I–limit superior and limit inferior

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Abstract. In this paper we extend concepts of statistical limit superior and inferior (as introduced by Fridy and Orhan) to I–limit superior and inferior and give some I–analogue of properties of statistical limit superior and inferior for a sequence of real numbers. Also we extend the concept of the statistical core to I–core for a complex number sequence and get necessary conditions for a summability matrix A to yield I–core \{Ax\} ⊆ I–core \{x\} whenever x is a bounded complex number sequence.

Key words: statistical limit superior and inferior, statistical core of a sequence, I–convergent sequence

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1. Introduction

If K is a subset of natural numbers \(\mathbb{N}\), \(K_n\) will denote the set \(\{k \in K : k \leq n\}\) and |\(K_n\)| will denote the cardinality of \(K_n\). Natural density of \(K\) [20], [13] is given by \(\delta(K) := \lim_{n \to \infty} \frac{1}{n} |K_n|\), if it exists. Fast introduced the definition of a statistical convergence using the natural density of a set. The number sequence \(x = (x_k)\) is statistically convergent to \(L\) provided that for every \(\varepsilon > 0\) the set \(K := K(\varepsilon) := \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}\) has natural density zero [7], [9]. Hence \(x\) is statistically convergent to \(L\) iff \((C_1 \chi_K(\varepsilon))_n \to 0, (as n \to \infty, for every \varepsilon > 0)\), where \(C_1\) is the Cesàro mean of order one and \(\chi_K\) is the characteristic function of the set \(K\). Properties of statistically convergent sequences have been studied in [1], [2], [9], [18], [21].

Statistical convergence can be generalized by using a nonnegative regular summability matrix \(A\) in place of \(C_1\).

Following Freedman and Sember [8], we say that a set \(K \subseteq \mathbb{N}\) has A–density if \(\delta_A(K) := \lim_n (A\chi_K)_n = \lim_n \sum_{k \in K} a_{nk}\) exists where \(A = (a_{nk})\) is a nonnegative regular matrix.

The number sequence \(x = (x_k)\) is A–statistically convergent to \(L\) provided that for every \(\varepsilon > 0\) the set \(K(\varepsilon)\) has A–density zero[2], [8], [18].

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Fridy [10] has introduced the notions of a statistical limit point and a cluster point. Fridy and Orhan [11] studied the idea of statistical limit superior and inferior. Connor and Kline [4] and Demirci [6] extended these concepts to $A$-statistical convergence using a nonnegative regular summability matrix $A$ in place of $C_1$. Also Connor has introduced a $\mu$-statistical analogue of these concepts using a finitely additive set function $\mu$ taking values in $[0, 1]$ defined on a field $\Gamma$ of subsets of $\mathbb{N}$ such that if $|A| < \infty$, then; if $A \subset B$ and $\mu(B) = 0$, then $\mu(A) = 0$; and $\mu(\mathbb{N}) = 1$ [3], [5].

The number sequence $x = (x_k)$ is $\mu$-statistically convergent to $L$ provided that $\mu(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0$ for every $\varepsilon > 0$ [3], [5].

Kostyrko, Maćaj and Šalát [15], [16] introduced the concepts of $I$-convergence, $I$-limit point and $I$-cluster point of sequences of real numbers based on the notion of the ideal of subsets of $\mathbb{N}$.

In this paper we extend concepts of statistical limit superior and inferior to $I$-limit superior and inferior and give some properties of $I$-limit superior and inferior for a sequence of real numbers. We also extend the concept of a statistical core to $I$-core for a complex number sequence and get necessary conditions for a summability matrix $A$ to yield $I$-core $\{Ax\} \subseteq I$-core $\{x\}$ whenever $x$ is a bounded complex number sequence.

2. Definition and notations

We first recall the concepts of an ideal and a filter of sets.

**Definition 1.** Let $X \neq \emptyset$. A class $S \subseteq 2^X$ of subsets of $X$ is said to be an ideal in $X$ provided that $S$ is additive and hereditary, i.e. if $S$ satisfies the conditions:

(i) $\emptyset \in S$,
(ii) $A, B \in S \Rightarrow A \cup B \in S$,
(iii) $A \in S, B \subseteq A \Rightarrow B \in S$

[17], p.34).

An ideal is called non-trivial if $X \notin S$.

**Definition 2.** Let $X \neq \emptyset$. A non-empty class $F \subseteq 2^X$ of subsets of $X$ is said to be a filter in $X$ provided that:

(i) $\emptyset \in F$,
(ii) $A, B \in F \Rightarrow A \cap B \in F$,
(iii) $\emptyset \in F, A \subseteq B \Rightarrow B \in F$

[19], p.44).

