A note on a Whitney map for continua

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Abstract. Let \( X \) be a non-metric continuum, and \( C(X) \) the hyperspace of subcontinua of \( X \). It is known that there is no Whitney map on the hyperspace \( 2^X \) for non-metrizable Hausdorff compact spaces \( X \). On the other hand, there exist non-metrizable continua which admit and the ones which do not admit a Whitney map for \( C(X) \). In this paper we investigate the properties of non-metrizable continua which admit a Whitney map and the ones which do not admit a Whitney map for \( C(X) \). It is shown that there is no Whitney map on the hyperspace \( C(X) \) if \( X \) is a non-metrizable locally connected or rim-metrizable continuum.

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1. Introduction

Let \( X \) be a space. We denote by \( 2^X \) the set of all nonempty closed subsets of \( X \), by \( C(X) \) the set of all nonempty closed connected subsets of \( X \) and by \( X(n) \), \( n \) a positive integer, the set of all nonempty subsets consisting of at most \( n \) points [5]. We consider \( C(X) \) and \( X(n) \) as a subset of \( 2^X \).

Let \( X \) and \( Y \) be compact spaces and let \( f : X \to Y \) be a continuous map. Define \( 2^f : 2^X \to 2^Y \) by \( 2^f(F) = f(F) \) for \( F \in 2^X \). By [9, 5.10] \( 2^f \) is continuous and \( 2^f(C(X)) \subseteq C(Y) \) and \( 2^f(X(n)) \subseteq Y(n) \). The restriction \( 2^f|C(X) \) is denoted by \( C(f) \).

Let \( X = \{ X_a, p_{ab}, A \} \) be an inverse system of compact spaces with the natural projections \( p_a : \lim X \to X_a \), \( a \in A \). Then \( 2^X = \{ 2^{X_a}, 2^{p_{ab}}, A \}, C(X) = \{ C(X_a), C(p_{ab}), A \} \) and \( X(n) = \{ X_a(n), 2^{p_{ab}}|X_b(n), A \} \) form inverse systems. For each \( F \in 2^{\lim X} \), i.e., for each closed \( F \subseteq \lim X \), \( p_a(F) \subseteq X_a \) is closed and compact. Thus, we have a mapping \( 2^{p_a} : 2^{\lim X} \to 2^X \) induced by \( p_a \) for each \( a \in A \). Define a mapping \( M : 2^{\lim X} \to 2^X \) by \( M(F) = \{ p_a(F) : a \in A \} \) since \( \{ p_a(F) : a \in A \} \) is a thread of the system \( 2^X \). The mapping \( M \) is continuous and 1-1. It is also an onto mapping since for each thread \( \{ F_a : a \in A \} \) of the system \( 2^X \) the set \( F' = \bigcap \{ p_a^{-1}(F_a) : a \in A \} \) is

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nonempty and \( p_a(F^{'}) = F_a \). Thus, \( M \) is a homeomorphism. If \( P_a : \lim 2^X \to 2^{X_a} \), \( a \in A \), are the projections, then \( P_a M = 2^{p_a} \). Identifying \( F \) by \( M(F) \) we have \( P_a = 2^{p_a} \).

**Lemma 1.** [5, Lemma 2.]. Let \( X = \lim X \). Then \( 2^X = \lim 2^X \), \( C(X) = \lim C(X) \) and \( X(n) = \lim X(n) \).

If \( F_a \in 2^{X_a} \), then \( p_a^{-1}(F_a) = (2^{p_a})^{-1}(F_a) = \{ F : F \) is a closed subset of \( \lim X \) and \( p_a(F) = F_a \} \in 2^{\lim X} \). Similarly, for the natural projection \( Q_a \) of the system \( C(X) = \{ C(X_a), C(p_{ab}), A \} \) we have \( Q_a = C(p_a) \). Moreover, if \( C_a \subset C(X_a) \), then \( Q_a^{-1}(C_a) = (C(p_a))^{-1}(C_a) = \{ C : C \) is a subcontinuum of \( \lim X \) and \( p_a(C) = C_a \} \in C(\lim X) \).

