right)^{-1}\left(\frac{\frac{2}{y-x}\int_x^y \phi_1(u)du - \frac{2t-x-y}{t-x}\phi_1(x) - \frac{y-x}{t-x}\phi_1(t)}{\frac{2}{y-x}\int_x^y \phi_2(u)du - \frac{2t-x-y}{t-x}\phi_2(x) - \frac{y-x}{t-x}\phi_2(t)}\right) \tag{4.11}$$

Now if we take $\phi_1(x) = x^{p-1}$, $\phi_2(x) = x^{q-1}$, in Remark 4.7 we can define new means. Suppose $0 < x < y < \infty$ and $p \neq q$, $p, q \neq 0, 1, 2$ are given. Let us define the following expressions

$$R^{1,t}_{p,q}(x,y) = \left(\frac{(q-1)(q-2)}{(p-1)(p-2)}\frac{\frac{2(y^p-x^p)}{p(y-x)} - \frac{2t-x-y}{t-x}x^{p-1} - \frac{y-x}{t-x}t^{p-1}}{\frac{2(y^q-x^q)}{q(y-x)} - \frac{2t-x-y}{t-x}x^{q-1} - \frac{y-x}{t-x}t^{q-1}}\right)^{\frac{1}{p-q}}, \tag{4.12}$$

where $t \in (x, \frac{x+y}{2}] \cup \{y\}$.
In order to deduce continuous extensions of (4.12), for $t \in (x, \frac{x+y}{2}] \cup \{y\}$ we consider the following function
(4.13)

$$\psi_1(p) = \begin{cases} \frac{1}{(p-1)(p-2)}\left(\frac{2(y^p-x^p)}{p(y-x)} - \frac{2t-x-y}{t-x}x^{p-1} - \frac{y-x}{t-x}t^{p-1}\right), & p \neq 0,1,2; \\ \frac{\ln y - \ln x}{y-x} - \frac{2t-x-y}{2x(t-x)} - \frac{y-x}{2t(t-x)}, & p = 0; \\ \frac{(2t-x-y)\ln x + (y-x)\ln t}{t-x} - \frac{2(y\ln y - x\ln x)}{y-x} + 2, & p = 1; \\ \frac{y^2\ln y - x^2\ln x}{y-x} - \frac{(2t-x-y)x\ln x}{t-x} - \frac{(y-x)t\ln t}{t-x} - \frac{x+y}{2}, & p = 2. \end{cases}$$

It is easy to see that ψ_1 is a continuous function and similar to Theorem 3.8 we can prove the following theorem.

Theorem 4.8. *Let $0 < x < y < \infty$.*

(i) *For every $t \in (x, \frac{x+y}{2}]$ function ψ_1 defined with* (4.13) *is exponentially convex on $\mathbb{R}$.*
(ii) *For every $t \in (x, \frac{x+y}{2}]$ and for all $t_k \in \mathbb{R}$, $k = 1, 2, ..., n$, matrix $\left[\psi_1\left(\frac{t_i+t_j}{2}\right)\right]_{i,j=1}^n$ is positive semi-definite matrix. Particularly*

$$\det\left[\psi_1\left(\frac{t_i+t_j}{2}\right)\right]_{i,j=1}^n \geq 0.$$

(iii) *For all $t \in (x, \frac{x+y}{2}] \cup \{y\}$ and $p \neq q$*

$$R^{1,t}_{p,q}(x,y) = \left(\frac{\psi_1(p)}{\psi_1(q)}\right)^{\frac{1}{p-q}}.$$

Now, all continuous extensions for $R^{1,t}_{p,q}(x,y)$ are obvious but the cases $p = q$:

$$R^{1,t}_{p,p}(x,y)=$$
$$\exp\left(\frac{3-2p}{(p-1)(p-2)}-\frac{\frac{2(y^p-x^p)}{p^2(y-x)}-\frac{2(y^p\ln y-x^p\ln y)}{p(y-x)}+\frac{2t-x-y}{t-x}x^{p-1}\ln x+\frac{y-x}{t-x}t^{p-1}\ln t}{\frac{2(y^p-x^p)}{p(y-x)}-\frac{2t-x-y}{t-x}x^{p-1}-\frac{y-x}{t-x}t^{p-1}}\right),$$

for $p\neq 0$;

