A CLOSED-FORM APPROXIMATE SOLUTION FOR COUPLED VIBRATIONS OF COMPOSITE THIN-WALLED BEAMS WITH MID-PLANE SYMMETRY

Aleksandar Prokić, Radomir Folić, Ilija Miličić

The purpose of this paper is to develop and present an approximate analysis procedure to estimate the natural frequencies and mode shapes of thin-walled beams with arbitrary open cross-section, made of laminates with mid-plane symmetry. Theory of thin-walled composite beams is based on assumptions consistent with the Vlasov's beam theory and classical lamination theory. The equations of motion for coupled bending-torsional vibrations are derived from the principle of virtual displacements. In the case of simply supported thin-walled beam the frequency equation, given in determinant form, is expanded in an explicit analytical form, which gives the natural frequencies of free harmonic vibrations. For verification, the analytical study by other researchers and the FEM analysis using ANSYS are compared with the current results.

Keywords: approximate analytical solutions, classical lamination theory, explicit closed form, thin–walled composite beam, vibration

1 Introduction

Thin-walled beams are widely used as structural elements in many types of structures. Compared to standard construction materials, composite materials present many advantages, e.g. light weight, corrosion resistance, low thermal expansion, and excellent fatigue characteristics in the direction of the fibers. To accurately predict the dynamic behavior of such constructions is an important part of engineering analysis.

Although a large amount of models have been developed to describe the behavior of composite thin-walled beams [1 ÷ 11], relatively fewer investigations are available that pertain to the vibration of laminated thin-walled beams [12 ÷ 17]. The accuracy of these models is strongly dependent on the different concepts and hypotheses used to formulate them. According to the authors' knowledge, there is no work in the literature studying coupled vibrations of composite thin-walled beams with arbitrary open cross-sections by means of an explicit closed-form solution.

The objective of this work is to develop an approximate analytical method for engineering practice that enables the prediction of natural frequencies of the free harmonic coupled vibrations of composite thin-walled beams with arbitrary open cross-sections accurately and efficiently. Mode shapes, which are coupled in bending and torsion, are expressed in explicit analytical form. The walls of the beam are symmetric laminates with different fiber orientations in different layers. The solution is limited to simply supported ends. The expressions are concise and very simple and as such convenient to be used by practicing engineer who does not need to go into detail of thin-walled beam theory. The method presented in this study is very useful for predesign purposes and also for verifying numerical results of complex and time-consuming computer procedures.

2 Theory

A straight thin-walled laminated composite beam of length $l$ with an open cross-section is considered (Fig. 1). In order to determine the geometry of the cross-section of the beam two coordinate systems are used. The first of these is an orthogonal Cartesian coordinate system $(x, y, z)$ for which the $z$-axis is parallel to the longitudinal axis of the beam. The second coordinate system is a local one $(e, s, z)$ where $e$ is the normal coordinate measured along the normal to contour (the midline of the cross-section) and $s$ is profile coordinates measured along the contour line from arbitrarily taken starting point $O_i$. The $(e, s, z)$ and $(x, y, z)$ coordinate systems are related through an
angle of orientation $\alpha$. The coordinates of the contour in the $(x, y, z)$ coordinate system are $(\xi, \eta, \zeta)$. Point $P$, with coordinates $x_P$ and $y_P$ with respect to the reference coordinate system, is called the pole.

2.1 Kinematics of the beam

Following Vlasov’s beam theory the following assumptions are made:

- The cross-section of the beam is not distorted during the deformation of the beam.
- The shear strains in the middle surface of the wall are negligible.
- The Kirchhoff-Love assumption in classical plate theory remains valid for laminated composite thin-walled beams.
- Each element is symmetric with respect to its mid-plane.

Based on assumption above, the displacements $u_z$, $v_z$ and $w$, at any point on the beam cross-section can be expressed by only four components, two translations $u_P$, $v_P$ of arbitrarily taken pole $P$, the cross-section rotation $\varphi$ about the pole $P$, and axial displacement $w$ of centroid

$$
\begin{align*}
    u_z &= u_P - (y - y_P)\varphi, \\
v_z &= v_P + (x - x_P)\varphi, \\
w &= w - u_P^r x - v_P^r y - \varphi^r \omega_P,
\end{align*}
\tag{1}
$$

where $\omega_P$ is generalized warping function with regard to pole $P$. In Eq. (1) the prime denotes the differentiation with respect to $z$.