We say that an inverse system \( X = \{ X_a, p_{ab}, A \} \) is \( \sigma \)-directed if for each sequence \( a_1, a_2, ..., a_k, ... \) of the members of \( A \) there is an \( a \in A \) such that \( a \geq a_k \) for each \( k \in \mathbb{N} \).

In the sequel we shall use the following theorem.

**Theorem 1.** [6, Lemma 2.2]. Let \( X = \{ X_a, p_{ab}, A \} \) be a \( \sigma \)-directed inverse system of compact spaces with surjective bonding mappings and limit \( X \). Let \( Y \) be a metric compact space. For each surjective mapping \( f: X \to Y \) there exists an \( a \in A \) such that for each \( b \geq a \) there exists a mapping \( g_b: X_b \to Y \) such that \( f = g_b p_b \).

If the bonding mappings are not surjective, then we consider the inverse system \( \{ p_a(X), p_{ab}|p_b(X), A \} \) which has surjective bonding mappings. Moreover, \( p_a(X) = \cap \{ p_{ab}(X_b) : b \geq a \} \). Applying Theorem 1 we obtain the following theorem.

**Theorem 2.** Let \( X = \{ X_a, p_{ab}, A \} \) be a \( \sigma \)-directed inverse system of compact spaces with limit \( X \). Let \( Y \) be a metric compact space. For each surjective mapping \( f: X \to Y \) there exists an \( a \in A \) such that for each \( b \geq a \) there exists a mapping \( g_b: p_b(X) \to Y \) such that \( f = g_b p_b \).

Let \( X = \{ X_a, p_{ab}, A \} \) be an inverse system. For each subset \( \Delta_0 \) of \( (A, \leq) \) we define sets \( \Delta_n, n = 0, 1, ..., \) by the inductive rule \( \Delta_{n+1} = \Delta_n \cup \{ m(x,y) : x,y \in \Delta_n \} \), where \( m(x,y) \) is a member of \( A \) such that \( x,y \leq m(x,y) \). Let \( \Delta = \bigcup \{ \Delta_n : n \in \mathbb{N} \} \). It is clear that \( \text{card}(\Delta) = \text{card}(\Delta_0) \). Moreover, \( \Delta \) is directed by \( \leq \) [12, Lemma 9.2].

For each directed set \( (A, \leq) \) we define

\[ A_\sigma = \{ \Delta : \emptyset \neq \Delta \subset A, \text{card}(\Delta) \leq \aleph_0 \text{ and } \Delta \text{ is directed by } \leq \}. \]

Then \( A_\sigma \) is \( \sigma \) - directed by inclusion [12, Lemma 9.3]. If \( \Delta \in A_\sigma \), let \( X^\Delta = \{ X_b, p_{ab}, A \} \) and \( X_\Delta = \lim X^\Delta \). If \( \Delta, \Gamma \in A_\sigma \) and \( \Delta \subseteq \Gamma \), let \( p_{\Delta \Gamma} : X_\Gamma \to X_\Delta \) denote the map induced by the projections \( p^\delta \Delta : X_\Gamma \to X_\delta, \delta \in \Delta, \) of the inverse system \( X_\Gamma \).

Now, we have the following theorem.

**Theorem 3.** [12, Theorem 9.4] If \( X = \{ X_a, p_{ab}, A \} \) is an inverse system, then \( X_\sigma = \{ X_\Delta, p_{\Delta \Gamma}, A_\sigma \} \) is a \( \sigma \)-directed inverse system and \( \lim X_\sigma \) and \( \lim X_\sigma \) are canonically homeomorphic.

**Theorem 4.** Let \( X \) be a compact space. There exists a \( \sigma \)-directed inverse system \( X = \{ X_a, p_{ab}, A \} \) of compact metric spaces \( X_a \) and surjective bonding mappings \( p_{ab} \) such that \( X \) is homeomorphic to \( \lim X \).

