Mann iteration for generalized pseudocontractive maps in Hilbert spaces

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Abstract. If $X$ is a real Hilbert space, $B$ is a nonempty, bounded, convex and closed subset, $T : B \to B$ is a generalized pseudocontraction; then the iteration

\begin{align}
    x_1 & \in B, \\
    x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Tx_n, \\
    (\alpha_n) &\subset (0, 1), \sum_{n=1}^{\infty} \alpha_n = \infty,
\end{align}

\[ \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \quad \lim_{n \to \infty} \alpha_n = 0, \]

strongly converges to the fixed point of $T$.

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1. Preliminaries

In this note we study the convergence of the Mann iteration process (1) for generalized pseudocontractions. According to [8] the generalized pseudocontractions are more general than the pseudocontractions introduced by Browder.

**Definition 1.** [8]. Let $X$ be a Hilbert space, let $B$ be a nonempty subset. A map $T : B \to B$ is said to be a generalized pseudocontraction if for $x, y \in B$ there exists $r > 0$ such that

\[ \langle Tx - Ty, x - y \rangle \leq r \|x - y\|^2. \]

Clearly, (2) is equivalent to

\[ ((I - T)x - (I - T)y, x - y) \geq (1 - r) \|x - y\|^2. \]

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The map $T$ is a strong pseudocontraction if there exists $k \in (0, 1)$ such that for all $x, y \in B$,
\[
\langle (I - T)x - (I - T)y, x - y \rangle \geq k \|x - y\|^2,
\]
see, for example [6]. Remark that both generalized pseudocontractivity and strong pseudocontractivity generalize the pseudocontractivity, but in a different manner. Iteration (1), where $T$ is a strong pseudocontraction in Banach spaces, was studied in [1], [2], [3], [4], [6], [9].

The following lemma can be found in [9] as Lemma 4. Also, it can be found in [4] as Lemma 1.2, with another proof. A more general case is in Lemma 2 from [5]. The proof from [5] is similar to the proof of Lemma 4 from [9].

**Lemma 1.** [9], [4]. Let $(\rho_n)_{n}$ be a nonnegative real sequence satisfying
\[
\rho_{n+1} \leq (1 - \lambda_n)\rho_n + \sigma_n,
\]
where $\lambda_n \in (0, 1), \forall n \in N, \sum_{n=1}^{\infty} \lambda_n = \infty$ and $\sigma_n = o(\lambda_n)$. Then $\lim_{n \to \infty} \rho_n = 0$.

The normalized duality mapping $J$ is the identity, when $X$ is a Hilbert space, see [4]. Thus Lemma 1.1 from [4] becomes:

**Lemma 2.** [4]. If $X$ is a Hilbert space, then
\[
\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, (x + y) \rangle,
\]
for all $x, y \in X$.

The following result is a corollary of Lemma 1 from [7]:

**Lemma 3.** [7]. If $X$ is a real Hilbert space, $B$ is a nonempty, bounded, convex and closed subset, and $T : B \to B$ is a generalized pseudocontraction, then the sequence given by (1) satisfies
\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.
\]

In [7], the map $T$ is nonexpansive. If we consider the proof of Lemma 1 from [7], we see that the result is true, when our assumptions are fulfilled.

2. **Main result**

We are now able to give the following result:

**Theorem 1.** If $X$ is a real Hilbert space, $B$ is a nonempty, bounded, convex and closed subset, and $T : B \to B$ is a generalized pseudocontraction, then the iteration (1):
\[
x_1 \in B,

x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n,
\]

$(\alpha_n)_{n} \subset (0, 1), \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, \[\lim_{n \to \infty} \alpha_n = 0 \].
strongly converges to the fixed point of $T$.

**Proof.** Theorem 2.1 from [8] gives us the existence and the uniqueness of the fixed point of $T$. Let us denote this fixed point by $q$. Using Lemma 3 and (2), we have

$$
\|x_{n+1} - q\|^2 = \|(1 - \alpha_n)(x_n - q) + \alpha_n(Tx_n - q)\|^2 \\
\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \langle Tx_n - q, x_{n+1} - q \rangle \\
= (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \langle Tx_n - q, x_n - q \rangle + 2\alpha_n \langle Tx_n - q, x_{n+1} - x_n \rangle \\
\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n r \|x_n - q\|^2 \\
+ 2\alpha_n \langle Tx_n - q, x_{n+1} - x_n \rangle \\
\leq [1 - \alpha_n (2(1 - r) - \alpha_n)] \|x_n - q\|^2 \\
+ 2\alpha_n \langle Tx_n - q, x_{n+1} - x_n \rangle.
$$

Let us denote

$$A_n : = \langle Tx_n - q, x_{n+1} - x_n \rangle,$$
$$\lambda_n : = \alpha_n (2(1 - r) - \alpha_n),$$
$$\rho_n : = \|x_n - q\|^2,$$
$$\sigma_n : = 2\alpha_n A_n.$$

Thus, we have

$$\rho_{n+1} \leq (1 - \lambda_n)\rho_n + \sigma_n.$$

We observe that

$$\lim_{n \to \infty} \frac{\sigma_n}{\lambda_n} = \lim_{n \to \infty} \frac{2\alpha_n \langle Tx_n - q, x_{n+1} - x_n \rangle}{\alpha_n (2(1 - r) - \alpha_n)} = 2 \lim_{n \to \infty} \frac{\langle Tx_n - q, x_{n+1} - x_n \rangle}{(2(1 - r) - \alpha_n)} = 0;$$

the last equality is true. From Lemma 4, we have $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$. The sequence $(\|Tx_n - q\|)_n$ is bounded, being in the bounded set $B$. Hence we have $\lim_{n \to \infty} \langle Tx_n - q, x_{n+1} - x_n \rangle = 0$. The assumptions from Lemma 2 are fulfilled. Hence $\rho_n \to 0$ as $n \to \infty$. Thus $x_n \to q$ as $n \to \infty$.

A prototype for $(\alpha_n)_n$ is $(1/\sqrt{n})_{n \geq 1}$.

**References**