$h_p$ and $h_{np}$, perpendicular distance from tangent and normal at arbitrary point of cross-section to the point $P$, are positive when normal $\vec{n}$ and tangent $\vec{t}$, respectively, are rotating counterclockwise about the pole $P$, when observed from the positive $z$ direction. The second term on the right-hand side of Eq. (2) determines the relative warping in relation to the midline of cross-section.

Consistent with displacement field, Eq. (1), the non-vanishing strain components are

$$
\begin{align*}
    \varepsilon_{zz} &= w' - u_P^r x - v_P^r y - \varphi^r \omega_P, \\
    \gamma_{xz} &= 2\varphi^r \omega_P.
\end{align*}
\tag{3}
$$

2.2 Constitutive equations

The constitutive equations of a $k$-th unidirectionally reinforced lamina in the $s-z$ co-ordinate system are given by

$$
\begin{bmatrix}
    \sigma_z \\
    \tau_{sz}
\end{bmatrix}_k = \begin{bmatrix}
    q_{11} & q_{12} & q_{16} \\
    q_{12} & q_{22} & q_{26} \\
    q_{16} & q_{26} & q_{66}
\end{bmatrix}_k
\begin{bmatrix}
    \varepsilon_z \\
    \varepsilon_{yz}
\end{bmatrix}_k,
\tag{4}
$$

where the terms $q_{ij}$ are transformed reduced stiffness for a plane stress state, Jones [21].

By using free stress in contour direction, $\sigma_s = 0$, the above equation can be simplified as

$$
\begin{bmatrix}
    \sigma_z \\
    \tau_{sz}
\end{bmatrix}_k = \begin{bmatrix}
    q_{11} & q_{12} & q_{16} \\
    q_{12} & q_{22} & q_{26} \\
    q_{16} & q_{26} & q_{66}
\end{bmatrix}_k
\begin{bmatrix}
    \varepsilon_z \\
    \varepsilon_{yz}
\end{bmatrix}_k,
\tag{5}
$$

where

$$
\begin{align*}
    q_{11} &= q_{11} - \frac{q_{12}^2}{q_{22}}, \\
    q_{16} &= q_{16} - \frac{q_{12} q_{26}}{q_{22}}, \\
    q_{66} &= q_{66} - \frac{q_{26}^2}{q_{22}}.
\end{align*}
\tag{6}
$$

Stress resultants at the cross-section can be derived by integrating the corresponding stresses over the cross-sectional area, as given by

$$
\begin{align*}
    N &= \iint_A \sigma_z dA, \\
    M_x &= \iint_A \sigma_{yz} dA, \\
    M_y &= -\iint_A \sigma_{xz} dA, \\
    M_{zP} &= \iint_A \sigma_{xP} dA, \\
    T_s &= 2 \iint_A \tau_{sz} dA.
\end{align*}
\tag{7}
$$

In Eq. (7), $N$ represents the axial force, $M_x$ and $M_y$ the bending moments with respect to the $x$ and $y$ axis, $T_s$ the Saint Venant torque, $M_{zP}$ the bimoment and $F$ the area of the cross-section. Combining Eqs. (3), (5) and (7) the forces may be defined in terms of componential displacements as