**Proof.** Apply [8, pp. 152, 164] and Theorem 3. □

**Theorem 5.** If \( X \) is a locally connected compact space, then there exists a \( \sigma \)-directed inverse system \( X = \{ X_a, p_{ab}, A \} \) such that each \( X_a \) is a metric locally connected compact space, each \( p_{ab} \) is a monotone surjection and \( X \) is homeomorphic
to \( \lim X \). Conversely, the inverse limit of such system is always a locally connected compact space.

**Proof.** Apply Theorem 4 and [8, p. 163, Theorem 2]. \( \square \)

2. Whitney map for \( C(X) \)

Let \( \Lambda \) be a subspace of \( 2^X \). By a Whitney map for \( \Lambda \) [10, p. 24, (0.50)] we will understand any mapping \( g : \Lambda \to [0, +\infty) \) satisfying

a) if \( A, B \in \Lambda \) such that \( A \subset B \), \( A \neq B \), then \( g(A) < g(B) \) and

b) \( g(\{x\}) = 0 \) for each \( x \in X \).

It is known that there is no Whitney map on the hyperspace \( 2^X \) for non-metrisable Hausdorff compact spaces \( X \) [1]. On the other hand, there exist non-metrisable continua which admit and the ones which do not admit a Whitney map for \( C(X) \) [1].

A continuous mapping \( f : X \to Y \) is light (zero-dimensional) if all fibers \( f^{-1}(y) \) are hereditarily disconnected (zero-dimensional or empty) [3, p. 450], i.e., if \( f^{-1}(y) \) does not contain any connected subset of cardinality larger than one (\( \dim f^{-1}(y) \leq 0 \)). Every zero-dimensional mapping is light, and in the realm of mappings with compact fibers the two classes of mappings coincide.

The key theorem of this section is the following theorem.

**Theorem 6.** Let \( X = \{X_a, p_{ab}, A\} \) be a \( \sigma \)-directed inverse system of compact spaces \( X_a \) and surjective bonding mappings \( p_{ab} \). Let \( X \) be the limit of \( X \). If there exists a Whitney map \( g \) for \( C(X) \) then there exists an \( a \in A \) such that \( p_a : X \to X_a \) is a light mapping for every \( b \geq a \).

**Proof.** From Lemma 1 it follows that \( C(X) = \{C(X_a), C(p_{ab}), A\} \) is an inverse system whose limit is homeomorphic to \( C(X) \). Because of Theorem 2, for a Whitney map \( g : C(X) \to [0, +\infty) \) there exists an \( a \in A \) such that for every \( b \geq a \) there exists a mapping \( g_b : Q_b(C(X)) \to [0, +\infty) \) such that \( g = g_bQ_b \), where \( Q_b \) is the natural projection \( Q_b : \lim C(X) \to C(X_a) \). Let \( b \geq a \) be fixed. Now we will prove that the natural projection \( p_b : \lim X \to X_b \) is a light mapping. Suppose that there exists a point \( x_b \in X_b \) such that \( p_b^{-1}(x_b) \) contains a non-degenerate component \( C \). Let \( x \) be a point of \( C \). Then \( \{x\} \subset C \) and \( \{x\} \neq C \). This means that \( g(\{x\}) < g(C) \), i.e., \( 0 \neq g(C) \). On the other hand, we have \( Q_b(\{x\}) = Q_b(C) \). This means that \( g_bQ_b(\{x\}) = g_bQ_b(C) \). From this it follows that \( g(\{x\}) = g(C) \). This contradicts \( 0 \neq g(C) \). We infer that \( p_b \) is a light mapping.

**Theorem 7.** Let \( X = \{X_a, p_{ab}, A\} \) be a \( \sigma \)-directed inverse system of compact spaces \( X_a \) and monotone surjections \( p_{ab} \). Let \( X \) be the limit of \( X \). If there exists a Whitney map \( g \) for \( C(X) \) then there exists an \( a \in A \) such that \( p_a : X \to X_a \) is a homeomorphism for every \( b \geq a \).

