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# CERTAIN RELATIONS BETWEEN TRIANGLES AND BICENTRIC HEXAGONS

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*Abstract.* In this article we prove, using relatively very elementary mathematical facts, that every triangle completely determines a bicentric hexagon. Obtained relations can be interesting.

## 1. Introduction

A polygon which is both chordal and tangential is called bicentric polygon. The following notation will be used.

If $A_1 \dots A_n$ is a considered bicentric polygon, then
$C_1$ is incircle of $A_1 \dots A_n$,
$C_2$ is circumcircle of $A_1 \dots A_n$,
$r$ is radius of $C_1$ and $R$ is radius of $C_2$,
$I$ is center of $C_1$ and $O$ is center of $C_2$, $d = |IO|$.

In the following we shall deal with bicentric hexagons and triangles. First let us remark that German mathematician Nicolaus Fuss(1755-1826) has proved that a tangential hexagon $A_1 \dots A_6$ will also be a chordal one iff holds relation

$$3(R^2-d^2)^4 - 4r^2(R^2+d^2)(R^2-d^2)^2 - 16R^2r^4d^2 = 0. \tag{1.1}$$

Concerning triangle, holds Euler's relation

$$R^2 - d^2 = 2Rr. \tag{1.2}$$

In the following will also be used Poncelet's closure theorem. This theorem for bicentric polygons can be stated as follows.

Let $C_1$ and $C_2$ be any given two circles in a plane such that one is completely inside of the other. Then only one of the following two assertions is true:

(a) There is no bicentric $n$-gon whose incircle is $C_1$ and circumcircle $C_2$.

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(b) There are infinitely many bicentric $n$-gons whose incircle is $C_1$ and circumcircle $C_2$. For every point $A_1$ on $C_2$ there is bicentric $n$-gon $A_1 \dots A_n$ whose incircle is $C_1$ and circumcircle $C_2$.

For example, if $C_1$ and $C_2$ are circles such that holds Euler's relation (1.2), then for any point $A_1$ on $C_2$ there is triangle whose incircle is $C_1$ and circumcircle $C_2$ (Figure 1.1).

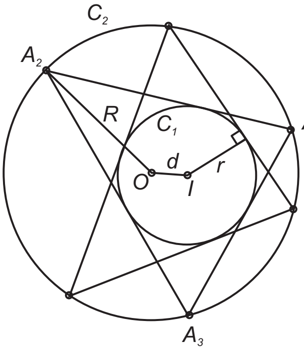



*Figure 1.1*

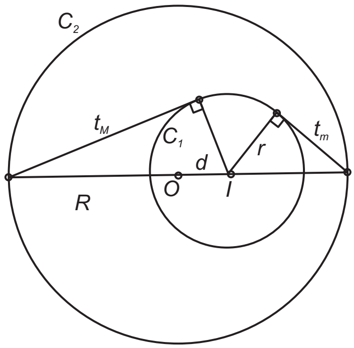



*Figure 1.2*

Although the Poncelet's closure theorem date from nineteenth century, many mathematicians have been working on number of problems in connection with it. Here let us remark that Richolet in [5], using some results given in [1], showed how some relations for bicentric $2n$-gon can be obtained from relations for bicentric $n$-gon. For this purpose elliptic functions are used.

Important role in the following will play tangent lengths $t_m$ and $t_M$ given by

$$t_m = \sqrt{(R-d)^2 - r^2}, \qquad t_M = \sqrt{(R+d)^2 - r^2}. \tag{1.3}$$

By $t_m$ and $t_M$ are denoted, respectively, the lengths of the least and the largest tangent that can be drawn from $C_2$ to $C_1$ (see Figure 1.2).

Instead of $t_m t_M$ will be shorter written $t^2$.