$$
\begin{align*}
    N &= \iint_A \left( w' - u_P^r x - v_P^r y - \varphi^r \omega_P \right) + \bar{q}_{11} (w' - u_P^r x - v_P^r y - \varphi^r \omega_P) + \\
    &\quad \bar{q}_{16} 2\varphi^r \omega_P dA, \\
    M_x &= \iint_A \left( w' - u_P^r x - v_P^r y - \varphi^r \omega_P \right) y + \bar{q}_{11} (w' - u_P^r x - v_P^r y - \varphi^r \omega_P) x + \\
    &\quad \bar{q}_{16} 2\varphi^r \omega_P x dA, \\
    M_y &= -\iint_A \left( w' - u_P^r x - v_P^r y - \varphi^r \omega_P \right) x + \bar{q}_{12} (w' - u_P^r x - v_P^r y - \varphi^r \omega_P) y + \\
    &\quad \bar{q}_{16} 2\varphi^r \omega_P y dA, \\
    M_{zP} &= \iint_A \left( w' - u_P^r x - v_P^r y - \varphi^r \omega_P \right) x + \bar{q}_{12} (w' - u_P^r x - v_P^r y - \varphi^r \omega_P) y + \\
    &\quad \bar{q}_{66} 2\varphi^r \omega_P dA,
\end{align*}
\tag{8}
$$

or written in matrix form

$$
\begin{bmatrix}
    N \\
    M_x \\
    M_y \\
    M_{zP} \\
    T_s
\end{bmatrix} =
\begin{bmatrix}
    A^c & -S_x^c & -S_y^c & -S_{zP}^c & S_z^c \\
    -S_x^c & I_{xx} & I_{xy} & I_{xP} & -I_{xx} \\
    -S_y^c & I_{xy} & I_{yy} & I_{yP} & -I_{yy} \\
    -S_z^c & I_{xP} & I_{yP} & I_{PP} & -I_{PP} \\
    S_z^c & -I_{xx} & -I_{yy} & -I_{xy} & I_{ee}
\end{bmatrix}
\begin{bmatrix}
    u_P^r \\
    v_P^r \\
    \varphi^r \omega_P \\
    q^r \omega_P \\
    \varphi^r \omega_P
\end{bmatrix}.
\tag{9}
$$

The sectional quantities in Eq. (9) are defined as a function of the geometry and material properties of cross-section.
where

\[ x = \bar{x} + e \cdot \cos \alpha, \]
\[ y = \bar{y} + e \cdot \sin \alpha, \]
\[ \omega_p = \bar{\omega}_p + h_n \cdot e, \]

and

\[ A_{ij} = \int \bar{Q}_{ij} \, dx \, dy, \]
\[ B_{ij} = \int \bar{Q}_{ij} \, dx \, dy, \]
\[ D_{ij} = \int \bar{Q}_{ij} \, dx \, dy. \]

By appropriate selection of Cartesian coordinate system, pole P and starting point O₁, we can achieve that

\[ S_x^c = S_y^c = I_{xy}^c = 0, \]
\[ S_{\omega_p}^c = I_{\omega_p}^c = I_{\omega_p}^c = 0. \]

So, introducing ideal center of gravity and shear center D, we get the simplified expressions for stress resultants:

\[
\begin{bmatrix}
A^c & 0 & 0 & 0 & S_x^c \\
0 & I_{xx}^c & 0 & 0 & -S_y^c \\
0 & 0 & I_{yy}^c & 0 & -S_{\omega_p}^c \\
0 & 0 & 0 & I_{\omega_p}^c & -I_{\omega_p}^c \\
0 & 0 & 0 & 0 & I_{\omega_p}^c \\
\end{bmatrix}
\begin{bmatrix}
w' \\
u'_D \\
u''_D \\
v''''_D \\
\phi'' \\
\end{bmatrix}
= \int \begin{bmatrix}
w'' \\
u''_D \\
u''''_D \\
u'''_D \\
\phi'' \\
\end{bmatrix} \, dx \, dy. \tag{14}
\]

2.3 Equations of motion and free vibration analysis

Using the principle of virtual work the following four equations are obtained, Prokić [19] and [20].

\[ N' - \rho \int \bar{w} \, da + P_x = 0, \]
\[ M''_y + \rho \int \bar{w} \, dx \, da + \rho \int \bar{\omega}_p \, dy \, da - \rho \int \bar{\omega}_p \, dx \, da + P_y + M'_y = 0, \]
\[ M''_x + \rho \int \bar{w} \, dy \, da + \rho \int \bar{\omega}_p \, dx \, da + P_x + P_y + M'_x = 0, \]
\[ M''_{\omega_p} + T_x + \rho \int \bar{w} \, dx \, da + + \rho \int \bar{\omega}_p \, dy \, da] \bar{u}_x - (x - x_D) \bar{v}_y \] \[ da + m_D + m''_{\omega_p} = 0, \tag{15}
\]

where \( p_x, p_y, m_x, m_y \) and \( m_{\omega_p} \) represent external distributed loads per unit length in \( x \) and \( y \) directions, externally applied distributed moments per unit length about \( x, y \) and \( z \) axes, and external distributed warping moment (bimoment), respectively.