**Proof.** From Lemma 1 it follows that \( C(X) = \{C(X_a), C(p_{ab}), A\} \) is an inverse system whose limit is homeomorphic to \( C(X) \). Moreover, every \( C(p_{ab}) \) is a surjection. By Theorem 1 for a Whitney map \( g : C(X) \to [0, +\infty) \) there exists an \( a \in A \) such that for every \( b \geq a \) there exists a mapping \( g_b : C(X_b) \to [0, +\infty) \) such that \( g = g_bQ_b \), where \( Q_b \) is the natural projection \( Q_b : \lim C(X) \to C(X_b) \). Let \( b \geq a \) be fixed.
Now we shall prove that the natural projection $p_b \colon \lim X \to X_b$ is a homeomorphism. It suffices to prove that $p_b$ is 1-1. From Theorem 6 it follows that there exists an $a_1 \in A$ such that for each $b \geq a_1$ $p_b$ is light. Now, $p_b$ for $b \geq a$, $a_1$ is light and monotone. This means that $p_b$ is 1-1. Hence, $p_b$ is a homeomorphism.

**Corollary 1.** If $X$ is a limit of a $\sigma$-directed inverse system $X = \{X_a, p_{ab}, A\}$ of compact metric spaces $X_a$ and monotone surjections $p_{ab}$, then there exists a Whitney map for $C(X)$ if and only if $X$ is metrizable.

**Proof.** If $X$ is metrizable, then there exists a Whitney $g$ map for $2^X$ [10, pp. 25-27]. The restriction $g|_{C(X)}$ is a Whitney map for $C(X)$. Conversely, if there exists a Whitney map for $C(X)$, then there exists an $a \in A$ such that for every $b \geq a$ mapping $p_b$ is a homeomorphism. Hence, $X$ is metrizable.

The following Theorem generalizes Observation 3 from [1] which states that for any non-metrizable dendron (i.e., a Hausdorff continuum such that any two of its distinct points are separated by a third one) $X$ there is no Whitney map for $C(X)$ since there is a canonical embedding of $X(2)$ in $C(X)$ (which maps any pair $\{x, y\}$ with $x \neq y$ to the unique arc $xy$).

**Theorem 8.** Let $X$ be a locally connected compact space. Then there exists a Whitney map for $C(X)$ if and only if $X$ is metrizable.

**Proof.** If $X$ is metrizable, then there exists a Whitney $g$ map for $2^X$ [10, pp. 25-27]. The restriction $g|_{C(X)}$ is a Whitney map for $C(X)$. Conversely, let $X$ be a locally connected compact space for which there exists a Whitney map $g : C(X) \to [0,+\infty)$. By virtue of Theorem 5 there exists a $\sigma$-directed inverse system $X = \{X_a, p_{ab}, A\}$ such that every $X_a$ is a locally connected metric space, every $p_{ab}$ is a monotone surjection and $X$ is homeomorphic to $\lim X$. From Corollary 1.

**Corollary 2.** If $X$ is a non-metric locally connected compact space, then there is no Whitney map for $C(X)$.

Let $\tau$ be an infinite cardinal. A space $X$ is said to be rim-$\tau$ if it has a basis $B$ such that the weight $w(Bd(U)) \leq \tau$ for each $U \in B$. In the sequel we shall use the following theorem.

**Theorem 9.** [15, Theorem 1.4]. Let $f : X \to Y$ be a light mapping of a non-degenerate continuum $X$ onto a space $Y$. If $X$ admits a basis of open sets whose boundaries have weight $\leq w(Y)$, then $w(X) = w(Y)$.

**Theorem 10.** Let $X$ be a rim-$\tau$ continuum with $w(X) > \tau$. Then there is no Whitney map for $C(X)$.

**Proof.** There exists an inverse system $Y = \{Y_a, q_{ab}, A\}$ of metric continua $X_a$ such that $X$ is homeomorphic to $\lim Y$. From Theorem 2.7 of [6] (see also the proof of Theorem 3) it follows that there exists a $\tau$-directed inverse system $X = \{X_a, p_{ab}, A_\tau\}$ such that each $X_a$ is homeomorphic to the limit of an inverse subsystem of $Y$ of cardinality $\tau$ and $X$ is homeomorphic to $\lim X$. We infer that $w(X_a) = \tau$. If we suppose that there exists a Whitney map for $C(X)$, then the projection $p_b$ must be light for every $b \geq a$ for some $a \in A$ (Theorem 6). From Theorem 9 it follows that $w(X) = w(X_a) = \tau$. This is impossible since $w(X) > \tau$.