In the following will also be used relation between two consecutive tangents of a bicentric $n$-gon, see Figure 1.3. If $t_1$ is given, then $t_2$ can be calculated in the following way. From rectangular triangles $A_1IT_1$ and $A_2IT_1$ it follows

$$t_1^2 + r^2 = (x_1 - d)^2 + y_1^2 = R^2 + d^2 - 2dx_1, \quad t_2^2 + r^2 = R^2 + d^2 - 2dx_2 \tag{1.4}$$

or

$$x_1 = \frac{-t_1^2 + R^2 - r^2 + d^2}{2d}, \quad x_2 = \frac{-t_2^2 + R^2 - r^2 + d^2}{2d}. \tag{1.5}$$

Since for area of triangle $A_1A_2I$ it holds

$$(t_1 + t_2)^2 r^2 = [x_1(y_2 - 0) + x_2(0 - y_1) + d(y_1 - y_2)]^2, \tag{1.6}$$

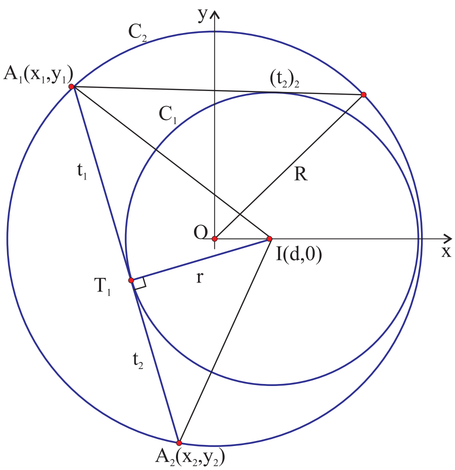



*Figure 1.3*

we can write

$$
\begin{aligned}
(t_1+t_2)^2 r^2 &= [y_1(d-x_2)-y_2(d-x_1)]^2, \\
[y_1y_2(d-x_1)(d-x_2)]^2 &= \left[y_1^2(d-x_2)^2+y_2^2(d-x_1)^2-(t_1+t_2)^2r^2\right]^2, \\
(t_1+t_2)^4-4R^2(t_1+t_2)^2+4x_1x_2(t_1+t_2)^2+4R^2(x_1-x_2)^2 &= 0, \\
(t_1+t_2)^4-4R^2(t_1+t_2)^2+4x_1x_2(t_1+t_2)^2+4R^2\frac{(t_1-t_2)^2(t_1+t_2)^2}{4d^2} &= 0,
\end{aligned}
\tag{1.7}
$$

from which we get the following equation for $t_2$

$$(r^2+t_1^2)t_2^2-2t_1t_2(R^2-d^2)-4R^2d^2+r^2t_1^2+(R^2+d^2-r^2)^2=0. \tag{1.8}$$

Thus, we have

$$(t_2)_{1,2}=\frac{t_1(R^2-d^2)\pm\sqrt{D}}{r^2+t_1^2} \tag{1.9a}$$

where

$$D=t_1^2(R^2-d^2)^2+(r^2+t_1^2)\left[4R^2d^2-r^2t_1^2-(R^2+d^2-r^2)^2\right]. \tag{1.9b}$$

(The length $(t_2)_1$ in Figure 1.3 is denoted by $t_2$.)

## 2. Certain relations between triangles and bicentric hexagons

First about notation which will be used in the following theorem.
Let $ABC$ be any given triangle and let the following notation be used:

$\hat{C}_1$ is incircle of $ABC$, $\hat{C}_2$ is circumcircle of $ABC$

$I$ is center of $\hat{C}_1$, $O$ is center of $\hat{C}_2$, $d_T=|IO|$

$r_T$ is radius of $\hat{C}_1$, $R_T$ is radius of $\hat{C}_2$

$$t_1+t_3=|AB|,\ t_3+t_5=|BC|,\ t_5+t_1=|CA| \tag{2.1}$$

$$\hat{t}_m=\sqrt{(R_T-d_T)^2-r_T^2},\quad \hat{t}_M=\sqrt{(R_T+d_T)^2-r_T^2} \tag{2.2}$$

$$\hat{t}^2=\hat{t}_m\hat{t}_M. \tag{2.3}$$

THEOREM 1. *There are lengths (in fact positive numbers) $R,\ d,\ r$ such that holds*

