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Family of triangles and related curves

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Abstract. The article observes a one-parameter triangle family $\mathcal{T}$. We prove that the sets of the orthocenters, centroids, circumcenters and the midpoints of the variable triangle side of the triangle family $\mathcal{T}$ lie on four different hyperbolae. Furthermore, there is constructed an envelope k_3^4 of the perpendicular bisectors of the variable triangle sides. Also it is constructed a bicircular rational quartic as an envelope of the circumcircles of the triangle family $\mathcal{T}$.

Key words: Triangle centers, family of triangles, envelope of lines, envelope of circles

MSC2010: 51M04, 51M15

1. Introduction

Many mathematicians have been occupied by theorems that imply the connection between a triangle and a line, or a circle or an another curve of triangle points [1]. Although it seems that this subject is explored, it is possible to extend some of the questions to the families of triangles and curves that are associated with them. These geometrical problems connected to the families of figures or curves can show the benefits of using dynamic software in geometry. In this paper we will express and prove a few theorems about one triangle family and curves associated with it.

2. A family of triangles and related curves

Let a and b be two lines and M their intersection point. Let (O) be a pencil of lines where its vertex O is an arbitrary point that does not lie on the line a or b. The lines of the pencil (O) are denoted by x_i for $i \in I$, (card $I = \aleph_1$). Hereafter the intersections between the line x_i and the given lines a and b are denoted by $P_i = x_i \cap a$ and $N_i = x_i \cap b$ (see Fig. 1).

A one-parameter family $\mathcal{T}$ of the triangles MN_iP_i, $i \in I$, will be studied in this paper. To each triangle of the family $\mathcal{T}$ we can assign

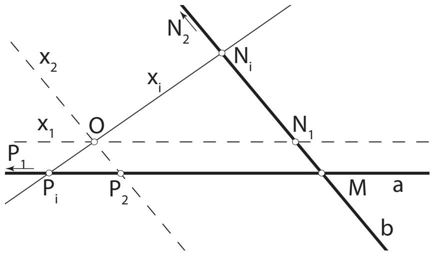

Figure 1.

the triangle centers (triangle centroid, incenter, circumcenter, orthocenter...) as well as some special circles and lines. The sets of such elements of the triangle family $\mathcal{T}$ create interesting curves and envelopes of circles and lines.

Before we proceed to the theorems let us point out and mark three special triangles of $\mathcal{T}$ which will be important for the following proofs. The line of the pencil (O) passing through the point M defines the triangle of $\mathcal{T}$ which degenerates into the point M. Furthermore, the line x_1 of the pencil (O) parallel to the line a defines the triangle MN_1P_1 whose vertex P_1 is the point at infinity of the line a. Similarly, the line x_2 of the pencil (O) parallel to the line b defines the triangle MN_2P_2 whose vertex N_2 is the point at infinity of the line b (see Fig. 1).

Theorem 1. *The orthocenters H_i of the triangles MN_iP_i of the family $\mathcal{T}$ lie on a hyperbola h (see Fig. 2).*

Proof: Every line x_i of the pencil (O) determines the triangle MN_iP_i of $\mathcal{T}$, and its orthocenter H_i is the intersection of the perpendiculars to the lines a and b from the vertices N_i and P_i, respectively. The pairs of such altitudes of all triangles MN_iP_i of $\mathcal{T}$ are the lines of two projective pencils of lines whose vertices are the points at infinity of the altitudes. The result of this projectivity is a hyperbola h. It is obvious that the hyperbola h passes through the point M and intersects the lines a and b in the points that are the vertices at the right angle of the triangles from $\mathcal{T}$ when $x_i \in (O)$ is perpendicular on a or b.

