CERTAIN CLASSES OF POLYGONS IN $\mathbb{R}^2$
AND AREAS OF POLYGONS

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In this article we consider certain classes of polygons in $\mathbb{R}^2$ and areas of polygons. The classes are connected with definitions like the following.

Let $A_1 \ldots A_n$ be a polygon in $\mathbb{R}^2$ and let $k$ be a positive integer such that $1 < k < n$. A polygon $P_1 \ldots P_n$ (if such exists) will be called $k$-outscribed polygon to the polygon $A_1 \ldots A_n$ if it holds

$$P_i + \ldots + P_{i+k-1} = kA_i, \quad i = 1, \ldots, n.$$ 

First we prove the following theorem.

THEOREM 1. Let $h, k, n$ be positive integers such that

$$hk = n - 2, \quad \text{GCD}(k, n) = 2 \quad (1)$$

and let $A_1 \ldots A_n$ be any given polygon in $\mathbb{R}^2$ such that (1) is satisfied and that

$$\sum_{i=1}^{n} (-1)^i A_i = 0. \quad (2)$$

Then for every point $P_1 \in \mathbb{R}^2$ there are points $P_2, \ldots, P_n$ in $\mathbb{R}^2$ such that

$$P_i + \ldots + P_{i+k-1} = kA_i, \quad i = 1, \ldots, n \quad (3)$$

and that area of the polygon $P_1 \ldots P_n$ is a constant, that is, does not depend of $P_1$ and is given by

$$2 \text{ area of } P_1 \ldots P_n = |U_1, \sum_{i=2}^{n} (-1)^i U_i| + |U_2, \sum_{i=3}^{n} (-1)^{i+1} U_i| +$$

$$|U_3, \sum_{i=4}^{n} (-1)^i U_i| + |U_4, \sum_{i=5}^{n} (-1)^{i+1} U_i| + \ldots + |U_{n-1}, U_n| \quad (4)$$


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where $U_i = P_i + P_{i+1}$ and $P_i + P_{i+1}$ for each $i = 1, \ldots, n$ is given by

\[
P_i + P_{i+1} = 2A_i - (k - 2)A_{i+2} + 2A_{i+4} + 2A_{i+6} + \ldots + 2A_{i+k} -
\]

\[
(k - 2)A_{i+2+k} + 2A_{i+4+k} + 2A_{i+6+k} + \ldots + 2A_{i+2k} -
\]

\[
(k - 2)A_{i+2+2k} + 2A_{i+4+2k} + 2A_{i+6+2k} + \ldots + 2A_{i+3k} -
\]

\[\vdots \]

\[
(k - 2)A_{i+2+(h-1)k} + 2A_{i+4+(h-1)k} + 2A_{i+6+(h-1)k} + \ldots + 2A_{i+hk}.
\]

Here let us remark that by $|K, L|$ is denoted determinant $\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$, where $K(x_1, y_1)$, $L(x_2, y_2)$.

**Proof.** The proof that holds (5) is as follows. First let $n = 14$ and $k = 6$. Thus, in this case, $h = 2$. Supposing that

\[
P_i + \ldots + P_{i+5} = 6A_i, \quad i = 1, \ldots, 14
\]

we can write the following equations

\[
P_i - P_{i+6} = 6(A_i - A_{i+1}), \quad i = 1, \ldots, 14
\]

from which follows

\[
P_7 = P_1 - 6(A_1 - A_2),
\]

\[
P_{13} = P_1 - 6(A_1 - A_2) - 6(A_7 - A_8),
\]

\[
P_5 = P_1 - 6(A_1 - A_2) - 6(A_7 - A_8) - 6(A_{13} - A_{14}),
\]

\[
P_{11} = P_1 - 6(A_1 - A_2) - 6(A_7 - A_8) - 6(A_{13} - A_{14}) - 6(A_5 - A_6),
\]

\[
P_3 = P_1 - 6(A_1 - A_2) - 6(A_7 - A_8) - 6(A_{13} - A_{14}) - 6(A_5 - A_6) - 6(A_1 - A_2),
\]

\[
P_9 = P_1 - 6(A_1 - A_2) - 6(A_7 - A_8) - 6(A_{13} - A_{14})
\]

\[\ldots - 6(A_5 - A_6) - 6(A_{11} - A_{12}) - 6(A_3 - A_4).
\]

Here let us remark that $P_{13} = P_{7+6}$, $P_5 = P_{13+6} = P_{7+2.6}$ and so on. Analogously we have

\[
P_8 = P_2 - 6(A_2 - A_3), \quad P_{14} = P_2 - 6(A_2 - A_3) - 6(A_8 - A_9)
\]

and so on.

Let us remark that in expressions of $P_8, P_{14}, \ldots, P_{10}$, in relation to expressions of $P_7, P_{13}, \ldots, P_9$, each index grow up to 1.

