GENERALIZATION OF PERTURBED TRAPEZOID FORMULA AND RELATED INEQUALITIES

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Abstract. We derive some new inequalities for perturbed trapezoid formula and give some sharp and best possible constants.

1. Introduction

A.McD. Mercer has proved the following identity ([1])

\[ \int_{-1}^{1} f(x)dx = \frac{2^n n!}{(2n)!} \sum_{q=0}^{n-1} (-1)^q + 1 \left\{ \left[ f^{(q)}(1) + (-1)^q f^{(q)}(-1) \right] P_n^{(n-1-q)}(1) \right\} \]

(1.1)

\[ = \frac{(-1)^k}{(2n)!} \int_{-1}^{1} f^{(2n-k)}(x)D^k[(x^2 - 1)^n]dx, \]

with \( k = 0, 1, \ldots, n, \) where \( f : [-1, 1] \rightarrow \mathbb{R} \) possesses continuous derivatives of all orders which appear, \( D \) denotes differentiation with respect to \( x, \) and \( P_n(x) \) is the Legendre polynomial of degree \( n. \)

Pečarić and Varošanec ([3]) have considered the following. Let \( \sigma = \{a = x_0 < x_1 < \cdots < x_m = b\} \)

be a subdivision of the interval \([a, b]\) for some \( m \in \mathbb{N}. \) Set

\[ S_n(t, \sigma) = \left\{ \begin{array}{ll}
P_{1n}(t), & t \in [a, x_1] \\
P_{2n}(t), & t \in (x_1, x_2) \\
\vdots \\
P_{mn}(t), & t \in (x_{m-1}, b], 
\end{array} \right. \]

(1.2)

where \( \{P_{jn}\}_n \) are the sequences of harmonic polynomials, i.e. \( P_{jk}'(t) = P_{j,k-1}(t), \) for \( k = 1, \ldots, n \) and \( P_{j0}(t) = 1. \) By successive integration by
parts they have proved that

\[-1^n \int_a^b S_n(t, \sigma) df^{(n-1)}(t) = \int_a^b f(t)dt + \sum_{k=1}^n (-1)^k \left[ P_{mk}(b) f^{(k-1)}(b) \right. \]

\[+ \sum_{j=1}^{m-1} \left( P_{jk}(x_j) - P_{j+1,k}(x_j) \right) f^{(k-1)}(x_j) - P_{1k}(a) f^{(k-1)}(a) \]

whenever the integrals exist. Formula (1.3) is generalized in the following way in [2]. Let us consider subdivision

\[\sigma = \{a = x_0 < x_1 < \cdots < x_m = b\}\]

of the interval \([a, b]\). Further, set

\[T_n(t, \sigma) = \left\{ \begin{array}{ll}
M_{1n}(t), & t \in [a, x_1] \\
M_{2n}(t), & t \in (x_1, x_2] \\
& \vdots \\
M_{mn}(t), & t \in (x_{m-1}, b], 
\end{array} \right. \]

where \(M_{jn}\) are monic polynomials of degree \(n\), for \(j = 1, \ldots, m\). The next theorem has been proved.

**Theorem 1.** Let \(f : [a, b] \rightarrow \mathbb{R}\) be \((n-1)\)-times differentiable function, for some \(n \in \mathbb{N}\). Then the next identity holds

\[\int_a^b f(t)dt + \frac{1}{n!} \sum_{k=0}^{n-1} (-1)^k \cdot \left[ M_{mn}^{(n-k-1)}(b) f^{(k)}(b) + \sum_{j=1}^{m-1} M_{jn}^{(n-k-1)}(x_j) \right] \]

\[- M_{j+1,n}^{(n-k-1)}(x_j) f^{(k)}(x_j) - M_{1n}^{(n-k-1)}(a) f^{(k)}(a) \]

\[= \frac{(-1)^n}{n!} \int_a^b T_n(t, \sigma) df^{(n-1)}(t), \]

whenever the integrals exist.

If we put in (1.5) \(M_{jn} = n! \cdot P_{jn}\), where \(\{P_{jn}\}\) are harmonic polynomials with leading coefficient \(\frac{1}{n!}\), then we will recover relation (1.3), since

\[P_{jn}^{(n-k-1)}(t) = P_{j,k+1}(t),\]

for \(0 \leq k \leq n - 1\).

In this paper we will use the Gamma function

\[\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt,\]
where $x \in \mathbb{R}_+$ and the incomplete Beta function
\[
B(x, a, b) = \int_0^x t^{a-1}(1-t)^{b-1}dt,
\]
where $x, a, b > 0$. In this paper we will show that identity (1.1) is a special case of Theorem 1. Further, we will obtain some sharp and best possible $L_p$ inequalities for quadrature formula in (1.1).

