PRIMJENA MATEMATIČKIH MODELA U PLANIRANJU LOGISTIČKIH OPERACIJA
APPLYING MATHEMATICAL MODELS IN PLANNING LOGISTICS OPERATIONS

Ratko Stanković, Siniša Radulović, Sladana Hrupački

Sažetak: Matematički modeli linearnog programiranja mogu se primjenjivati u planiranju logističkih operacija, kako bi se time pomoglo rješavanje logističkih problema. Taj je pristup demonstriran na dva osnovna logistička problema, problemu alokacije resursa i problemu distribucijske mreže. Optimalno rješenje koje zadovoljava postavljena ograničenja dobiveno je primjenom programskog alata na matematičkim modelima problema. Daljnjim poboljšavanjem prikazanih modela omogućila bi se njihova primjena u praksi.

Ključne riječi: planiranje, logističke operacije, linearno programiranje, matematički modeli

Abstract: Linear programming mathematical models can be applied in planning logistics operations, in order to facilitate solving logistics problems. This approach is demonstrated on two basic logistics problems – the resource allocation problem and the distribution network problem. The optimal solution that meets the given constraints is generated using a software tool applied to the mathematical models of the problems. Further improvement of the outlined models would enable their practical implementation.

Key words: planning, logistics operations, linear programming, mathematical models

1. INTRODUCTION

Supply chain is a system which generates value by providing for customer’s needs. It consists of interactions between parties involved, such as customers, suppliers of raw material, manufacturers, distributers (wholesalers), retailers, logistics operators, carriers, etc. Those interactions generate flows of products (material), information and funds between and within different stages of supply chain [1].

Planning logistics operations, whether considered from the aspect of a particular product or a company, always involves a trade-off between efficiency and effectiveness in compliance with the corporate competitive strategy, subject to a series of constraints, while finding an optimal solution [2]. Solving such type of problems can be facilitated by applying linear programming mathematical models, as discussed in the following paragraphs.

2. PLANNING OF LOGISTICS OPERATIONS

Planning of logistics operations is a part of supply chain management, mainly focused on the following issues:

- Exploiting resources in compliance with the given trade-off between effectiveness and efficiency.

With respect to these issues, applying the linear programming mathematical models in resource allocation planning and distribution network planning will be outlined in further paragraphs.

2.1. Resource allocation planning

Particular resources of the logistics systems could be or could become insufficient to meet requirements, whether the different processes share the same resource or some processes require several different resources, such as:

- quantity of products to be distributed via DC exceeds its capacity,
- quantity of cargo to be transported exceeds max. payload of the vehicle,
- available workforce is insufficient to get the job done within the given time,
- particular equipment act as a bottle-neck in the production line,
- limited amount of certain ingredient or raw material, etc.

Optimal solution in such cases is to allocate available resources in a way that the whole system generates the best total outcome, while the following rule should be taken into account: ...optimimum of the whole is not necessarily a sum of particular optimums of its parts... [3].
Due to insufficient resources it is not possible to meet the demand of all processes, so those contributing less to the result of the whole are sacrificed for the benefit of those contributing more, but only to the extent that the operating of the whole is not put at question [4]. In this respect, the following issues should be considered:
- How to define relations between processes and resources?
- How to identify bottle-necks?
- How to evaluate outcome of a process?

2.2. Distribution network planning

Performance of a distribution network should be evaluated against customers’ needs to be met (competitive strategy, efficiency) and costs of meeting those needs (effectiveness) [5]. Optimal solution in this case means to connect source nodes (production plants, distribution centers, etc.) and destination nodes (warehouses, points of sale (POS), etc.) by transport routes, so that the demand is met at minimum distribution costs.

In planning a distribution network [6], the following issues should be considered:
- Where to locate source nodes and which capacity to allocate?
- Which sources should supply which destinations?
- Which quantity should be shipped?
- Which transport routes to select?

Mathematical models used to facilitate distribution network planning are based on assumption that supply chain can be depicted as a set of nodes and arcs connecting them into a functional network [1]. Nodes represent resources, while arcs represent physical or logical links between them, enabling flows of material, funds and information throughout the supply chain.

3. APPLYING LINEAR PROGRAMING IN PLANNING OF LOGISTICS OPERATIONS

In planning their logistics operations to satisfy customers demand and maximize profit, companies are faced with a series of constraints, such as capacities of production plants, raw material supply, distribution network design, etc. Linear programming is a powerful tool for obtaining optimal solution which would maximize utility while meeting the constraints given [5].

Applying linear programming mathematical models in planning of logistics operations is outlined by the following example which involves both, resource allocation and distribution network design.