From (6) we see that $P_9 - P_1 = 6(A_9 - A_{10})$, and from the expression of $P_9$ we see that

\[
P_9 - P_1 = -6(A_1 - A_2 + A_3 - A_4 + A_5 - A_6 + A_7 - A_8 + A_{11} - A_{12} + A_{13} - A_{14}).
\]

Thus, the system (6) will be consistent iff $\sum_{i=1}^{14} (-1)^i A_i = 0$. 

8
In the same way can be seen that the system (3) will be consistent iff holds (2).

Using, for example, the equation

\[ P_1 + P_2 + P_3 + P_4 + P_5 + P_6 = 6A_1 \]

and expressions of \( P_3, P_4, P_5, P_6 \) we find that

\[ P_1 + P_2 = 2A_1 - 4A_3 + 2A_5 + 2A_7 - 4A_9 + 2A_{11} + 2A_{13}. \]

Since each equation of the system (6) can be used, it holds

\[ P_i + P_{i+1} = 2A_i - 4A_{i+2} + 2A_{i+4} + 2A_{i+6} - 4A_{i+8} + 2A_{i+10} + 2A_{i+12}, \quad i = 1, \ldots, 14. \]

Now, in relation to (5), let us remark that on the right side of (5) there are \( h \) negative terms and that between two negative terms there are \( \frac{k-2}{2} \) positive terms. Thus, it holds

\[ 1 + h + h \cdot \frac{k-2}{2} = \frac{n}{2} \quad \text{or} \quad hk + 2 = n. \]

If right side of (5) is denoted by \( U_i \), then

\[
\begin{align*}
  P_2 &= -P_1 + U_1, \\
  P_3 &= -P_2 + U_2 = P_1 - U_1 + U_2, \\
  &\quad \hdots \hdots \hdots \hdots \\
  P_n &= -P_{n-1} + U_{n-1} = -P_1 + U_1 - U_2 + \ldots + U_{n-1},
\end{align*}
\]

where \( P_1 \) can be taken arbitrary. Hence

\[
\begin{align*}
  P_1 + \ldots + P_k &= U_1 + U_3 + \ldots + U_{k-1}, \\
  P_2 + \ldots + P_{k+1} &= U_2 + U_4 + \ldots + U_k, \\
  &\quad \hdots \hdots \hdots \hdots \\
  P_n + \ldots + P_{k-1} &= U_n + U_2 + \ldots + U_{k-2}.
\end{align*}
\]

Thus, we have to prove that

\[ U_i + U_{i+2} + \ldots + U_{i+k-2} = kA_i, \quad i = 1, \ldots, n. \]

First let us consider the case where \( n = 14 \) and \( k = 6 \). In this case we have

\[
\begin{align*}
  P_1 + P_2 &= U_1 = 2A_1 - 4A_3 + 2A_5 + 2A_7 - 4A_9 + 2A_{11} + 2A_{13}, \\
  P_3 + P_4 &= U_3 = 2A_3 - 4A_5 + 2A_7 + 2A_9 - 4A_{11} + 2A_{13} + 2A_1, \\
  P_5 + P_6 &= U_5 = 2A_5 - 4A_7 + 2A_9 + 2A_{11} - 4A_{13} + 2A_1 + 2A_3,
\end{align*}
\]

from which, by adding, we get

\[ P_1 + P_2 + P_3 + P_4 + P_5 + P_6 = U_1 + U_3 + U_5 = 6A_1. \]

Since analogously holds for \( i = 2, \ldots, 14 \), also we have

\[ P_2 + P_3 + P_4 + P_5 + P_6 + P_7 = U_2 + U_4 + U_6 = 6A_2 \quad \text{and so on.} \]
It is not difficult to see that analogously holds generally for the case where positive integers \( h, k, n \) are such that \( n - 2 = hk \). To see this, it is important to see that

\[-(k - 2)A_i,
2A_i - (k - 2)A_{i+2}
\ldots
2A_i + \ldots - (k - 2)A_{i+k-2}\]

and \(-(k - 2)A_i + \frac{2k^2}{2} \cdot 2A_i = 0\).

Concerning area of the polygon \( P_1 \ldots P_n \), first let us remark that, as it is known, area of a polygon \( A_1 \ldots A_n \) in \( \mathbb{R}^2 \) is given by

\[2 \text{ area of } A_1 \ldots A_n = \sum_{i=1}^{n} |A_i, A_{i+1}|.\]

Using expressions for \( P_2, \ldots, P_n \) given by (8) it can be found that holds (4). This completes the proof of Theorem 1.

In the following theorem will be shown that area of the polygon \( P_1 \ldots P_n \) can be written in a much simpler and interesting form. For this purpose will be used determinant of rectangular matrix. In short about this.