2. **Perturbed Trapezoid Identity**

Let us define polynomial
\[
(2.1) \quad M_{1n}(t) = \frac{(n!)^2}{(2n)!} 2^n P_n(t), \quad t \in [-1, 1].
\]

Since the leading coefficient of $P_n(t)$ equals to $\frac{(2n)!}{2^n(n!)^2}$, the polynomial $M_{1n}$ is monic, so we can apply Theorem 1 with $m = 1$ for some function $f : [-1, 1] \to \mathbb{R}$ with continuous $n$–th derivative. Using the property of the Legendre polynomials
\[
P_n^{(k)}(-t) = (-1)^{n+k} P_n^{(k)}(t),
\]
and Rodrigues formula
\[
D^n[(t^2-1)^n] = 2^n n! P_n(t),
\]
we get from the relation (1.5)
\[
\int_{-1}^{1} f(x)dx + \frac{2^n n!}{(2n)!} \sum_{q=0}^{n-1} (-1)^{q+1} \left\{ \left[ f^{(q)}(1) + (-1)^q f^{(q)}(-1) \right] P_n^{(n-1-q)}(1) \right\}
\]
\[
= \frac{(-1)^n}{(2n)!} \int_{-1}^{1} f^{(n)}(x) D^n[(x^2-1)^n]dx.
\]
In ([1]) is obtained that
\[
(-1)^{k} \int_{-1}^{1} f^{(2n-k)}(x) D^k[(x^2-1)^n]dx = \int_{-1}^{1} f^{(2n)}(x)(x^2-1)^n dx,
\]
for $k = 0, 1, \ldots, n$, so (2.2) becomes (1.1).

3. **Some Inequalities**

**Theorem 2.** Let us suppose $f : [-1, 1] \to \mathbb{R}$ is $(2n-k)$–times differentiable function for some $n \in \mathbb{N}$ and some $k = 0, 1, 2, \ldots, n$. Further,
Let us assume that \( f^{(2n-k)} \in L_p[-1, 1] \), for some \( 1 \leq p \leq \infty \). Then the following inequality holds

\[
\left| \int_{-1}^{1} f(x) \, dx \right| + \frac{2^n n!}{(2n)!} \sum_{j=0}^{n-1} (-1)^{j+1} \left\{ \left[ f^{(j)}(1) \right] + (-1)^j f^{(j)}(-1) \right\} P_{n}^{(n-1-j)}(1) \right\} \right| \leq C(n, k, q) \| f^{(2n-k)} \| _p,
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \) and

\[
C(n, k, q) = \begin{cases} 
\frac{1}{(2n)!} \left[ \int_{-1}^{1} \left| D^k [(x^2 - 1)^n] \right| \, dx \right]^{\frac{1}{q}}, & 1 \leq q < \infty \\
\frac{1}{(2n)!} \sup_{x \in [-1, 1]} \left| D^k [(x^2 - 1)^n] \right|, & q = \infty.
\end{cases}
\]

The inequality is the best possible for \( p = 1 \) and sharp for \( 1 < p \leq \infty \). In the last case equality is attained for the functions of the form

\[
f(x) = M f_*(x) + r_{2n-k-1}(x),
\]

where \( M \in \mathbb{R} \), \( r_{2n-k-1} \) is an arbitrary polynomial of degree at most \( 2n - k - 1 \) and function \( f_* : [-1, 1] \to \mathbb{R} \) is defined by

\[
f_*(x) := \int_{-1}^{x} \frac{(x - \xi)^{2n-k-1}}{(2n - k - 1)!} \text{sgn} D^k [(\xi^2 - 1)^n] d\xi, \text{ for } p = \infty
\]

and for \( 1 < p < \infty \)

\[
f_*(x) := \int_{-1}^{x} \frac{(x - \xi)^{2n-k-1}}{(2n - k - 1)!} \text{sgn} D^k [(\xi^2 - 1)^n] D^k [(\xi^2 - 1)^n]^{\frac{1}{p-1}} d\xi
\]

Proof. We apply H"{o}lder inequality to the relation (1.1) to get

\[
\left| \int_{-1}^{1} f(x) \, dx \right| + \frac{2^n n!}{(2n)!} \sum_{j=0}^{n-1} (-1)^{j+1} \left\{ \left[ f^{(j)}(1) \right] + (-1)^j f^{(j)}(-1) \right\} P_{n}^{(n-1-j)}(1) \right\} \right| \leq \frac{1}{(2n)!} \| D^k [(x^2 - 1)^n] \| _q \| f^{(2n-k)} \| _p.
\]

Obviously, \( C(n, k, q) = \frac{1}{(2n)!} \| D^k [(x^2 - 1)^n] \| _q \), so we obtain relation (3.1).