For the special triangle MN_1P_1 of $\mathcal{T}$, the altitude from the vertex P_1 is the line at infinity, and it intersects the altitude through the vertex N_1 in the point at infinity which is the orthocenter of the triangle MN_1P_1. Thereby the altitude through the vertex N_1 is one asymptote of the hyperbola h. The other asymptote is similarly determined by the special triangle MN_2P_2 of $\mathcal{T}$. □

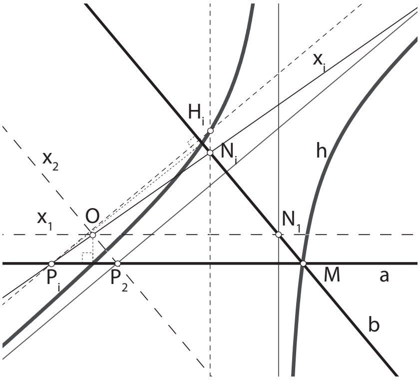

Figure 2.

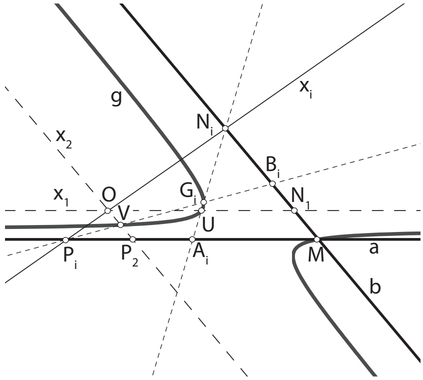

Figure 3.

Theorem 2. *The triangle centroids G_i of the triangles MN_iP_i of the family $\mathcal{T}$ lie on a hyperbola g (see Fig. 3).*

Proof: The correspondence between the ranges of points (a) and (b) is established in the following way: the vertex N_i of the triangle MN_iP_i corresponds with the midpoint A_i of the opposite side MP_i. It is obvious that these two ranges (a) and (b) are in perspectivity, and let the center of this perspective transformation be denoted by U. Similarly, there is a correspondence between the vertex P_i and the midpoint B_i of the opposite triangle side MN_i that define one other perspectivity between the ranges (a) and (b). Let its center be denoted by V. Since it is obvious that the mentioned midpoints A_i and B_i of the triangles MN_iP_i are defining the third perspectivity between the ranges (a) and (b), it follows that the medians are the lines of two projective pencils of lines with the vertices U and V, therefore the resulting locus is a conic. The medians of the special triangle MN_1P_1 intersect in the point at infinity of the line a. Similarly, the pair of medians associated with the other special triangle MN_2P_2 intersect in the point at infinity of the line b. So the resulting conic is a hyperbola g whose asymptotes are parallel with the given lines a and b. □

Theorem 3. *The midpoints L_i of the variable side N_iP_i of the triangles MN_iP_i of the family $\mathcal{T}$ lie on a hyperbola p (see Fig. 4).*

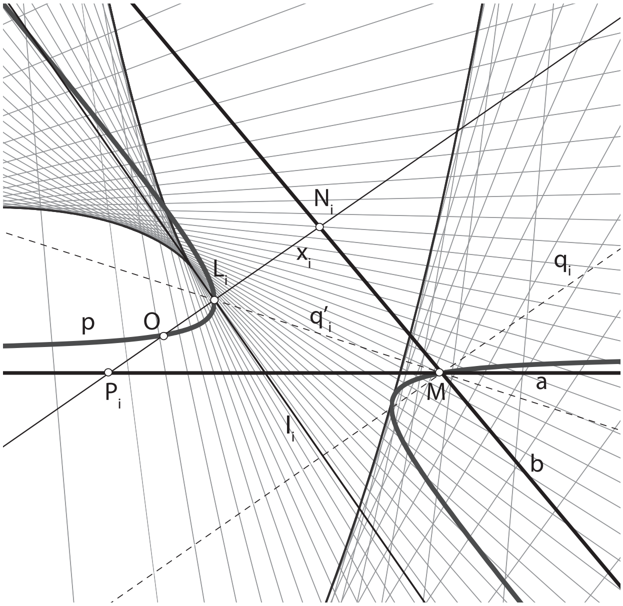

Figure 4.