The Equations of motion can be obtained in matrix form, by substituting for the stress resultants from (14) into (15), with generalized displacements as primary unknowns

\[
\begin{bmatrix}
A^c & 0 & 0 & 0 & S_x^c \\
0 & I_{xx}^c & 0 & 0 & -S_y^c \\
0 & 0 & I_{yy}^c & 0 & -S_{\omega_p}^c \\
0 & 0 & 0 & I_{\omega_p}^c & -I_{\omega_p}^c \\
0 & 0 & 0 & 0 & I_{\omega_p}^c \\
\end{bmatrix}
\begin{bmatrix}
w' \\
u'_D \\
u''_D \\
u''''_D \\
\phi'' \\
\end{bmatrix}
= \int \begin{bmatrix}
w'' \\
u''_D \\
u''''_D \\
u'''_D \\
\phi'' \\
\end{bmatrix} \, dx \, dy + \rho \int \begin{bmatrix}
w''_D \\
u''''_D \\
u'''_D \\
\phi'' \\
\end{bmatrix} \, dx \, dy. \tag{16}
\]

where

\[
S_x = \int \bar{w} \, dx \, dy; \quad S_y = \int \bar{y} \, da; \quad S_{\omega_p} = \int \bar{\omega}_p \, da;
\]
\[
I_{xx} = \int x^2 \, dx \, dy; \quad I_{yy} = \int y^2 \, dx \, dy; \quad I_{xy} = \int xy \, dx \, dy;
\]
\[
I_{\omega_p} = \int \omega_p \, dx \, dy; \quad I_{\omega_p} = \int \omega_p \, dx \, dy; \tag{17}
\]

To achieve the compact form we artificially raise the order of Eq. (16–1) by one.

If we now examine the expressions for \( I_{xe}^c \) and \( I_{ye}^c \)
\[ I_{xx} = 2 \int (B_1 x^2 + B_3 \cos \alpha) \, ds, \]
\[ I_{yy} = 2 \int (B_1 y^2 + B_3 \sin \alpha) \, ds, \]  

(18)

we can assume that the second terms on the r.h.s. of Eq. (18) are, in general, negligible as compared to the first one.

Introducing this simplifying assumption we neglect the change of point coordinates along the wall thickness that is, we adopt their mean value. It is justified since the wall thickness is small enough with the dimensions of the cross section. Also, according to initial assumption each laminate is symmetric about its mid-plane, thus all the coupling terms \( B_{ij} \) are zero, so we can write

\[ S_{xx} = 0, \]
\[ I_{xx} = 0, \]
\[ I_{yy} = 0, \]

(19)

\[
\begin{bmatrix}
A^c & 0 & 0 & 0 \\
0 & I_{xx} & 0 & 0 \\
0 & 0 & I_{yy} & 0 \\
0 & 0 & 0 & I_{\omega_0 \omega_0}
\end{bmatrix}
\begin{bmatrix}
w'''
p'' \\
x_{x}'' \\
x_{y}'' \\
\omega_0'' \\
\end{bmatrix}
= 
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
w'''
p'' \\
x_{x}'' \\
x_{y}'' \\
\omega_0'' \\
\end{bmatrix}
\]

(21)

The first equation in (21), describing axial vibration, is uncoupled from the rest of the system and may be analyzed independently.