For the proof of sharpness we need to find function \( f \) such that

\[
\frac{1}{(2n)!} \left| \int_{-1}^{1} D^k [(x^2 - 1)^n] f^{(2n-k)}(x) \, dx \right| = C(n, k, q) \cdot \| f^{(2n-k)} \| _p,
\]
where $1 < p \leq \infty$. The function $f_*$ defined by (3.2) and (3.3) is $(2n - k)$-times differentiable and $f_*^{(2n-k)} \in L_p[-1, 1]$. Further, $f_*$ is a solution of the differential equation

$$D^k[(x^2 - 1)^n]f^{(2n-k)}(x) = |D^k[(x^2 - 1)^n]|^q,$$

so the above identity holds.

For $p = 1$ we shall prove that

$$(3.4)$$

$$\left| \int_{-1}^{1} D^k[(x^2 - 1)^n] f^{(2n-k)}(x) dx \right| \leq \sup_{x \in [-1,1]} |D^k[(x^2 - 1)^n]| \cdot \int_{-1}^{1} |f^{(2n-k)}(x)| dx$$

is the best possible inequality. Suppose that $|D^k[(x^2 - 1)^n]|$ attains its maximum at point $x_0 \in [-1, 1]$. First, let us assume that $D^k[(x_0^2 - 1)^n] > 0$. For $\epsilon$ small enough define $f^{(2n-k-1)}_\epsilon(x)$ by

$$f^{(2n-k-1)}_\epsilon(t) = \begin{cases} 0, & x \leq x_0 \\ \frac{x-x_0}{\epsilon}, & x \in [x_0, x_0 + \epsilon] \\ 1, & x \geq x_0 + \epsilon. \end{cases}$$

Then, for $\epsilon$ small enough,

$$\left| \int_{-1}^{1} D^k[(x^2 - 1)^n] f^{(2n-k)}_\epsilon dx \right| = \left| \int_{x_0}^{x_0+\epsilon} D^k[(x^2 - 1)^n] \frac{1}{\epsilon} dx \right| = \frac{1}{\epsilon} \int_{x_0}^{x_0+\epsilon} D^k[(x^2 - 1)^n] dx.$$

Now, relation (3.4) implies

$$\frac{1}{\epsilon} \int_{x_0}^{x_0+\epsilon} D^k[(x^2 - 1)^n] dx \leq \frac{1}{\epsilon} D^k[(x_0^2 - 1)^n] \int_{x_0}^{x_0+\epsilon} dt = D^k[(x_0^2 - 1)^n].$$

Since

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{x_0}^{x_0+\epsilon} D^k[(x^2 - 1)^n] dx = D^k[(x_0^2 - 1)^n],$$

the statement follows. The case $D^k[(x_0^2 - 1)^n] < 0$ follows similarly.

**Remark 1.** For $n \in \mathbb{N}$ we have by direct calculation

$$C(n, 0, q) = \frac{1}{(2n)!} \left[ \frac{\sqrt{\pi} \Gamma(nq + 1)}{\Gamma(\frac{3}{2} + nq)} \right]^{\frac{1}{q}}, \quad 1 \leq q < \infty, \quad C(n, 0, \infty) = \frac{1}{(2n)!}$$
and
\[ C(n, n, 2) = \frac{2n+1\, n!}{(2n + 1)!}, \quad C(n, n, \infty) = \frac{2^n \, n!}{(2n)!}. \]

Further,
\[ C(1, 1, q) = \left( \frac{2}{q + 1} \right)^{\frac{1}{q}}, \quad 1 \leq q < \infty, \quad C(1, 1, \infty) = 1, \]
\[ C(2, 1, q) = \frac{1}{3 \cdot 2^{1/q}} \left( \frac{\Gamma \left( \frac{1+q}{2} \right) \Gamma (1 + q)}{\Gamma \left( \frac{3(1+q)}{2} \right)} \right)^{\frac{1}{q}}, \quad 1 \leq q < \infty, \quad C(2, 1, \infty) = \frac{\sqrt{3}}{27}, \]
and
\[ C(2, 2, q) = \frac{1}{6} \left( (-1)^q \left( (-1 + (-1)^q) \sqrt{\pi} \Gamma (1 + q) + B(3, \frac{1}{2}, 1 + q) \Gamma \left( \frac{3}{2} + q \right) \right) \right)^{\frac{1}{q}}, \]
for \( 1 \leq q < \infty, \) and
\[ C(2, 2, \infty) = \frac{1}{3}. \]