The following proposition expresses a relation between the notions of an ideal and a filter:

**Proposition 1.** Let $S$ be non-trivial in $X$, $X \neq \phi$. Then the class

$F(S) = \{M \subseteq X : \exists A \in S : M = X \setminus A\}$

is a filter on $X$ (we will call $F(S)$ the filter associated with $S$) [15].

**Definition 3.** A non-trivial ideal $S$ in $X$ is called admissible if $\{x\} \in S$ for each $x \in X$ [15].

As usual, $\mathbb{R}$ will denote real numbers and $\mathbb{C}$ complex numbers.
Definition 4. Let $\mathcal{I}$ be a non-trivial ideal in $\mathbb{N}$. Then

(i) A sequence $x = (x_n)$ of real numbers is said to be $\mathcal{I}$–convergent to $L \in \mathbb{R}$ if for every $\varepsilon > 0$ the set $A(\varepsilon) = \{n : |x_n - L| \geq \varepsilon\}$ belongs to $\mathcal{I}$ [15]. In this case we write $\mathcal{I}$–lim $x = L$.

(ii) An element $\xi \in \mathbb{R}$ is said to be $\mathcal{I}$–limit point of the real number sequence $x = (x_n)$ provided that there exists a set $M = \{m_1 < m_2 < \ldots\} \subset \mathbb{N}$ such that $M \notin \mathcal{I}$ and $\lim_{k} x_{m_k} = \xi$ [16].

(iii) An element $\xi \in \mathbb{R}$ is said to be $\mathcal{I}$–cluster point of the real number sequence $x = (x_n)$ iff for each $\varepsilon > 0$ we have $\{k : |x_k - \xi| < \varepsilon\} \notin \mathcal{I}$ [16].

Note that the set of $\mathcal{I}$–cluster points of $x$ is a closed point set in $\mathbb{R}$ where $\mathcal{I}$ is an admissible ideal [15].

Some results on $\mathcal{I}$–convergence, $\mathcal{I}$–limit point and $\mathcal{I}$–cluster point may be found in [15],[16].

Throughout the paper $\mathcal{I}$ will be an admissible ideal.

3. $\mathcal{I}$–limit superior and inferior

In this section we study the concepts of $\mathcal{I}$–limit superior and inferior for a real number sequence.

For a real number sequence $x = (x_k)$ let $B_x$ denote the set

$$B_x := \{b \in \mathbb{R} : \{k : x_k > b\} \notin \mathcal{I}\}.$$ 

Similarly,

$$A_x := \{a \in \mathbb{R} : \{k : x_k < a\} \notin \mathcal{I}\}.$$ 

We begin with a definition.

Definition 5. Let $\mathcal{I}$ be an admissible ideal and $x$ a real number sequence. Then the $\mathcal{I}$–limit superior of $x$ is given by

$$\mathcal{I}$–lim sup x := \begin{cases} \sup B_x, & \text{if } B_x \neq \emptyset, \\ -\infty, & \text{if } B_x = \emptyset. \end{cases}$$

Also, the $\mathcal{I}$–limit inferior of $x$ is given by

$$\mathcal{I}$–lim inf x := \begin{cases} \inf A_x, & \text{if } A_x \neq \emptyset, \\ +\infty, & \text{if } A_x = \emptyset. \end{cases}$$

Note that if we define $\mathcal{I} = \{K \subseteq \mathbb{N} : \delta_A(K) = 0\}$, $\mathcal{I} = \{K \subseteq \mathbb{N} : \delta(K) = 0\}$ and $\mathcal{I} = \{K \in \Gamma : \mu(K) = 0\}$ in Definition 5, then we get Definition 1 of [6], Definition 1 of [11] and Connor’s definitions [5] of $\mu$–statistical superior and inferior, respectively. This observation suggests the following result which can be proved by a straightforward least upper bound argument.

Theorem 1. If $\beta = \mathcal{I}$–lim sup $x$ is finite, then for every positive number $\varepsilon$

$$\{k : x_k > \beta - \varepsilon\} \notin \mathcal{I}$$

and

$$\{k : x_k > \beta + \varepsilon\} \in \mathcal{I}.$$ 

Conversely, if (1) holds for every positive $\varepsilon$, then $\beta = \mathcal{I}$–lim sup $x$.

The dual statement for $\mathcal{I}$–lim inf $x$ is as follows.
Theorem 2. If $\alpha = \mathcal{I}^-$lim inf $x$ is finite, then for every positive $\varepsilon$

$$\{k : x_k < \alpha + \varepsilon\} \notin \mathcal{I} \text{ and } \{k : x_k < \alpha - \varepsilon\} \in \mathcal{I}. \quad (2)$$

Conversely, if (2) holds for every positive $\varepsilon$, then $\alpha = \mathcal{I}^-$lim inf $x$.