The free harmonic transverse and torsional vibrations are defined by the coupled homogeneous Eq. (21, 22, 23, 24). In the case of a beam with simply supported ends (fork supports at each end which prevent rotation and can warp freely) the end conditions are

\[
[u_{D_x}, v_{D_x}, \varphi, u''_{D_x}, v''_{D_x}, \varphi''] = 0.
\]  

(22)

Setting the determinant of the above system equal to zero

\[
| \begin{bmatrix}
l_{xx} & l_{xy} & l_{x\omega_0} \\
l_{xy} & l_{yy} & l_{y\omega_0} \\
l_{x\omega_0} & l_{y\omega_0} & l_{\omega_0 \omega_0}
\end{bmatrix} | = 0.
\]

(25)

where

\[ p = \rho \pi^2, \]

(26)

yields the following algebraic frequency equation

\[ a p^4 + b p^2 + c p + d = 0. \]  

(27)

If all laminates are equal, with the laminas arranged symmetrically about the middle surface of the laminate, from both a geometric and a material property standpoint, then center of gravity coincides with ideal center of gravity \( D \), i.e.

\[ S_{xx} = S_{yy} = 0, \]

\[ S_{\omega_0} = S_{\omega_0} = 0. \]  

(20)

Now, taking into account (19) and (20), matrix equation (16) becomes form (21), i.e.

The solution may be expressed in the form

\[
\begin{bmatrix}
u_D(x, t) \\
\varphi(x, t)
\end{bmatrix} = \begin{bmatrix}
C_u \\
C_\varphi
\end{bmatrix} \sin \lambda_n z \sin pt t, \]

(23)

where \( p \) is the radian frequency, \( \lambda_n = n \pi / L, n=1,2,... \) and \( C_u, C_\varphi \) are unknown constants.

Substituting (23) into homogeneous Eq. (21) yields

\[
\begin{bmatrix}
u_D(x, t) \\
\varphi(x, t)
\end{bmatrix} = \begin{bmatrix}
C_u \\
C_\varphi
\end{bmatrix} \sin \lambda_n z \sin pt t, \]

(24)

The coefficients \( a, b, c \) and \( d \) are given in Appendix.

For every \( n \), from Eq. (27), three natural frequencies \( p_{n,i} (i = 1, 2 \text{ and } 3) \) of a simply supported beam are derived.

Substituting \( p_{n,i} \) into Eq. (24) the three vectors \( C \) may be determined to the multiplicative constant factor.
The current theoretical predictions are correlated with results obtained by Cortinez and Piovan [18] in Tab. 3, for flexural–torsional vibration (xz plane), and in Tab. 4, for symmetric flexural vibration (xy plane).

3.3 Example 3

In order to have an additional comparison, for the case of simply supported composite thin–walled beam with span length of 10, 15 and 20 m and with open unsymmetrical cross–section shown in Fig. 3, the closed–form solutions presented in this paper are compared with a finite element results.

The finite element software ANSYS was used in this comparison, modeling the beam by means of elements SOLID 185.

The material properties are the same as previous example. The cross section is formed by four laminates with eight layers of equal thickness /75, 60, 45, 30/.

The results are presented in Tab. 5.

\[ C_i = \begin{bmatrix} C_{ix} \\ C_{iy} \end{bmatrix} = \frac{1}{\lambda_i} \begin{bmatrix} \lambda_i^2 I_{xx} - (\lambda_i^2 I_{xx} + A) p_i \left( \lambda_i^2 I_{ww} - x D A \right) + (\lambda_i^2 I_{ww} + y D A) \lambda_i^2 I_{xy} p_i \\ \lambda_i^2 I_{yy} - (\lambda_i^2 I_{yy} + A) p_i \left( \lambda_i^2 I_{ww} + y D A \right) + (\lambda_i^2 I_{ww} - x D A) \lambda_i^2 I_{xy} p_i \end{bmatrix} \]
Table 1 Variation of sectional properties for different laminate stacking sequences