$$R^{1,t}_{0,0}(x,y)=$$
$$\exp\left(\frac{\frac{\ln^2 y-\ln^2 x}{2(y-x)}+\frac{3}{2}\frac{\ln y-\ln x}{y-x}-\frac{2t-x-y}{2x(t-x)}(\ln x+\frac{3}{2})-\frac{y-x}{2t(t-x)}(\ln t+\frac{3}{2})}{\frac{\ln y-\ln x}{y-x}-\frac{2t-x-y}{2x(t-x)}-\frac{y-x}{2t(t-x)}}\right)$$

$$R^{1,t}_{1,1}(x,y)=\exp\left(\frac{\frac{(2t-x-y)\ln^2 x+(y-x)\ln^2 t}{2(t-x)}-\frac{y\ln^2 y-x\ln^2 x}{y-x}}{\frac{(2t-x-y)\ln x+(y-x)\ln t}{t-x}-\frac{2(y\ln y-x\ln x)}{y-x}+2}\right)$$

$$R^{1,t}_{2,2}(x,y)=$$
$$\exp\left(\frac{\frac{(2t-x-y)x(2\ln x-\ln^2 x)+(y-x)t(2\ln t-\ln^2 t)}{2(t-x)}-\frac{3(y^2\ln y-x^2\ln x)}{2(y-x)}+\frac{3(x+y)}{4}}{\frac{y^2\ln y-x^2\ln x}{y-x}-\frac{(2t-x-y)x\ln x}{t-x}-\frac{(y-x)t\ln t}{t-x}-\frac{x+y}{2}}\right)$$

We now continue with means generated from (4.10).

Remark 4.9. *If ϕ_1''/ϕ_2'' has inverse, from* (4.10) *we have*

$$\xi=\left(\frac{\phi_1''}{\phi_2''}\right)^{-1}\left(\frac{\frac{2}{y-x}\int_x^y\phi(t)dt-\frac{y-x}{y-v}\phi(v)-\frac{y+x-2v}{y-v}\phi(y)}{\frac{2}{y-x}\int_x^y\phi(t)dt-\frac{y-x}{y-v}\psi(v)-\frac{y+x-2v}{y-v}\psi(y)}\right)\tag{4.14}$$

We put $\phi_1(x)=x^{p-1},\ \phi_2(x)=x^{q-1}$, in Remark 4.9 and we can define new means. Suppose $x,y>0,\ x\neq y$ and $p\neq q,\ p,q\neq 0,1,2$ are given. Let us define the following expressions

$$R^{2,v}_{p,q}(x,y)=\left(\frac{(q-1)(q-2)}{(p-1)(p-2)}\frac{\frac{2(y^p-x^p)}{p(y-x)}-\frac{y-x}{y-v}v^{p-1}-\frac{y+x-2v}{y-v}y^{p-1}}{\frac{2(y^q-x^q)}{q(y-x)}-\frac{y-x}{y-v}v^{q-1}-\frac{y+x-2v}{y-v}y^{q-1}}\right)^{\frac{1}{p-q}},\tag{4.15}$$

where $v\in[\frac{x+y}{2},y)\cup\{x\}$.
In order to deduce continuous extensions of (4.15), for $v\in[\frac{x+y}{2},y)\cup\{x\}$ we consider the following function

(4.16)

$$\psi_2(p) = \begin{cases} \frac{1}{(p-1)(p-2)}\left(\frac{2(y^p-x^p)}{p(y-x)} - \frac{y-x}{y-v}v^{p-1} - \frac{y+x-2v}{y-v}y^{p-1}\right), & p \neq 0,1,2; \\ \frac{\ln y - \ln x}{y-x} - \frac{y-x}{v(y-v)} - \frac{y+x-2v}{y(y-v)}, & p=0; \\ \frac{(y-x)\ln v + (y+x-2v)\ln y}{y-v} - \frac{2(y\ln y - x\ln x)}{y-x} + 2, & p=1; \\ \frac{y^2\ln y - x^2\ln x}{y-x} - \frac{(y-x)v\ln v}{y-v} - \frac{(y+x-2v)y\ln y}{y-v} - \frac{x+y}{2}, & p=2. \end{cases}$$

It is easy to show that ψ_2 is a continuous function and similar to Theorem 3.8 we can prove the following theorem.