A space $X$ is said to be rim-metrizable if it has a basis $B$ such that $Bd(U)$ is metrizable for each $U \in B$. Equivalently, a space $X$ is rim-metrizable if and only if for each pair $F, G$ of disjoint closed subsets of $X$ there exists a metrizable closed subset of $X$ which separates $F$ and $G$. 
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Lemma 2. [15, Theorem 1.2]. Let $X$ be a nondegenerate rim-metrizable continuum and let $Y$ be a continuous image of $X$ under a light mapping $f : X \to Y$. Then $w(X) = w(Y)$.

Lemma 3. [15, Theorem 3.2]. Let $X$ be a rim-metrizable continuum and let $f : X \to Y$ be a monotone mapping onto $Y$. Then $Y$ is rim-metrizable.

It is clear that rim-metrizable continua are rim-$\tau$ for $\tau = \aleph_0$. Hence, we have the following theorem.

Theorem 11. Let $X$ be a rim-metrizable continuum. Then there exists a Whitney map for $C(X)$ if and only if $X$ is metrizable.

Proof. Theorem 11 follows from Theorem 10. We shall give an independent proof. There exists a $\sigma$-directed inverse system $X = \{X_a, p_{ab}, A\}$ of metric continua $X_a$ such that $X$ is homeomorphic to $\lim X$. From Theorem 6 it follows that there exists an $a \in A$ such that for each $b \geq a$ the projection $p_b$ is light. By Lemma 2 we infer that $w(X) = w(X_b)$. This means that $X$ is metrizable since $w(X_b) = \aleph_0$. $\square$

If $X$ is a continuous image of an ordered compact space, then $X$ is rim-metrizable [7, Theorem 5.]. Hence we have the following corollary.

Corollary 3. If a continuum $X$ is a continuous image of an ordered compact space, then there exists a Whitney map for $C(X)$ if and only if $X$ is metrizable.

In 1973 Heath, Lutzer and Zenor [4] introduced the concept of monotone normality which is a strengthening of normality.

A space $X$ is monotonically normal [4] if points are closed and, for each $x \in X$ and an open set $U$ with $x \in U$, there is an open $H(x, U)$ with $x \in H(x, U) \subseteq U$ such that:

1. (normality) $H(x, U) \cap H(y, V) = \emptyset$ unless $x \in V$ or $y \in U$,
2. (monotonicity) if $x \in U \subseteq V$, then $H(x, U) \subseteq H(x, V)$.

Every metrizable space is monotonically normal and every linearly ordered space is monotonically normal [4]. An arbitrary subspace of monotonically normal space is monotonically normal and a closed image of a monotonically normal space is a monotonically normal space [4]. It follows that every continuous image of an ordered compactum is monotonically normal. Moreover, we have the following excellent recent result of M.E. Rudin [13].

Theorem 12. A space is compact and monotonically normal if and only if it is the continuous image of some compact, linearly ordered space.

Thus, we have the following corollary.

Corollary 4. Let $X$ be a monotonically normal continuum. Then there exists a Whitney map for $C(X)$ if and only if $X$ is metrizable.

Theorem 13. Let $X$ be a continuum which admits a Whitney map for $C(X)$. Then each arc $L$ in $X$ is metrizable.

Proof. It is clear that there exists a Whitney map for $C(L)$. From Theorem 11 it follows that $L$ is metrizable. $\square$

A dendroid is a hereditarily unicoherent continuum which is arcwise connected. If $X$ is a dendroid and $x, y \in X$, then there exists a unique arc $[x, y]$ in $X$ with endpoints $x$ and $y$.

Corollary 5. If $X$ is a dendroid which admits a Whitney map for $C(X)$, then each arc in $X$ is a metric arc.
The dendroids in which every arc is a metric arc play an interesting role as the following theorem shows.