Proof: The lines a and b are the lines of the pencil (M). Suppose they are the double lines of an involution. For each line $q_i \in (M)$ there is a line $x_i \in (O)$ parallel to it which defines the triangle MN_iP_i of $\mathcal{T}$. It is well-known that a pair of corresponding lines are harmonically related to the fixed lines of an involution [2]. Therefore, the line $q_i \in (M)$ is associated to the line q_i' which passes through the midpoint L_i of the side N_iP_i of the triangle MN_iP_i. It follows that the set of the midpoints L_i of the segments N_iP_i is the result of the projectivity between the pencils of lines (O) and (M). Hence it is a conic through the points O and M. Since in that projectivity the lines $a, b \in (M)$ are related to the lines of the pencil (O) which are respectively parallel to the lines a and b, it follows that the points at infinity of the lines a and b belong to the conic. Thereby the resulting conic is a hyperbola p. $\square$

Remark. The hyperbola p from Theorem 3. is related to the hyperbola g from Theorem 2. due to homothecy with the coefficient $\frac{2}{3}$ and the centerpoint at the point M. Those hyperbolae have four common elements: two points at infinity and the point M counted twice because they have a common tangent at the point M. Hence they determine a pencil of hyperbolae.

Theorem 4. *The envelope of the perpendicular bisectors l_i of the variable sides N_iP_i of the triangles MN_iP_i of the family $\mathcal{T}$ is a curve of the fourth order and third class (see Fig. 4). This curve coincides with the negative pedal curve of the hyperbola p in Theorem 3. The pole of this negative pedal transformation is the point O of the hyperbola p.*

The proof of this theorem can be found in [5]. A similar curve, an envelope of so-called butterfly lines associated with a complete quadrangle inscribed in a circle has been studied in [4].

Theorem 5. *The circumcenters O_i of the triangles MN_iP_i of the family $\mathcal{T}$ lie on a hyperbola s (see Fig. 5).*

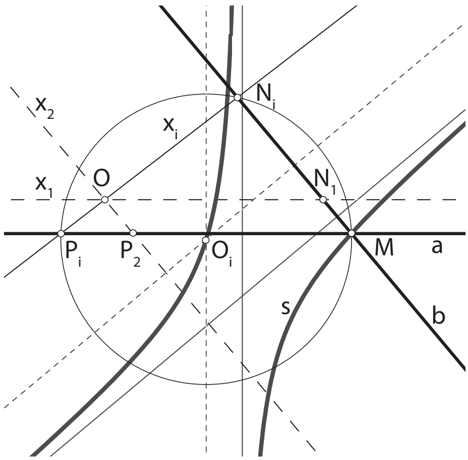

Figure 5.

Proof: The circumcenter O_i of the triangles MN_iP_i are the intersection of the perpendicular bisectors of the triangle sides MN_i and MP_i. Since the lines a and b are fixed, all those bisectors are the rays of two projective pencils of lines. The vertices of this pencils are the points at infinity of the perpendiculars on the fixed lines a and b. The result of this projectivity is a conic that passes through the vertices of the projective pencils. Hence it is a hyperbola s. It is obvious that the point M is a point of the hyperbola as the circumcenter of the degenerated triangle of $\mathcal{T}$ which degenerates into the point M. The hyperbola also passes through the points of the lines a and b which are the circumcenteres of those right triangles of $\mathcal{T}$ whose hypotenuse lies on a or b. Analogously to the proof of Theorem 1. it is obvious that the circumcenters of the special triangles MN_1P_1 and MN_2P_2 are the points at infinity of the perpendicular bisectors of the regular sides of those triangles respectively. Therefore those bisectors are the asymptotes of the resulting hyperbola s. □

Corollary 1. *If the given lines a and b are perpendicular then the curves g, p and s are equilateral hyperbolae. In this case the curve h degenerates into the lines a and b.*