$$(R-d)^2-r^2=(R_T-d_T)^2-r_T^2, \tag{2.4}$$

$$(R+d)^2-r^2=(R_T+d_T)^2-r_T^2, \tag{2.5}$$

$$r^2=4R_Tr_T+d_T^2. \tag{2.6}$$

*Proof.* It is easy to see that above relations are satisfied if

$$R=\frac{1}{2}\left(\sqrt{(R_T+d_T)^2+4R_Tr_T}+\sqrt{(R_T-d_T)^2+4R_Tr_T}\right), \tag{2.7}$$

$$d=\frac{1}{2}\left(\sqrt{(R_T+d_T)^2+4R_Tr_T}-\sqrt{(R_T-d_T)^2+4R_Tr_T}\right), \tag{2.8}$$

$$r=\sqrt{4R_Tr_T+r_T^2}. \tag{2.9}$$

So, using relation (2.6), the relations (2.4) and (2.5) can be written as

$$(R-d)^2=(R_T-d_T)^2+4R_Tr_T,\quad (R+d)^2=(R_T+d_T)^2+4R_Tr_T.$$

In connection with relation (2.6) let us remark that, according to Theorem 2.1 in [2], for triangle $ABC$ holds relation

$$4R_Tr_T+r_T^2=t_1t_3+t_3t_5+t_5t_1.$$

From the following theorem it will be clear that there is a bicentric hexagon whose incircle has radius $r$ and it holds

$$r^2=t_1t_3+t_3t_5+t_5t_1.$$

THEOREM 2. *There is a bicentric hexagon $A_1\dots A_6$ such that*

$$t_i+t_{i+1}=|A_iA_{i+1}|,\quad i=1,\dots,6 \tag{2.10}$$

*where $t_1,\ t_3,\ t_5$ are given by* (2.1) *and*

$$t_2=\frac{\hat{t}^2}{t_5},\quad t_4=\frac{\hat{t}^2}{t_1},\quad t_6=\frac{\hat{t}^2}{t_3}. \tag{2.11}$$

*The corresponding values $R,\ d,\ r$ for bicentric hexagon $A_1\dots A_6$ are given by* (2.7), (2.8) *and* (2.9).

*Proof.* First, using computer it is easy to check that $R$, $d$, $r$ given by (2.7)-(2.9) satisfy Fuss' relation (1.1) for bicentric hexagon. Namely, it can be found that (1.1) can be written as

$$64(R_T^2-d_T^2-2R_Tr_T)r_TR_T^2(r_T+R_T)(r_T+4R_T)=0, \tag{2.12}$$

where only second factor is zero since holds Euler's relation

$$R_T^2-d_T^2=2R_Tr_T. \tag{2.13}$$

From relations (2.4) and (2.5) we see that

$$t_m=\hat{t}_m, \quad t_M=\hat{t}_M \tag{2.14}$$

where

$$t_m=\sqrt{(R-d)^2-r^2}, \quad t_M=\sqrt{(R+d)^2-r^2}. \tag{2.15}$$

Thus, it holds

$$t^2=\hat{t}^2, \tag{2.16}$$

where

$$t^2=t_mt_M. \tag{2.17}$$

It is easy to prove that

$$r_T=\frac{t^2}{r}, \tag{2.18}$$

$$R_T=\frac{r^4-t^4}{4rt^2}. \tag{2.19}$$

So, using relations (2.9) and (2.16), the proof that holds (2.18) can be written as

$$\begin{aligned}
r^2r_T^2-t^4 &= (4R_Tr_T+r_T^2)r_T^2-\hat{t}^4\\
&= 4R_Tr_T^3-(R_T^2-d_T^2)^2+2r_T^2(R_T^2+d_T^2)\\
&= 4R_Tr_T^3-(2R_Tr_T)^2+2R_T^2r_T^2+2r_T^2d_T^2\\
&= 2r_T^2(-R_T^2+d_T^2+2R_Tr_T)=0, \text{ since holds (2.13)}
\end{aligned}$$

Using relations (2.9) and (2.18), the proof that holds (2.19) can be written as

$$R_T=\frac{r^2-r_T^2}{4r_T}=\frac{r^2-\frac{t^4}{r^2}}{4\frac{t^2}{r}}=\frac{r^4-t^4}{4rt^2}.$$