In [1] the following definition of a determinant of rectangular matrix is given: The determinant of a \( m \times n \) matrix \( A \) with columns \( A_1, \ldots, A_n \) and \( m \leq n \), is the sum

\[\sum_{1 \leq j_1 < j_2 < \ldots < j_m \leq n} (-1)^{r+s} |A_{j_1}, \ldots, A_{j_m}|,\]

where \( r = 1 + \ldots + m \), \( s = j_1 + \ldots + j_m \).

This determinant is a skew-symmetric multilinear functional with respect to the rows and therefore has many well known standard properties, for example, the general Laplace’s expansion along rows.

In particular, if \( m = 2 \), then

\[|A_1, \ldots, A_n| = \sum_{1 \leq i < j \leq n} (-1)^{3+i+j} |A_i, A_j|.\]  

(11)

In [2] the following theorem (Theorem 3) is proved.

Let \( A_1 \ldots A_n \) be any given polygon in \( \mathbb{R}^2 \). Then

\[2 \text{ area of } A_1 \ldots A_n = |A_1 + A_2, A_2 + A_3, \ldots, A_n + A_1|.\]  

(12)

It is easy to see that, according to (11), relation (4) can be written as

\[\sum_{i=1}^{n} |P_i, P_{i+1}| = \sum_{1 \leq i < j \leq n} (-1)^{3+i+j} |U_i, U_j|.\]

In the following theorem we shall use the following two theorems given in [2].
Theorem 7. Let $A_1 \ldots A_n$ be a polygon in $\mathbb{R}^2$ with even $n$ and let $\sum_{i=1}^{n} (-1)^i A_i = 0$. Then for every point $X \in \mathbb{R}^2$ it holds

$$|A_1 + X, \ldots, A_n + X| = |A_1, \ldots, A_n|.$$ \hspace{1cm} (13)

Theorem 8. Let $A_1 \ldots A_n$ be as in Theorem 7. Then for each $i = 1, \ldots, n$ it holds

$$|A_{i+1}, \ldots, A_n, A_1, \ldots, A_i| = |A_1, A_2, \ldots, A_n|.$$ \hspace{1cm} (14)

Now we can prove the following theorem.

**Theorem 2.** Let $P_1 \ldots P_n$ be polygon as in Theorem 1. If $k = 2$, then

$$2 \text{ area of } P_1 \ldots P_n = 4|A_1, \ldots, A_n|,$$ \hspace{1cm} (15)

and if $k > 2$, then

$$2 \text{ area of } P_1 \ldots P_n = k^2|V_1, \ldots, V_n|,$$ \hspace{1cm} (16)

where

$$V_i = A_i + A_{i+k} + \ldots + A_{j+(k-1)k}, \quad i = 1, \ldots, n.$$ \hspace{1cm} (17)

**Proof.** If $k = 2$, then from (5) can be seen that $P_i + P_{i+1} = 2A_i$. Since determinant has two rows, it holds

$$|2A_1, \ldots, 2A_n| = 4|A_1, \ldots, A_n|.

The proof that holds (16) if $k > 2$ is as follows. Since $U_i = P_i + P_{i+1}$, it is easy to see that $\sum_{i=1}^{n} (-1)^i U_i = 0$. Thus, we can use Theorem 7 given in [2] and take

$$X = -2(A_1 + A_3 + \ldots + A_{n-1}) \quad \text{or} \quad X = -2(A_2 + A_4 + \ldots + A_n).$$

Here let us remark that from (2) follows $A_1 + A_3 + \ldots + A_{n-1} = A_2 + A_4 + \ldots + A_n$. Thus, relation

$$2 \text{ area of } P_1 \ldots P_n = |U_1, \ldots, U_n|$$

can be written as

$$2 \text{ area of } P_1 \ldots P_n = |U_1 - X, \ldots, U_n - X|$$

or

$$2 \text{ area of } P_1 \ldots P_n = |-kV_1 + 2S, \ldots, -kV_n + 2S|,$$ \hspace{1cm} (18)

where $S = \sum_{i=1}^{n} A_i$. Now, according to the properties expressed by (13) and (14), the relation (18) can be written as (16).

For example, let $n = 14$ and $k = 6$. Then, as can be seen from the considered example (see (10)), it holds

$$P_1 + P_2 - 2S = -6A_3 - 6A_9, \quad P_2 + P_3 - 2S = -6A_4 - 6A_{10}, \ldots,$$

$$P_{13} + P_{14} - 2S = -6A_1 - 6A_7, P_{14} + P_{1} - 2S = -6A_2 - 6A_8,$$

$$| -6A_3 - 6A_9, -6A_4 - 6A_{10}, \ldots, -6A_1 - 6A_7, -6A_2 - 6A_8| = 6^2|A_1 + A_7, \ldots, A_{14} + A_6|.$$

This proves Theorem 2.

The proof that holds (16) seems to be far from to be easy without using properties of determinant given by (11).
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