**Theorem 14.** Let $X$ be a dendroid. There exists an inverse system $X = \{X_a, p_{ab}, A\}$ such that each $X_a$ is a dendroid with metrizable arcs, every $p_{ab}$ is monotone and $X$ is homeomorphic to $\lim X$.

**Proof.** There exists an inverse system $Y = \{Y_a, q_{ab}, A\}$ of metric continua $X_a$ such that $X$ is homeomorphic to $\lim Y$. Let $q_a$ be the natural projection of $X$ onto $Y_a$. Applying the monotone-light factorization [16] to $q_a$, we get compact spaces $X_a$, monotone surjection $m_a : X -\rightarrow X_a$ and light surjection $l_a : X_a -\rightarrow Y_a$ such that $q_a = l_a \circ m_a$. By [8, Lemma 8] there exist monotone surjections $p_{ab} : Y_b -\rightarrow X_a$ such that $p_{ab} \circ m_a = m_a$, $a \leq b$. It follows that $X = \{X_a, p_{ab}, A\}$ is an inverse system such that $X$ is homeomorphic to $\lim X$. Let us prove that $X_a$ is a dendron. The space $X_a$ is hereditarily unicoherent since $m_a$ is monotone. Moreover, $X_a$ is arcwise connected. Namely, if $x_a, y_a$ are distinct points of $X_a$, then there exists a pair $x, y$ of points of $X$ such that $x_a = m_a(x)$ and $y_a = m_a(y)$. Let $L$ be the arc with endpoints $x$ and $y$. Now, $m_a(L)$ is a continuous image of an arc and, consequently, arcwise connected [14]. Hence, $X_a$ is a dendroid. Since every map $l_a$ is light, we infer that each arc in $X_a$ is metrizable (Theorem 2).

**Theorem 15.** Let $X$ be a rim-metrizable dendroid. There exists an inverse system $X = \{X_a, p_{ab}, A\}$ such that each $X_a$ is a metric dendroid, every $p_{ab}$ is monotone and $X$ is homeomorphic to $\lim X$.

**Theorem 16.** Let $X = \{X_a, p_{ab}, A\}$ be an inverse system of dendroids and monotone surjective bonding mappings $p_{ab}$. The $X = \lim X$ is a dendroid.

**Proof.** It is well known that $X$ is hereditarily unicoherent [11, Theorem 3]. Let us prove that $X$ is arcwise connected. Let $x, y$ be a pair of distinct points in $X$. There exists an $a \in A$ such that $p_a(x) \neq p_a(y)$ for every $b \geq a$. There exists a unique arc $L_b$ which contains $p_b(x)$ and $p_b(y)$. Let us prove that $p_{bc}(L_c) = L_b$. Now, $p_{bc}(L_c)$ is a continuous image of an arc and, consequently, arcwise connected [14]. It follows that there exists an arc $M_b$ with endpoints $p_b(x)$ and $p_b(y)$. It follows that $M_b = L_b$ since $X_b$ is hereditarily unicoherent. Moreover, $p_{bc}^{-1}(M_b) = p_b^{-1}(L_b)$ is a continuum containing $L_c$ since $X_c$ is hereditarily unicoherent. This means that $p_{bc}(L_c) \subseteq L_b$. Finally, $p_{bc}(L_c) = L_b$ since $L_b$ is the arc and $p_{bc}(L_c)$ contains $p_b(x)$ and $p_b(y)$.

Now we consider the existence of transfinite sequences of subcontinua in a continuum $X$.

**Theorem 17.** If a continuum $X$ contains a transfinite increasing (decreasing) sequence of subcontinua, then $X$ admits no Whitney map for $C(X)$.

**Proof.** Suppose that $X$ is a continuum which contains the transfinite increasing sequence $C_0 \subset C_1 \subset \ldots \subset C_\xi \subset \ldots$, $\xi < \omega_1$ (decreasing sequence $C_0 \supset C_1 \supset \ldots \supset C_\xi \supset \ldots$, $\xi < \omega_1$) of subcontinua of $X$. Then $\omega(C_0) < \omega(C_1) < \ldots < \omega(C_\xi) < \ldots$, $\xi < \omega_1$ is an increasing transfinite sequence of real numbers $(\omega(C_0) > \omega(C_1) > \ldots \omega(C_\xi) > \ldots$, $\xi < \omega_1$ is a decreasing transfinite sequence of real numbers). This is impossible since $w(\mathbb{R}) = \omega_0$.