Now, let $t_1$, $t_3$, $t_5$ be given by (2.1). The tangent lengths $t_3$ and $t_5$, according to Theorem 2.2 in [2], can be expressed as

$$t_3=\frac{2R_Tr_Tt_1+\sqrt{D}}{r_T^2+t_1^2}, \quad t_5=\frac{2R_Tr_Tt_1-\sqrt{D}}{r_T^2+t_1^2} \tag{2.20}$$

where

$$D=4R_T^2r_T^2t_1^2-r_T^2(r_T^2+t_1^2)(4R_Tr_T+r_T^2+t_1^2). \tag{2.21}$$

Using relations for $R_T$ and $r_T$ given by (2.18) and (2.19), the tangent lengths $t_3$ and $t_5$ given by (2.20) can be expressed as

$$t_3 = \frac{(r^4 - t^4)t_1 + 2r^2\sqrt{D_1}}{2(t^4 + r^2t_1^2)}, \quad t_5 = \frac{(r^4 - t^4)t_1 - 2r^2\sqrt{D_1}}{2(t^4 + r^2t_1^2)} \tag{2.22}$$

where

$$D_1 = \frac{(r^4 - t^4)^2t_1^2 - 4t^4(t^4 + r^2t_1^2)(r^2 + t_1^2)}{4r^4}. \tag{2.23}$$

Now, using expression (1.9) it is not difficult to show that $t_2$ given by (2.11) can be written as $t_2$ given by (1.9a), that is

$$\frac{t^2}{t_5} = \frac{t_1(R^2 - d^2) + \sqrt{D}}{r^2 + t_1}, \tag{2.24}$$

where $D$ is given by (1.9b). Namely, starting from the above relation and using computer, we find that

$$\frac{t^2}{t_5} - \frac{t_1(R^2 - d^2) + \sqrt{D}}{r^2 + t_1} = 0 \Leftrightarrow \left[(R-d)^2 - r^2\right]\left[(R+d)^2 - r^2\right]\phi_6 = 0, \tag{2.25}$$

where $\phi_6$ is Fuss' relation given by (1.1) and $(R-d)^2 - r^2 = t_m^2$, $(R+d)^2 - r^2 = t_M^2$.

In the same way can be found that $t_3$, $t_4$, $t_5$ and $t_6$ are also tangent lengths of bicentric hexagon whose Fuss' relation $\phi_6 = 0$ is given by (1.1).

In this connection let us remark that it is easy to check that

$$t_1t_3 + t_3t_5 + t_5t_1 = t_2t_4 + t_4t_6 + t_6t_2 = r^2. \tag{2.26}$$

Also let us remark that from

$$\hat{t}_m \leqslant t_1 \leqslant \hat{t}_M \tag{2.27}$$

and from (2.14) it follows that

$$t_m \leqslant t_1 \leqslant t_M \tag{2.28}$$

Since holds (2.16), the relations (2.11) can be rewritten as

$$t_2 = \frac{t^2}{t_5}, \quad t_4 = \frac{t^2}{t_1}, \quad t_6 = \frac{t^2}{t_3}. \tag{2.29}$$

Thus, there is bicentric hexagon $A_1 \ldots A_6$ such that holds (2.10), where $t_1$, $t_3$, $t_5$ are given by (2.1) and $t_2$, $t_4$, $t_6$ are given by (2.29).

This completes the proof of Theorem 2.

COROLLARY 2.1. *It holds*

$$t^2 = r\sqrt{\frac{t_1t_3t_5}{t_1 + t_3 + t_5}}. \tag{2.30}$$

*Proof.* The above relation follows from $t_2t_4 + t_4t_6 + t_6t_2 = r^2$, using relations (2.11)

COROLLARY 2.2. *It holds*

$$t^2 = r\sqrt{\frac{t_2t_4t_6}{t_2+t_4+t_6}}. \tag{2.31}$$

THEOREM 3. *There is triangle $PQR$ whose incircle and circumcircle are the same as of the triangle $ABC$ considered in Theorem 2 and it holds*

$$t_2+t_4 = |PQ|, \quad t_4+t_6 = |QR|, \quad t_6+t_2 = |RP| \tag{2.32}$$

*where $t_2$, $t_4$, $t_6$ are given by* (2.11).