<table>
<thead>
<tr>
<th>Sectional quantities</th>
<th>/0/45</th>
<th>/15/−15/45</th>
<th>/30/−30/45</th>
<th>/45/−45/45</th>
<th>/60/−60/45</th>
<th>/75/−75/45</th>
<th>/90/−90/45</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_{x}^c$ / kN-mm²</td>
<td>2.33248x10^6</td>
<td>2.10736x10^6</td>
<td>1.5648x10^6</td>
<td>1.08322x10^6</td>
<td>8.57074x10^5</td>
<td>7.89597x10^5</td>
<td>7.77639x10^5</td>
</tr>
<tr>
<td>$I_{y}^c$ / kN-mm²</td>
<td>8.16067x10^6</td>
<td>7.37302x10^6</td>
<td>5.47475x10^6</td>
<td>3.78985x10^6</td>
<td>2.9965x10^6</td>
<td>2.76257x10^6</td>
<td>2.72073x10^6</td>
</tr>
<tr>
<td>$I_{x}^c_{e_o}$ / kN-mm²</td>
<td>1.45780x10^6</td>
<td>1.31710x10^6</td>
<td>9.77997x10^5</td>
<td>6.7755x10^5</td>
<td>4.9347x10^5</td>
<td>4.85082x10^4</td>
<td>4.86024x10^4</td>
</tr>
<tr>
<td>$I_{x}^c$ / kN-mm²</td>
<td>4.03151x10^4</td>
<td>4.77099x10^3</td>
<td>5.93268x10^3</td>
<td>5.93279x10^3</td>
<td>5.07283x10^3</td>
<td>4.30859x10^3</td>
<td>4.03151x10^3</td>
</tr>
<tr>
<td>$A$ / mm²</td>
<td>3.1200x10^6</td>
<td>4.33708x10^4</td>
<td>1.51742x10^5</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>$I_{xx}$ / mm⁴</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>$I_{yy}$ / mm⁴</td>
<td>4.03151x10^4</td>
<td>4.77099x10^3</td>
<td>5.93268x10^3</td>
<td>5.93279x10^3</td>
<td>5.07283x10^3</td>
<td>4.30859x10^3</td>
<td>4.03151x10^3</td>
</tr>
<tr>
<td>$I_{xz_o}$ / mm⁴</td>
<td>2.71068x10^4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2 Natural frequencies (Hz) of a simply supported composite I-beam with /±θ/45° angle-ply laminations

<table>
<thead>
<tr>
<th>Mode</th>
<th>Formulation</th>
<th>/0/16</th>
<th>/15/−15/45</th>
<th>/30/−30/45</th>
<th>/45/−45/45</th>
<th>/60/−60/45</th>
<th>/75/−75/45</th>
<th>/90/−90/45</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Ref. [14]</td>
<td>24,194</td>
<td>22,997</td>
<td>19,816</td>
<td>16,487</td>
<td>14,666</td>
<td>14,077</td>
<td>13,970</td>
</tr>
<tr>
<td>2</td>
<td>Ref. [14]</td>
<td>35,233</td>
<td>36,247</td>
<td>37,051</td>
<td>30,827</td>
<td>27,420</td>
<td>26,319</td>
<td>26,119</td>
</tr>
<tr>
<td>3</td>
<td>Ref. [17]</td>
<td>45,235</td>
<td>42,997</td>
<td>37,864</td>
<td>37,915</td>
<td>35,372</td>
<td>31,313</td>
<td>29,175</td>
</tr>
<tr>
<td>4</td>
<td>Ref. [14]</td>
<td>96,726</td>
<td>91,940</td>
<td>79,225</td>
<td>65,916</td>
<td>58,633</td>
<td>56,278</td>
<td>55,850</td>
</tr>
<tr>
<td>5</td>
<td>Ref. [17]</td>
<td>109,441</td>
<td>107,655</td>
<td>102,159</td>
<td>94,884</td>
<td>87,051</td>
<td>79,330</td>
<td>75,767</td>
</tr>
</tbody>
</table>

Table 3 Flexural–torsion frequencies (Hz) of U beams (xz plane)

<table>
<thead>
<tr>
<th>Laminate</th>
<th>L / m</th>
<th>Formulation</th>
<th>Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>/0/0/0/</td>
<td>12</td>
<td>9.85, 38.73</td>
<td>51.67, 86.76</td>
</tr>
<tr>
<td>/0/90/0/</td>
<td>12</td>
<td>9.82, 38.63</td>
<td>52.07, 86.64</td>
</tr>
<tr>
<td>/45/−45/</td>
<td>12</td>
<td>38.73, 153.80</td>
<td>201.94, 344.01</td>
</tr>
<tr>
<td>/0/90/0/</td>
<td>6</td>
<td>38.63, 153.86</td>
<td>207.98, 345.90</td>
</tr>
<tr>
<td>/45/−45/</td>
<td>4</td>
<td>38.76, 344.02</td>
<td>438.99, 765.09</td>
</tr>
<tr>
<td>/45/−45/</td>
<td>4</td>
<td>86.64, 345.90</td>
<td>467.84, 777.95</td>
</tr>
</tbody>
</table>