Using Theorem 17 we obtain the following theorem.

**Theorem 18.** Let $X$ be a continuum such that there exists a point $x$ of $X$ with the property that for each $y \neq x$ there exists a locally connected compact subspace
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(or rim-metrizable subcontinuum) \( C(x, y) \) containing \( x \) and \( y \). If the density \( d(X) > \aleph_0 \), then there is no Whitney map for \( C(X) \).

**Proof.** Suppose that there exists a Whitney map for \( C(X) \). Consider the subset \( E = X \setminus \{x\} \). For each \( e \in E \) we consider a subcontinuum \( C(e, x) \). It is clear that \( X = \bigcup \{C(e, x) : e \in E\} \). If there exists a Whitney map for \( C(X) \), then there exists a Whitney map for hyperspace \( C(C(e,x)) \). From Theorems 8 and 11 it follows that every subcontinuum \( C(e, x) \) is separable since it is metrizable. Hence if \( E \) is countable, then \( X \) is separable. This means that if \( d(X) > \aleph_0 \), then \( E \) is uncountable.

Now, we define a separable subcontinuum \( C_\alpha \subseteq X \) for every countable ordinal \( \alpha < \omega_1 \) such that \( C_\alpha \subseteq C_\beta \) if \( \alpha < \beta \). Let \( e_1 \) be any point of \( E \). Set \( C_1 = C(e_1, x) \). There exists a point \( e_2 \in E \setminus C_1 \) since \( C_1 \) is separable and \( d(X) > \aleph_0 \). Set \( C_2 = C_1 \cup C(e_2, x) \). Suppose that \( C_\alpha \) is defined for every \( \alpha < \beta \) and define \( C_\beta \). If \( \beta \) is a non-limit ordinal, then there exists a point \( e_\beta \in E \setminus C_{\beta-1} \) since \( C_{\beta-1} \) is separable and \( d(X) > \aleph_0 \). Set \( C_\beta = C_{\beta-1} \cup C(e_\beta, x) \). If \( \beta \) is a limit ordinal, then we set \( C_\beta = \bigcup \{C_\alpha : \alpha < \beta\} \). It is clear that \( C_\beta \) is separable. We have strictly increased a transfinite sequence \( C_1 \subset C_2 \subset ... \subset C_\alpha \subset ... \alpha < \omega_1 \). From Theorem 17 it follows that there is no Whitney map for \( C(X) \), which is a contradiction.

**Corollary 6.** Let \( X \) be a continuum. If there exists a point \( x \) of \( X \) such that for each \( y \neq x \) there exists a locally connected compact subspace (or rim-metrizable subcontinuum) \( C(x, y) \) containing \( x \) and \( y \) and if there is a Whitney map for \( C(X) \), the density \( d(X) = \aleph_0 \).

Theorem 18 implies the following corollary.

**Corollary 7.** Let \( X \) be an arcwise connected continuum. If the density \( d(X) > \aleph_0 \), then there is no Whitney map for \( C(X) \).

An arc \( L \) in a space \( X \) is said to be a free arc in \( X \) provided that \( L \) without its endpoints is open in \( X \).

**Corollary 8.** Let \( X \) be an arcwise connected continuum. If \( X \) contains uncountable many free arcs, then there is no Whitney map for \( C(X) \).