*Proof.* First we shall prove that there is triangle $PQR$ such that holds (2.32) and that incircle and circumcircle of $PQR$ are congruent to the incircle and circumcircle of the triangle $ABC$, namely, we shall prove that

$$(t_2+t_4+t_6)r_T^2 = t_2t_4t_6, \tag{2.33}$$

$$t_2t_4+t_4t_6+t_6t_2 = 4R_Tr_T + r_T^2. \tag{2.34}$$

The proof that holds (2.33), using relations (2.11), (2.16) and (2.18), can be written as

$$\begin{aligned}
(t_2+t_4+t_6)r_T^2 &= \left(\frac{t^2}{t_5}+\frac{t^2}{t_1}+\frac{t^2}{t_3}\right)r_T^2 \\
&= \frac{t^2(t_1t_3+t_3t_5+t_5t_1)r_T^2}{t_1t_3t_5} \\
&= \frac{t^2r^2r_T^2}{t_1t_3t_5} = \frac{t_6}{t_1t_3t_5}, \\
t_2t_4t_6 &= \frac{t^2}{t_5}\cdot\frac{t^2}{t_1}\cdot\frac{t^2}{t_3} = \frac{t^6}{t_1t_3t_5}.
\end{aligned}$$

Now, using relations (2.29) and (2.18), the proof that holds (2.34), can be written as

$$t_2t_4+t_4t_6+t_6t_2 = \frac{t^4(t_1+t_3+t_5)}{t_1t_3t_5} = \frac{t^4}{r_T^2} = r^2 = 4R_Tr_T + r_T^2,$$

since from (2.33) it follows

$$\frac{t_1+t_3+t_5}{t_1t_3t_5} = \frac{1}{r_T^2}.$$

Now we shall prove that triangle $PQR$ has property that holds

$$\text{area of } ABC \cdot \text{area of } PQR = r^2t^2, \tag{2.35}$$

that is

$$(t_1+t_3+t_5)(t_2+t_4+t_6)r_T^2 = (4R_Tr_T + r_T^2)\hat{t}^2$$

since by Theorem 2 it holds

$$r^2 = 4R_Tr_T + r_T^2 = t_1t_3 + t_3t_5 + t_5t_1, \quad t^2 = \hat{t}^2.$$

The proof, using relations (2.11), can be written as

$$(t_1+t_3+t_5)(t_2+t_4+t_6) = (t_1+t_3+t_5)\frac{\hat{t}^2(t_1t_3+t_3t_5+t_5t_1)r_T^2}{t_1t_3t_5} = r^2\hat{t}^2.$$

Besides we have to prove the following two lemmas.

LEMMA 1. *Let $EFG$ and $KLM$ be axial symmetric triangles whose incircle and circumcircle are the same as of the triangle $ABC$. Then*

$$\text{area of } EFG \cdot \text{area of } KLM = r^2t^2. \tag{2.36}$$

*(See Figure 2.1 For easy reference we have drawn two figures.)*

*Proof.* Let

$$u_1+u_3 = |EF|, \quad u_3+u_5 = |FG|, \quad u_5+u_1 = |GE|$$
$$u_2+u_4 = |KL|, \quad u_4+u_6 = |LM|, \quad u_6+u_2 = |MK|.$$