Considering the definition of $\mathcal{I}^-$cluster point in Definition 4 we see that Theorems 1 and 2 can be interpreted as saying that $\mathcal{I}^-$lim sup $x$ and $\mathcal{I}^-$lim inf $x$ are the greatest and the least $\mathcal{I}^-$cluster points of $x$.

Now we have the following

Theorem 3. For any real number sequence $x$,

$$\mathcal{I}^- \text{lim inf } x \leq \mathcal{I}^- \text{lim sup } x.$$ 

Proof. First consider the case in which $\mathcal{I}^- \text{lim sup } x = -\infty$. Hence we have $B_x = \phi$, so for every $b$ in $\mathbb{R}$, $\{k : x_k > b\} \in \mathcal{I}$ which implies that $\{k : x_k \leq b\} \in \mathcal{F}(\mathcal{I})$ so for every $a \in \mathbb{R}$, $\{k : x_k \leq a\} \notin \mathcal{I}$. Hence $\mathcal{I}^-$lim inf $x = -\infty$.

The case in which $\mathcal{I}^- \text{lim sup } x = +\infty$ needs no proof, so we next assume that $\beta = \mathcal{I}^- \text{lim sup } x$ is finite, and $\alpha := \mathcal{I}^- \text{lim inf } x$. Given $\varepsilon > 0$ we show that $\beta + \varepsilon \in A_\alpha$, so that $\alpha \leq \beta + \varepsilon$. By Theorem 1, $\{k : x_k > \beta + \varepsilon\} \in \mathcal{I}$ because $\beta = \text{ lub } B_x$. This implies $\{k : x_k \leq \beta + \frac{\varepsilon}{2}\} \subseteq \{k : x_k < \beta + \varepsilon\}$ and $\mathcal{F}(\mathcal{I})$ is a filter on $\mathbb{R}$, $\{k : x_k < \beta + \varepsilon\} \in \mathcal{F}(\mathcal{I})$. This implies $\{k : x_k < \beta + \varepsilon\} \notin \mathcal{I}$. Hence $\beta + \varepsilon \in A_\alpha$. By definition $\alpha = \text{ inf } A_\alpha$, so we conclude that $\alpha \leq \beta + \varepsilon$; and since $\varepsilon$ is arbitrary this proves that $\alpha \leq \beta$.

From Theorem 3 and Definition 5, it is clear that

$$\text{lim inf } x \leq \mathcal{I}^- \text{lim inf } x \leq \mathcal{I}^- \text{lim sup } x \leq \text{ lim sup } x$$

for any real number sequence $x$.

$\mathcal{I}^-$limit point of a sequence $x$ is defined in (ii) of Definition 4 as the limit of a subsequence of $x$ whose indices do not belong to $\mathcal{I}$. We cannot say that $\mathcal{I}^- \text{lim sup } x$ is equal to the greatest $\mathcal{I}^- \text{limit points of } x$. This can be seen from Example 4 in [11] where $\mathcal{I} = \{K \subseteq \mathbb{N} : \delta(K) = 0\}$.

Definition 6. The real number sequence $x = (x_k)$ is said to be $\mathcal{I}^-$bounded if there is a number $B$ such that $\{k : |x_k| > B\} \notin \mathcal{I}$.

Note that $\mathcal{I}^-$boundedness implies that $\mathcal{I}^- \text{lim sup } x$ and $\mathcal{I}^-$lim inf are finite, so properties (1) and (2) of Theorems 1 and 2 hold.

Theorem 4. The $\mathcal{I}^-$bounded sequence $x$ is $\mathcal{I}^-$convergent if and only if

$$\mathcal{I}^-$lim inf $x = \mathcal{I}^-$lim sup $x.$$

Proof. Let $\alpha := \mathcal{I}^- \text{lim inf } x$ and $\beta := \mathcal{I}^- \text{lim sup } x$. First suppose that $\mathcal{I}^- \text{lim } x = L$ and $\varepsilon > 0$. Then $\{k : |x_k - L| \geq \varepsilon\} \in \mathcal{I}$, so $\{k : x_k > L + \varepsilon\} \in \mathcal{I}$, which implies that $\beta \leq L$. We also have $\{k : x_k < L - \varepsilon\} \in \mathcal{I}$, which yields that $L \leq \alpha$. Therefore $\beta \leq \alpha$. Combining this with Theorem 3 we conclude that $\alpha = \beta$.