Specially,
\[ C(2, 2, 1) = \frac{4\sqrt{3}}{27}, \]
which coincides with constants obtained in [4]. For \( n = 3 \) we have the following constants
\[ C(3, 1, q) = \frac{1}{120} \left( \frac{\Gamma \left( \frac{1+q}{2} \right) \Gamma (1 + q)}{\Gamma \left( \frac{3+3q}{2} \right)} \right)^{\frac{1}{q}}, \quad 1 \leq q < \infty \]
and \( C(3, 1, \infty) = \frac{2\sqrt{5}}{1875}. \)

The case \( k = 0 \) in (1.1) is of special interest since function \( (x^2 - 1)^n \) doesn’t change sign on \([-1, 1]\) for every \( n \in \mathbb{N} \). More precisely, \( (x^2 - 1)^n \geq 0 \) for even \( n \) and \( (x^2 - 1)^n \leq 0 \) for odd \( n \). So we have the following

**Theorem 3.** Let us suppose \( f : [-1, 1] \to \mathbb{R} \) is such that \( f^{(2n)} \) is continuous function on \([-1, 1]\) for some \( n \in \mathbb{N} \). Then there exists \( \eta \in (-1, 1) \) such that
\[
\int_{-1}^{1} f(x) \, dx + \frac{2^n \, n!}{(2n)!} \sum_{q=0}^{n-1} (-1)^{q+1} \left\{ [f^{(q)}(1) + (-1)^q f^{(q)}(-1)] P_n^{(n-1-q)}(1) \right\} = (-1)^n \sqrt{\pi n!} \frac{1}{(2n)! \Gamma \left( \frac{3}{2} + n \right)} \cdot f^{(2n)}(\eta).
\]

(3.5)
Proof. The proof follows from the integral mean value theorem applied to the right-hand side of (1.1) with $k = 0$, since $(x^2 - 1)^n$ does not change sign on $[-1, 1]$. So there exists some $\eta \in (-1, 1)$ such that

$$\frac{1}{(2n)!} \int_{-1}^{1} f^{(2n)}(x)(x^2 - 1)^n dx = \frac{f^{(2n)}(\eta)}{(2n)!} \cdot \int_{-1}^{1} (x^2 - 1)^n dx$$

$$= \frac{(-1)^n \sqrt{\pi}n!}{(2n)!\Gamma(\frac{3}{2} + n)} \cdot f^{(2n)}(\eta).$$

Remark 2. Applying previous theorem for $n = 1, 2, 3$ respectively, we get the following identities:

$$\int_{-1}^{1} f(x)dx - [f(1) + f(-1)] = -\frac{2}{3} f''(\eta),$$

which is identity related to the famous trapezoid formula,

$$\int_{-1}^{1} f(x)dx - [f(1) + f(-1)] + \frac{1}{3}[f'(1) - f'(-1)] = \frac{2}{45} f^{(4)}(\eta),$$

and

$$\int_{-1}^{1} f(x)dx - [f(1) + f(-1)] + \frac{2}{5}[f'(1) - f'(-1)]$$

$$- \frac{1}{15} [f''(1) + f''(-1)] = -\frac{2}{1575} f^{(6)}(\eta).$$

References


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Abstract. In this article some new inequalities for convex functions are proved from which some other known inequalities for log-convex, 3-convex and 3-log-convex functions are derived.

Key words and phrases: Inequalities, convex function, log-convex function, 3-log-convex function.

Introduction

The function \( f \) is called \( n \)-convex on the interval \((a, b)\) if its \( n \)-th derivative \( f^{(n)}(t) \) is positive for all \( t \in (a, b) \). Especially, using this terminology, convex function is called 2-convex function. Moreover, the function is called \( n \)-log-convex function if \( f \) is positive and \((\ln f(t))^{(n)} \) is positive for all \( t \in (a, b) \).

Also, let's introduce the usual notation \( g(a+) \) for \( \lim_{x \to a^+} g(x) \) and \( g(b-) \) for \( \lim_{x \to b^-} g(x) \).

The aim of this article is to establish some basic result for convex functions which can be easily used for obtaining many other results.

In the introduction, let's remind on some results for 3-log-convex functions given in [1]:

**Theorem A.** Suppose that \( f(x) > 0 \) for \( x \in (a, b) \) and let \( h = f'f \) is twice differentiable and \( h''(x) > 0 \). Set \( R(x) = f(a+b-x)f(x) \). Then, for all \( x \in (a, b) \), the following inequalities hold

\[
R(b-2h(a+b)(b-x)) \leq R(x) \leq R(a+)2h(a+b)(a-x).
\]

and

\[
R(a+)e(h(a+)+h(b-))(a-x) \leq R(x) \leq R(b-)e(h(a+)+h(b-))(b-x).
\]

This result will be also obtained, in a different way, as a consequence of our main result theorem (Theorem 1) for convex function.