Theorem 4.10. *Let $0 < x < y < \infty$.*

(i) *For every $v \in [\frac{x+y}{2}, y)$ function ψ_2 defined with* (4.16) *is exponentially convex on $\mathbb{R}$.*

(ii) *For every $v \in [\frac{x+y}{2}, y)$ and for all $t_k \in \mathbb{R}$, $k = 1, 2, ..., n$, matrix $\left[\psi_2\left(\frac{t_i+t_j}{2}\right)\right]_{i,j=1}^n$ is positive semi-definite matrix. Particularly*

$$\det\left[\psi_2\left(\frac{t_i+t_j}{2}\right)\right]_{i,j=1}^n \geq 0.$$

(iii) *For all $v \in [\frac{x+y}{2}, y) \cup \{x\}$, and $p \neq q$*

$$R_{p,q}^{2,v}(x,y) = \left(\frac{\psi_2(p)}{\psi_2(q)}\right)^{\frac{1}{p-q}}.$$

Now, all continuous extensions for $R_{p,q}^{2,v}(x,y)$ are obvious but the cases $p = q$:

$$R_{p,p}^{2,v}(x,y) = \exp\left(\frac{3-2p}{(p-1)(p-2)} - \frac{\frac{2(y^q-x^q)}{q^2(y-x)} - \frac{2(y^p\ln y - x^p\ln x)}{p(y-x)} + \frac{y-x}{y-v}v^{q-1}\ln v + \frac{y+x-2v}{y-v}y^{q-1}\ln y}{\frac{2(y^p-x^p)}{p(y-x)} - \frac{y-x}{y-v}v^{p-1} - \frac{y+x-2v}{y-v}y^{p-1}}\right);$$

$$R_{0,0}^{2,v}(x,y) = \exp\left(\frac{\frac{\ln^2 y - \ln^2 x}{2(y-x)} + \frac{3(\ln y - \ln x)}{2(y-x)} - \frac{y-x}{2v(y-v)}(\ln v + 3) - \frac{y+x-2v}{2y(y-v)}(\ln y + 3)}{\frac{\ln y - \ln x}{y-x} - \frac{y-x}{v(y-v)} - \frac{y+x-2v}{y(y-v)}}\right);$$

$$R_{1,1}^{2,v}(x,y) = \exp\left(\frac{\frac{(y-x)\ln v + (y+x-2v)\ln y}{y-v} - \frac{y\ln^2 y - x\ln^2 x}{y-x} + 2}{\frac{(y-x)\ln v + (y+x-2v)\ln y}{y-v} - \frac{2(y\ln y - x\ln x)}{y-x} + 2}\right);$$

$$R^{2,v}_{2,2}(x,y) =$$
$$\exp\left(\frac{\frac{(y-x)v(2\ln v-\ln^2 v)+(y+x-2v)y(2\ln y-\ln^2 y)}{2(y-v)}-\frac{3(y^2\ln y-x^2\ln x)}{2(y-x)}+\frac{3(x+y)}{4}}{\frac{y^2\ln y-x^2\ln x}{y-x}-\frac{(y-x)v\ln v}{y-v}-\frac{(y+x-2v)y\ln y}{y-v}-\frac{x+y}{2}}\right).$$

Similarly to Theorem 3.10 we can prove monotonicity of the new means.

Theorem 4.11. *Let $p \le r,\ q \le s$ and $0 < x < y < \infty$. Then*

(i) *for every $t \in (x, \frac{x+y}{2}] \cup \{y\}$*

$$R^{1,t}_{p,q}(x,y) \le R^{1,t}_{r,s}(x,y); \tag{4.17}$$

(ii) *for every $v \in [\frac{x+y}{2}, y) \cup \{x\}$*

$$R^{2,v}_{p,q}(x,y) \le R^{2,v}_{r,s}(x,y). \tag{4.18}$$

Remark 4.12. *Observe that means $E^1_{p,q}(x,y)$ and $E^2_{p,q}(x,y)$ defined in* (1.3) *and* (1.4) *can be deduced from means $R^{1,t}_{p,q}(x,y)$ and $R^{2,v}_{p,q}(x,y)$:*

(i) *for $t = \frac{x+y}{2}$ we have $R^{1,t}_{p,q}(x,y) = E^1_{p,q}(x,y)$, and for $t = y$ we have $R^{1,t}_{p,q}(x,y) = E^2_{p,q}(x,y)$;*

(ii) *for $v = \frac{x+y}{2}$ we have $R^{2,v}_{p,q}(x,y) = E^1_{p,q}(x,y)$, and for $t = x$ we have $R^{2,v}_{p,q}(x,y) = E^2_{p,q}(x,y)$.*

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