**Proof.** It is clear that \( d(X) > \aleph_0 \). Apply Theorem 18. □

**Remark 1.** The cone over \( X \) [10, p. 19] is the decomposition space of the upper semi-continuous decomposition \( (X \times [0, 1])/(X \times \{1\}) \) of \( X \times [0, 1] \) obtained by “shrinking \( X \times \{1\} \) to a point”. The cone over \( X \) will be denoted by \( \text{Cone}(X) \), its base \( X \times \{0\} \) by \( B(X) \), and its vertex \( X \times \{1\} \) in \( \text{Cone}(X) \) by \( v \). Let \( \Omega_1 \) be the set of all ordinals \( \alpha \leq \omega_1 \), where \( \omega_1 \) is the first uncountable ordinal. The space \( X = \text{Cone}(\Omega_1) \) is a dendroid which contains uncountable many free arcs. Hence, there does not exist a Whitney map for \( C(X) \). Let us note that \( X \) is not locally connected. Moreover, \( X \) is not rim-metrizable. Thus, there exists a non locally connected and non rim-metrizable continuum \( X \) which admits no Whitney map for \( C(X) \).

A point \( e \) of a dendroid \( X \) is said to be an endpoint of \( X \) if there exists no arc \( [a, b] \) in \( X \) such that \( x \in [a, b] \setminus \{a, b\} \). The set of all endpoints of a dendroid \( X \) is denoted by \( E(X) \).

**Corollary 9.** [2, p. 317]. For every point \( x \) of a dendroid \( X \), \( X = \bigcup\{[ex] : e \in E(X)\} \).

**Corollary 10.** Let \( X \) be a dendroid. If the density \( d(X) > \aleph_0 \), then there is no Whitney map for \( C(X) \).
Corollary 11. If $X$ is a dendroid such that there is a Whitney map for $C(X)$, then $d(X) = \aleph_0$.

3. Whitney map for $2^X$

Now we shall give an alternate proof of Theorem 1 from [1] using the inverse system method.

Theorem 19. The following conditions are equivalent for a Hausdorff compact space $X$:

(1.1) $X$ is metrizable;
(1.2) there exists a Whitney map for $2^X$;
(1.3) there exists a Whitney map for $X$.

Proof. The implication from (1.1) to (1.2) is well known. The one from (1.2) to (1.3) is obvious. It remains to show that (1.3) implies (1.1). So, assume (1.3). Let $X$ be a Hausdorff compact space for which there exists a Whitney map $g : X(2) \rightarrow [0, +\infty)$. By virtue of Theorem 4 there exists an inverse system $X = \{X_a, p_{ab}, A\}$ such that every $X_a$ is a metric space, every $p_{ab}$ is a surjection and $X$ is homeomorphic to $\lim X$. From Lemma 1 it follows that $X_a(2) = \{X_a(2), 2^{p_{ab}} | X_b(2), A\}$ is an inverse system whose limit is homeomorphic to $X(2)$. Moreover, every $X_a(2)$ is a metric space and every $p_{ab} | X_b(2)$ is a surjection. By virtue of Theorem 1, for a Whitney map $g : X_a(2) \rightarrow [0, +\infty)$ there exists an $a \in A$ such that for every $b \geq a$ there exists a mapping $g_a : X_b(2) \rightarrow [0, +\infty)$ such that $g = g_a p_b$, where $p_b$ is the natural projection $p_b : \lim X \rightarrow X_b(2)$. Now we shall prove that every natural projection $p_b : \lim X \rightarrow X_b$ is a homeomorphism. It suffices to prove that $p_b$ is 1-1. Suppose that there exists a point $x_b \in X_b$ such that $p_b^{-1}(x_b)$ contains two different points $x$ and $y$. Then $\{x\} \subset p_{ab}^{-1}(x_b)$ and $\{x\} \neq p_{ab}(x_b)$. This means that $g(\{x\}) < g(\{p_{ab}^{-1}(x_b)\})$, i.e., $0 < g(\{p_{ab}^{-1}(x_b)\})$. On the other hand, $g(\{p_{ab}^{-1}(x_b)\}) = g_b p_b \{p_{ab}^{-1}(x_b)\} = g_b(\{x_b\})$ since $p_b \{p_{ab}^{-1}(x_b)\} = \{x_b\}$. This means that $g(\{p_{ab}^{-1}(x_b)\}) = 0$. This is impossible since $0 < g(\{p_{ab}^{-1}(x_b)\})$. We infer that $p_b$ is a homeomorphism. Hence, $X = \lim X$ is a metric space since every $X_b$ is a metric space.

References