3.1. Defining the resource allocation problem

A company assembles final product out of standardized parts (components), supplied by various vendors. Assembly operations are not demanding in terms of equipment nor infrastructure, but determined mainly by available workforce. The demand of the market is not constant, but can be predicted as listed in Table 1.

<table>
<thead>
<tr>
<th>Table 1: Demand forecast</th>
</tr>
</thead>
<tbody>
<tr>
<td>Month</td>
</tr>
<tr>
<td>-------</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
</tbody>
</table>

The company can handle these fluctuations by building-up inventory during low demand months, increasing production during high demand months or backlogging orders to be delivered late to the customers, but no backlog order must remain at the end of the planning horizon. In order to utilize its resources effectively and efficiently, the company has to set up optimal operational plan (six months period is long enough for this example).

3.1.1. Mathematical model of the problem

Mathematical model that encompasses all the elements of the resource allocation problem (i.e. operational planning problem) is defined by the following mathematical structure:

**Input Data**
- Selling price ........................................ 40.00 €/unit,
- Initial inventory ...................................... 1.200 units,
- Minimal inventory .................................... 800 units,
- Initial workforce ..................................... 100 workers.

**Objective Function**

Objective function is to minimize total cost of the planning period. Since all the demand must be satisfied by the end of the planning horizon with fixed selling price, minimizing cost is equal to maximizing profit. It encompasses the following elements:
- Costs of regular work ................................ 480.00 €/month,
- Costs of overtime work ................................ 4.50 €/hour,
- Costs of reassigning workers ...................... 300.00 €/worker,
- Costs of inventory .................................... 2.00 €/unit,
- Stockout/backlog costs .............................. 4.00 €/unit,
- Material costs ........................................ 10.00 €/unit.

**Objective Function** is given by the following equation:

\[
\text{min}\; F = \sum_{t=1}^{6} 480R_t + \sum_{t=1}^{6} 4.5S_t + \sum_{t=1}^{6} 300(pR_t + sR_t) + \sum_{t=1}^{6} 10P_t
\]

where:
- \( R_t \) = number of workers in month \( t, t = 1, \ldots, 6 \)
- \( pR_t \) = increase of the workforce in month \( t, t = 1, \ldots, 6 \)

(additional workers assigned in the beginning of month \( t \))
1) The number of workers \( R_t \) in month \( t \) is equal to the number of workers \( R_{t-1} \) in month \( t-1 \), minus \( sR_t \) (decrease of the number of workers in the beginning of month \( t \)), plus \( pR_t \) (increase of the number of workers in month \( t \)) respectively:
\[
R_t = R_{t-1} + pR_t - sR_t \quad (t = 1, \ldots, 6)
\]

2) Production \( P_t \) in month \( t \) is limited by available workforce (number of working hours – regular and overtime) in month \( t \). With respect to the norm, each worker can produce two units daily in regular hours (40 units monthly), and one unit for every four hours overtime.
\[
P_t \leq 40R_t + \frac{1}{4}S_t \quad (t = 1, \ldots, 6)
\]

3) Total demand in month \( t \) equals to the sum of current demand \( D_t \) in month \( t \) and backlogged orders \( N_{t-1} \) from the previous month \( t-1 \). This demand can be satisfied either by current production \( P_t \) and previous month inventory \( Z_{t-1} \) or can be partly backlogged \( N_t \).
\[
Z_{t-1} + P_t + N_{t-1} + Z_t - N_t \quad (t = 1, \ldots, 6)
\]

4) Overtime hours per worker are limited to maximum 10 hours monthly.
\[
S_t \leq 10R_t \quad (t = 1, \ldots, 6)
\]

### 3.1.2. Optimal solution of the problem

The optimal solution of the problem (i.e. operational plan) is the one that yields minimal operating costs over the planning horizon. It is generated out of the mathematical model of the problem, by MS Excel spreadsheet optimizer Solver, as shown in Table 2 and in Table 3. The equations (1) to (5) of the mathematical model correspond to the Excel formulas entered in the respective cells of the spreadsheet.