Now assume $\alpha = \beta$ and define $L := \alpha$. If $\varepsilon > 0$ then (1) and (2) of Theorem 1 and 2 imply $\{k : x_k > L + \frac{\varepsilon}{2}\} \in \mathcal{I}$ and $\{k : x_k < L - \frac{\varepsilon}{2}\} \in \mathcal{I}$. Hence $\mathcal{I}^- \text{lim } x = L$. \qed
4. \(I\)-core

In [11] Fridy and Orhan introduced the concept of the statistical core of a real number sequence, and proved the statistical core theorem. Those results have also been extended to the complex case too [12]. Using the same technique as in [12], we introduce the concept of \(I\)-core of a complex sequence and get necessary conditions for a summability matrix \(A\) to yield \(I\)-core \(\{Ax\} \subseteq I\)-core \(\{x\}\) whenever \(x\) is a bounded complex number sequence.

In this section \(x, y\) and \(z\) will denote complex number sequences and \(A = (a_{nk})\) will denote an infinite matrix of complex entries which transforms a complex number sequence \(x = (x_k)\) into the sequence \(Ax\) whose \(n\)-th term is given by \((Ax)_n = \sum_{k=1}^{\infty} a_{nk}x_k\).

In [14] the Knopp core of the sequence \(x\) is defined by

\[K-\text{core} \{x\} := \cap_{n \in \mathbb{N}} C_n(x),\]

where \(C_n(x)\) is the closed convex hull of \(\{x_k\}_{k \geq n}\). In [22] it is shown that for every bounded \(x\)

\[K-\text{core} \{x\} := \cap_{z \in \mathbb{C}} B_z^+(x),\]

where \(B_z^+(x) := \{ w \in \mathbb{C} : |w - z| \leq \limsup_k |x_k - z| \} \).

The next definition is an \(I\)-analogue of statistical core [12] of a sequence.

**Definition 7.** Let \(I\) be an admissible ideal in \(\mathbb{N}\). For any complex sequence \(x\) let \(H_I(x)\) be the collection of all closed half-planes that contain \(x_k\) for \(I\)-a.a. \(k\); i.e.,

\[H_I(x) := \{ H : \text{is a closed half-plane} \ \{ k \in \mathbb{N} : x_k \notin H \} \in I \},\]

then the \(I\)-core of \(x\) is given by

\[I-\text{core} \{x\} := \cap_{H \in H_I(x)} H.\]

It is clear that \(I\)-core \(\{x\} \subseteq K\)-core \(\{x\}\) for all \(x\). Also

\[I-\text{core} \{x\} = [I-\liminf x, I-\limsup x]\]

for any \(I\)-bounded real number sequence.

The next theorem is an \(I\)-analogue of the Lemma of [12].

**Theorem 5.** Let \(I\) be an admissible ideal in \(\mathbb{N}\) and assume that \(x\) is an \(I\)-bounded sequence; for each \(z \in \mathbb{C}\) let

\[B_z(x) := \left\{ w \in \mathbb{C} : |w - z| \leq I-\limsup_k |x_k - z| \right\};\]

then \(I\)-core \(\{x\} := \cap_{z \in \mathbb{C}} B_z(x)\).

**Proof.** From the definition of \(I\)-lim sup \(x\) and Theorem 1, observe that the disk \(B_z(x)\) is equal to the intersection of all closed disks centered at \(z\) that contain \(x_k\).
for $\mathcal{I} - \text{a.a.k.}$ First assume $w \notin \cap_{z \in \mathbb{C}} B_z(z)$, say $w \notin \cap_{z \in \mathbb{C}} B_z(z^*)$ for some $z^*$. Let $H$ be the half-plane containing $B_z(z^*)$ whose boundary line is perpendicular to the line containing $w$ and $z^*$ and tangent to the circular boundary of $B_z(z^*)$. Since $B_z(z^*) \subset H$ and $B_z(z^*)$ contains $x_k$ for $\mathcal{I} - \text{a.a.k.}$, it follows that $H \in \mathcal{H}_{\mathcal{I}}(x)$. Since $w \notin H$, this implies $w \notin \cap_{H \in \mathcal{H}_{\mathcal{I}}(x)} H$. Hence, $\mathcal{I} - \text{core} \{x\} \subset \cap_{z \in \mathbb{C}} B_z(z)$.