888 Technical Gazette 20, 5(2013), 883-890
4 Conclusion

In this paper an approximate analytical method for determining natural frequencies was developed for thin–walled beams with arbitrary open cross–section made of laminated composites, symmetric with respect to their mid plane. In each laminate, the fibers are continuous, unidirectional, and directed in an arbitrary orientation with respect to the longitudinal axis of the element. All of the possible vibration modes including axial mode, and fully coupled flexural–torsional modes are included in the analysis. At the example of simply supported beam, the simplified analysis procedure has been validated by comparing the results based on the simplified method with other results reported in literature and by comparing with the results obtained from finite element analysis using ANSYS. It is shown that approximations introduced have a small effect on the accuracy of results. The model presented is found to be appropriate and efficient in analyzing free vibration problem of a thin–walled laminated composite beam, so the method is useful
• to define a quick approximate method;
• to execute simple preliminary design considerations or fast final general checks of accuracy.

Although the paper deals only with simple support conditions it may be supposed that it is reasonable enough to extend all conclusions to the beams with other boundary conditions.

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5 References

A closed-form approximate solution for coupled vibrations of composite thin-walled beams with mid-plane symmetry

A. Prokić et al.


Appendix

The coefficients in the frequency equation (27) are:

\[ a = -2\left(\lambda_{n}^{2}I_{xy} - x_{D}A\right)\left(\lambda_{n}^{2}I_{xx} + y_{D}A\right)\lambda_{n}^{2}I_{xy} + \left(\lambda_{n}^{2}I_{xx} + y_{D}A\right)\lambda_{n}^{2}I_{xy} + \left(\lambda_{n}^{2}I_{xy} + l_{D}\right)\lambda_{n}^{2}I_{xy} - x_{D}A\right)^{2}\left(\lambda_{n}^{2}I_{xx} + y_{D}A\right)\left(\lambda_{n}^{2}I_{xy} + A\right)\left(\lambda_{n}^{2}I_{xy} + l_{D}\right)
\]

\[ b = -\lambda_{n}^{2}I_{xy} + y_{D}A\left(\lambda_{n}^{2}I_{xx} + x_{D}A\right)\lambda_{n}^{2}I_{xy} + \left(\lambda_{n}^{2}I_{xx} + x_{D}A\right)\lambda_{n}^{2}I_{xy} + \left(\lambda_{n}^{2}I_{xy} + l_{D}\right)\lambda_{n}^{2}I_{xy} + \left(\lambda_{n}^{2}I_{xy} + A\right)\left(\lambda_{n}^{2}I_{xy} + l_{D}\right)
\]

\[ c = -\lambda_{n}^{2}I_{xy} + x_{D}A\left(\lambda_{n}^{2}I_{xx} + y_{D}A\right)\lambda_{n}^{2}I_{xy} + \left(\lambda_{n}^{2}I_{xx} + y_{D}A\right)\lambda_{n}^{2}I_{xy} + \left(\lambda_{n}^{2}I_{xy} + l_{D}\right)\lambda_{n}^{2}I_{xy} + \left(\lambda_{n}^{2}I_{xy} + A\right)\left(\lambda_{n}^{2}I_{xy} + l_{D}\right)
\]

\[ d = \lambda_{n}^{2}I_{xy} + \left(\lambda_{n}^{2}I_{xy} + A\right)\lambda_{n}^{2}I_{xy} + \left(\lambda_{n}^{2}I_{xy} + l_{D}\right)\lambda_{n}^{2}I_{xy} + \left(\lambda_{n}^{2}I_{xy} + A\right)\lambda_{n}^{2}I_{xy} + \left(\lambda_{n}^{2}I_{xy} + l_{D}\right)
\]