Theorem 6. *The envelope of the circumcircles of the triangles MN_iP_i of the family $\mathcal{T}$ is a bicircular curve of the fourth order and fifth class. It touches the lines a and b and has a cusp at the point M (see Fig. 6).*

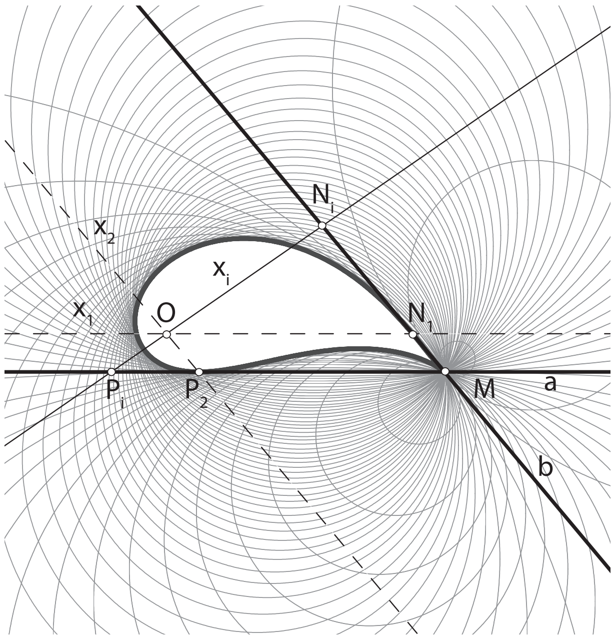

Figure 6.

Proof: To determine the order and class of the envelope of circumcircles we will use the inversion with respect to any circle c with the center at the point M. For every triangle MN_iP_i of $\mathcal{T}$ the circumcircle k_i passes through the point M. Hereafter it follows that the inverse image of the circumcircle k_i is a line $k_i^{'}$ through the intersections of the circumcircle k_i and the circle c. Thereby the envelope of circumcircles is transformed into a envelope of lines (see Fig. 7). The lines a and b are the rays of this inversion so they transform into themselves, and the point M as the degenerate circumcircle of $\mathcal{T}$ transforms into the line at infinity.

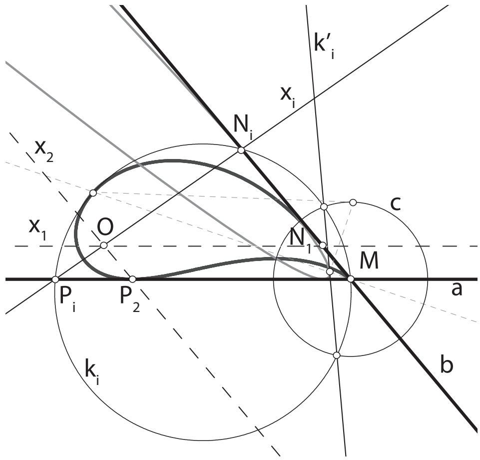

Figure 7.

Let the point X' be the inverse image of an arbitrary point X of the plain. Then the bisector of the line segment MX' intersects the hyperbola s in two points that are two circumcenters of $\mathcal{T}$. Therefore those two circumcircles of the circle envelope passes through the point X', and their inverse images are two lines through the point X. Hence the envelope of the line has two tangents through every point X of the plain, so it is a curve of the second class. Whereas the line at infinity is also a tangent to the line envelope, thus it is a parabola. It is known that an inversion is a quadratic transformation that transforms a parabola into a bicircular curve with a cusp at the pole of the inversion, so the envelope of circumcircles is a curve of the fourth order. Furthermore, this curve is a bicircular curve and it has only one cusp therefore according to the Plücker's equations it is the fifth class [3], [6].

Moreover, the lines a and b are the parts of two splitting circumcircles for the triangles MN_1P_1 and MN_2P_2. Therefore the lines a and b are the tangents to the envelope and the contact points are P_2 and N_1, respectively.

□

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