Conversely, $w \notin \cap_{H \in \mathcal{H}_{\mathcal{I}}(x)} H$, let $H$ be a plane in $\mathcal{H}_{\mathcal{I}}(x)$ such that $w \notin H$. Let be the line through $w$ that is perpendicular to the boundary of $H$ and let $p$ be the mid-point of the segment to $L$ between $w$ and $H$. Let $z$ be a point of $L$ such that $z \in H$ and consider the disk

$$B(z) := \{ \xi \in \mathbb{C} : |\xi - z| \leq |p - z| \}.$$ 

Since $x$ is $\mathcal{I}$--bounded and $x_k \in H \mathcal{I} - \text{a.a.k.}$, we can choose $z$ sufficiently far from $p$ so that $|p - z| = \mathcal{I} - \limsup_k |x_k - z|$. Thus $B(z)$ is one of the $B_z(z)$ disks, and since $w \notin B(z)$, we get that $w \notin \cap_{z \in \mathbb{C}} B_z(z)$. This establishes the proof. $\square$

We note that Theorem 5 is not necessarily valid if $x$ is not $\mathcal{I}$--bounded. This can be seen from Remark in [12] where $\mathcal{I} = \{ \delta \subseteq \mathbb{N} : \delta(K) = 0 \}$.

Throughout the remainder of this paper the set of bounded complex sequences will be denoted by $\ell^\infty$.

Now we give necessary conditions on matrix $A$ so that the Knopp core of $Ax$ is contained in the $\mathcal{I}$--core of $x$ for every bounded complex number sequence.

**Theorem 6.** Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. If matrix $A$ satisfies $\sup_n \sum_{k=1}^\infty |a_{nk}| < \infty$ and the following conditions

(i) $A$ regular and $\lim_n \sum_{k \in E} |a_{nk}| = 0$ whenever $E \in \mathcal{I}$;

(ii) $\lim_n \sum_{k=1}^\infty |a_{nk}| = 1$,

then $\mathcal{K}$--core $\{Ax\} \subseteq \mathcal{I}$--core $\{x\}$ for every $x \in \ell^\infty$.

**Proof.** Assume (i) and (ii) and let $w \in \mathcal{K}$--core $\{Ax\}$. For any $z \in \mathbb{C}$ we have

$$|w - z| \leq \limsup_n |z - (Ax)_n|$$

$$= \limsup_n \left| z - \sum_{k=1}^\infty a_{nk} x_k \right|$$

$$= \limsup_n \left| \sum_{k=1}^\infty a_{nk} (z - x_k) + z \left( 1 - \sum_{k=1}^\infty a_{nk} \right) \right|$$

$$\leq \limsup_n \left| \sum_{k=1}^\infty a_{nk} (z - x_k) \right| + \limsup_n |z| \left| 1 - \sum_{k=1}^\infty a_{nk} \right|$$

$$= \limsup_n \left| \sum_{k=1}^\infty a_{nk} (z - x_k) \right|.$$ (3)

Let $r = \mathcal{I} - \limsup_n |x_n - z|$, suppose $\varepsilon > 0$, and let $E := \{ k : |z_k - L| > r + \varepsilon \}$. Then $E \in \mathcal{I}$, and we have
\[
\sum_{k=1}^{\infty} a_{nk} (z - x_k) = \left| \sum_{k \in E} a_{nk} (z - x_k) + \sum_{k \notin E} a_{nk} (z - x_k) \right|
\leq \sum_{k \in E} |a_{nk}| |z - x_k| + \sum_{k \notin E} |a_{nk}| |z - x_k|
\leq \sup_k |z - x_k| \left( \sum_{k \in E} |a_{nk}| + (r + \varepsilon) \sum_{k \notin E} |a_{nk}| \right).
\]

Now (i) and (ii) imply that
\[
\limsup_n \left| \sum_{k=1}^{\infty} a_{nk} (z - x_k) \right| \leq r + \varepsilon.
\]

It follows from (3) that \( |w - z| \leq r + \varepsilon \); and since \( \varepsilon \) is arbitrary, this yields \( |w - z| \leq r \). Hence, \( w \notin B_r(z) \) so by the Theorem 5 we get \( w \in I^* - \text{core} \{ x \} \). Hence the proof is completed. \( \square \)

Since \( I^* - \text{core} \{ x \} \subseteq \mathcal{K} - \text{core} \{ x \} \), we have the following corollary.

**Corollary 1.** If matrix \( A \) satisfies \( \sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty \) and properties (i) and (ii) of Theorem 6, then
\[
I^* - \text{core} \{ Ax \} \subseteq I^* - \text{core} \{ x \}
\]
for every \( x \in \ell^\infty \).

Note that the converse of Corollary 1 does not hold. This can be seen from Example in [12] where \( I = \{ K \subseteq \mathbb{N} : \delta(K) = 0 \} \).

**References**