### Table 2: Operational plan for six months period

<table>
<thead>
<tr>
<th>Month</th>
<th>pR</th>
<th>sR</th>
<th>R</th>
<th>S</th>
<th>Z</th>
<th>N</th>
<th>P</th>
<th>Demand</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>0</td>
<td>100</td>
<td>0</td>
<td>1.200</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>37</td>
<td>0</td>
<td>137</td>
<td>0</td>
<td>3.065</td>
<td>0</td>
<td>5.465</td>
<td>3.600</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>137</td>
<td>0</td>
<td>2.329</td>
<td>0</td>
<td>5.465</td>
<td>6.200</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>137</td>
<td>0</td>
<td>894</td>
<td>0</td>
<td>5.465</td>
<td>6.900</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>137</td>
<td>1.366</td>
<td>800</td>
<td>800</td>
<td>5.806</td>
<td>6.700</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>17</td>
<td>120</td>
<td>0</td>
<td>800</td>
<td>800</td>
<td>4.800</td>
<td>4.800</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>8</td>
<td>112</td>
<td>0</td>
<td>800</td>
<td>0</td>
<td>4.500</td>
<td>3.700</td>
</tr>
</tbody>
</table>

### Table 3: Costs, Incomes and Profit over a six-month period

<table>
<thead>
<tr>
<th></th>
<th>Costs</th>
<th>Income</th>
<th>Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>137,335</td>
<td>144,000</td>
<td>6,665</td>
</tr>
<tr>
<td>2</td>
<td>124,880</td>
<td>248,000</td>
<td>123,120</td>
</tr>
<tr>
<td>3</td>
<td>122,009</td>
<td>276,000</td>
<td>153,991</td>
</tr>
<tr>
<td>4</td>
<td>134,585</td>
<td>267,200</td>
<td>132,615</td>
</tr>
<tr>
<td>5</td>
<td>115,385</td>
<td>192,000</td>
<td>76,615</td>
</tr>
<tr>
<td>6</td>
<td>102,850</td>
<td>148,800</td>
<td>45,950</td>
</tr>
<tr>
<td></td>
<td>737,044</td>
<td>1,276,000</td>
<td>538,956</td>
</tr>
</tbody>
</table>

### 3.2. Defining the distribution network problem

The demand (cf. Table 1) is generated by points of sale (POS) in nine cities, the company supplies from its production plant located in Bratislava. The demand is satisfied from the distribution centers (DC) the company operates in five of those nine cities. The product is not yet ready to be delivered to the point of sale, until finishing operations are done at DC (sorting, packing into commercial packing materials, labeling, etc.).

The company must decide which of five DCs to use for distribution and which quantity to supply from each DC to the cities, in order to meet the demand at minimum distribution costs.

The month with the highest demand (6,900 units in 3rd month) is taken as a reference, however any other month within the planning horizon could be referred to by the same model.

Relevant input data are given in Table 4. Entries are based on the tariffs of logistic operators, labor and infrastructure costs, as well as the market analysis.
### 3.2.1. Mathematical model of the problem

**Objective Function**

The mathematical model that encompasses relevant elements of the problem (issues not relevant to this problem are disregarded) is defined by the following mathematical expressions:

Let \( K = \sum_{i=1}^{n} k_i \) and \( P = \sum_{j=1}^{m} p_j \)

\[
K = \sum_{i=1}^{n} k_i \quad \text{and} \quad P = \sum_{j=1}^{m} p_j \tag{6}
\]

Problem can be solved if \( K \geq P \) \tag{7}

where

- \( n \) = total number of used DCs, \( n_{\text{max}} = 5 \),
- \( k_i \) = capacity of DC on location \( i \),
- \( K \) = total distribution capacity,
- \( m \) = total number of cities to be supplied, \( m = 9 \),
- \( p_j \) = demand of POS in city \( j \),
- \( P \) = total demand of the market.

**3.2.2. Optimal solution of the problem**

The optimal solution (i.e. distribution plan) is generated out of the mathematical model of the problem (equations 8, 9, 10, 11 and 12), by MS Excel spreadsheet optimizer Solver, as shown in Table 6.

<table>
<thead>
<tr>
<th>From / To</th>
<th>Wien</th>
<th>Bratislava</th>
<th>Budapest</th>
<th>Ljubljana</th>
<th>Munich</th>
<th>Nuernberg</th>
<th>Prague</th>
<th>Stuttgart</th>
<th>Zagreb</th>
<th>ft (€)</th>
<th>d (€/unit)</th>
<th>K (units)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wien</td>
<td>9</td>
<td>22</td>
<td>35</td>
<td>32</td>
<td>35</td>
<td>38</td>
<td>33</td>
<td>41</td>
<td>32</td>
<td>15.000</td>
<td>20</td>
<td>3500</td>
</tr>
<tr>
<td>Bratislava</td>
<td>22</td>
<td>35</td>
<td>19</td>
<td>32</td>
<td>35</td>
<td>33</td>
<td>32</td>
<td>32</td>
<td>32</td>
<td>10.000</td>
<td>8</td>
<td>3500</td>
</tr>
<tr>
<td>Munich</td>
<td>35</td>
<td>22</td>
<td>35</td>
<td>35</td>
<td>33</td>
<td>19</td>
<td>32</td>
<td>19</td>
<td>51</td>
<td>15.000</td>
<td>32</td>
<td>3500</td>
</tr>
<tr>
<td>Prague</td>
<td>32</td>
<td>32</td>
<td>32</td>
<td>57</td>
<td>32</td>
<td>51</td>
<td>9</td>
<td>51</td>
<td>57</td>
<td>11.000</td>
<td>30</td>
<td>3500</td>
</tr>
<tr>
<td>Zagreb</td>
<td>32</td>
<td>35</td>
<td>32</td>
<td>19</td>
<td>51</td>
<td>76</td>
<td>57</td>
<td>76</td>
<td>9</td>
<td>12.000</td>
<td>32</td>
<td>3500</td>
</tr>
</tbody>
</table>