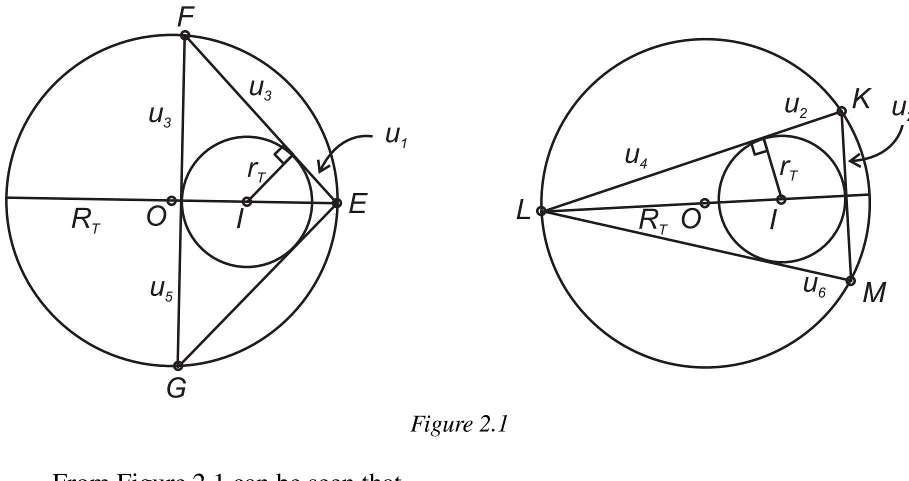



*Figure 2.1*

From Figure 2.1 can be seen that

$$u_1^2 = (R_T-d_T)^2 - r_T^2, \quad u_3^2 = R_T^2 - (r_T-d_T)^2, \quad u_5 = u_3 \tag{2.37}$$
$$u_2^2 = R_T^2 - (r_T+d_T)^2, \quad u_4^2 = (R_T+d_T)^2 - r_T^2, \quad u_6 = u_2. \tag{2.38}$$

In this connection let us remark that $\hat{t}_m = u_1$, $\hat{t}_M = u_4$.
It is easy to check that holds

$$\frac{\hat{t}^2}{u_5} = u_2, \quad \frac{\hat{t}^2}{u_1} = u_4, \quad \frac{\hat{t}^2}{u_3} = u_6.$$

So, for example, the check that $\frac{\hat{t}^2}{u_5} = u_2$ or $\hat{t^4} = u_2^2 u_5^2$ can be written as

$$\hat{t}_m^2\hat{t}_M^2 - u_2^2u_5^2 = \\ \left[(R_T^2 - d_T^2)^2 - 2r_T^2(R_T^2 + d_T^2 + r_T^2)\right] - \left[R_T^4 - 2R_T^2(r_T^2 + d_T^2) + (r_T^2 - d_T^2)\right] = 0.$$

LEMMA 2. *For given triangle $ABC$ there is triangle $PQR$ whose incircle and circumcircle are the same as of the triangle $ABC$ and it holds*

$$\textit{area of } ABC \cdot \textit{area of } PQR = r^2t^2.$$

*Proof.* Between all triangles whose incircle is $\hat{C}_1$ and circumcircle $\hat{C}_2$, the triangle $EFG$(shown in Figure 2.1a) has minimal area and triangle $KLM$ (shown in Figure 2.1b) has maximal area.(See [4, Theorem 1].) Since

$$\text{area of } ABC \leqslant \text{area of } KLM,$$

it is clear that there is triangle $PQR$ whose incircle and circumcircle are the same as of the triangle $ABC$ and it holds

$$\text{area of } ABC \cdot \text{area of } PQR = \text{area of } EFG \cdot \text{area of } KLM.$$

This completes the proof of Theorem 3.

REMARK 1. Triangles $ABC$ and $PQR$ can be called *conjugate triangles* in relation to bicentric hexagon $A_1 \dots A_6$.

In connection with important of Lemma 2 in Theorem 3 we shall prove the following theorem.

THEOREM 4. *Let $NPQ$ and $RST$ be any given two triangles whose incircle and circumcircle are the same as of the triangle $ABC$ considered in Theorem 2 and let it hold*

$$(v_1 + v_3 + v_5)(v_2 + v_4 + v_6)r_T^2 = r^2t^2, \tag{2.39}$$

*where*

$$v_1 + v_3 = |NP|, \quad v_3 + v_5 = |PQ|, \quad v_5 + v_1 = |QN|$$
$$v_2 + v_4 = |RS|, \quad v_4 + v_6 = |ST|, \quad v_6 + v_2 = |TR|.$$

*Then*

$$\{v_2, v_4, v_6\} = \left\{\frac{t^2}{v_1}, \frac{t^2}{v_3}, \frac{t^2}{v_5}\right\}. \tag{2.40}$$