### Table 6: Input data

**Unit transport costs (€/unit)**

- **Demand (units):** 900 600 700 500 1200 700 800 900 600

### The problem solvability condition

The problem can be solved if the total distribution capacity of the network (sum of individual DC capacities) is greater than or equal to the total market demand (sum of the demands of individual cities with their gravity zones) [6], as defined by the following mathematical expressions:

\[
k_i l_i \geq \sum_{j=1}^{m} q_{ij} \quad \text{for every } i = 1, \ldots, n \tag{10}
\]

### Simplification

1) There are five possible locations of DC to be used for distribution. Each location of DC can be used for distribution or not, as defined by location variable (binary variable, 0 means not used, 1 means used):

\[
l_i \in \{0,1\} \quad \text{for every } i = 1, \ldots, n \tag{11}
\]

### With simplification

Inbound flow of DC is equal to outbound flow, i.e. there are no local inventories in DCs:

\[
r_i l_i = \sum_{j=1}^{m} q_{ij} \quad \text{for every } i = 1, \ldots, n \tag{12}
\]

where

- \( l_i \) = location variable, binary: 1 = DC on location \( i \) is used, 0 = not used,
- \( f_i \) = fixed cost of DC on location \( i \) (€/month),
- \( r_i \) = realized turnover of DC on location \( i \), (units),
- \( d_i \) = unit transport cost from the production plant to DC on location \( i \) (€/unit),
- \( n \) = total number of DCs used, \( n_{\text{max}} = 5 \),
- \( m \) = total number of cities to be supplied, \( m = 9 \),
- \( t_{ij} \) = unit transport cost from DC on location \( i \) to city \( j \) (€/unit),
- \( q_{ij} \) = number of units delivered from DC on location \( i \) to city \( j \) (units),
- \( p_j \) = demand in city \( j \) (units/month),
- \( k_i \) = capacity of DC on location \( i \) (units/month).

### 3.2.2. Optimal solution of the problem

Decision variable \( q_{ij} \) may take any non-negative integer value, as there is no logical point to deliver negative number of units or less than a whole unit.

It is allowed to round-up fractional values of \( q_{ij} \) that may occur, because the values are large enough so that the tolerance corresponds to accuracy of the input data (rounding-up cannot cause major logical error). Defining decision variable \( l_i \) as binary (only takes values of 1 or 0), fractional values are excluded.
The demand of the market is satisfied through three DCs, located in Bratislava, Munich and Zagreb, according to the following distribution plan:

1. DC in Bratislava is used to supply local markets of Wien, Bratislava, Budapest, Ljubljana, and Prague;
2. DC in Munich is used to supply local markets of Munich, Nuernberg and Stuttgart;
3. DC in Zagreb is used to supply local market of Zagreb.

All the constraints (equations 9, 10 and 11) are satisfied at total distribution cost of 306,300 € for the given month. Graphical presentation of the optimal solution of the problem is shown in Figure 1.

### 4. CONCLUSION

Planning of logistics operations, as a part of supply chain management, deals with issues that can be mathematically expressed. Based on this assumption, linear programming is applied to obtain the optimal solution of the two basic logistics problems, which would maximize utility (benefit/cost ratio), while meeting the constraints given. For this purpose, two mathematical models are developed, each one representing the respective logistic problem.

Similarity with the real system is limited by accuracy of the input data and the simplifications made in the models. However, it is sufficient to demonstrate the basic principles of the solving method, which is outlined in this paper.

The mathematical models developed and presented in this paper could be improved by adding more elements representing relevant issues of the real system and by obtaining more accurate input data, in order to achieve higher level of similarity. It would enable implementation of this approach in solving practical logistics problems.
As the improvement of the models is only a technical issue, the main point here is to show the possibilities of how to facilitate planning of logistics operations by applying linear programming mathematical models.

5. REFERENCES


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