*Proof.* According to Theorem 2.2 in [2], the equality (2.39) can be written as

$$\left(v_1+\frac{4R_Tr_Tv_1}{r_T^2+v_1^2}\right)\left(v_2+\frac{4R_Tr_Tv_2}{r_T^2+v_2^2}\right)r_T^2=r^2t^2$$

or

$$v_2^3-kv_2^2+(4R_Tr_T+r_T^2)v_2-kr_T^2=0, \tag{2.41}$$

where

$$k=\frac{r^2t^2(r_T^2+v_1^2)}{v_1r_T^2(v_1^2+4R_Tr_T+r_T^2)}. \tag{2.42}$$

Since

$$v_2+v_4+v_6=v_2+\frac{4R_Tr_Tv_2}{r_T^2+v_2^2}=v_4+\frac{4R_Tr_Tv_4}{r_T^2+v_4^2}=v_6+\frac{4R_Tr_Tv_6}{r_T^2+v_6^2},$$

it is clear that equation (2.41) has roots $v_2$, $v_4$, $v_6$, that is

$$\{(v_2)_1,(v_2)_2,(v_2)_3\}=\{v_2,v_4,v_6\}.$$

Thus, we have to prove that holds (2.40), that is

$$\begin{aligned}
&v_2+v_4+v_6=\frac{r^2t^2}{v_1v_3v_5},\\
&v_2v_4+v_4v_6+v_6v_2=\frac{t^4(v_1+v_3+v_5)}{v_1v_3v_5}=\frac{t^4}{r_T^2}=r^2=4R_Tr_T+r_T^2,\\
&v_2v_4v_6=\frac{t^6}{v_1v_3v_5}.
\end{aligned}$$

Using expressions

$$v_3=\frac{2R_Tr_Tv_1+\sqrt{D}}{r_T^2+v_1^2},\quad v_5=\frac{2R_Tr_Tv_1-\sqrt{D}}{r_T^2+v_1^2} \tag{2.43}$$

$$D=4R_T^2r_T^2v_1^2-r_T^2(r_T^2+v_1^2)(4R_Tr_T+r_T^2+v_1^2),$$

we find that

$$\frac{1}{v_1v_3v_5}=\frac{r_T^2+v_1^2}{r_T^2(r^2+v_1^2)v_1}. \tag{2.44}$$

It is easy to see that holds

$$\frac{r^2t^2}{v_1v_3v_5}=k,\quad \frac{t_6}{v_1v_3v_5}=kr_T^2$$

where $k$ is given by (2.42).

Of course, the same can be proved if instead of relations (2.43) we use relations

$$v_3=\frac{(r^4-t^4)v_1+2r^2\sqrt{D_1}}{2(t^4+r^2v_1^2)},\quad v_5=\frac{(r^4-t^4)v_1-2r^2\sqrt{D_1}}{2(t^4+r^2v_1^2)} \tag{2.45}$$

$$D_1 = \frac{(r^4-t^4)^2v_1^2 - 4t^4(t^4+r^2v_1^2)(r^2+v_1^2)}{4r^4}$$

where $v_3$ and $v_5$ are expressed as tangent lengths of a bicentric hexagon. In this case instead of (2.44) we have

$$\frac{1}{v_1v_3v_5} = \frac{t^4+r^2v_1^2}{v_1t^4(r^2+v_1^2)}\,.$$

It is easy to see that

$$\frac{t^4+r^2v_1^2}{v_1t^4(r^2+v_1^2)} = \frac{r_T^2+v_1^2}{r_T^2(r^2+v_1^2)v_1}\,,$$

since holds relation (2.18).

This completes the proof of Theorem 4.

Of course, proving this theorem we have proved(according to Theorem 2) that there is a bicentric hexagon whose tangent lengths are $v_1,\dots,v_6$.

THEOREM 5. *Let $A_1 \dots A_6$ be any given bicentric hexagon whose incircle is $C_1$ and circumcircle $C_2$. Then*

$$\frac{|A_1A_3|}{t_1+t_3} = \frac{4t^2Rr}{r^4-t^4}. \tag{2.46}$$

*Proof.* First we see (Figure 2.2) that

$$|A_1A_3|^2 = (t_1+t_2)^2 + (t_2+t_3)^2 - 2(t_1+t_2)(t_2+t_3)\cos 2\beta_2,$$

where $\beta_2 =$ measure of $\measuredangle IA_2A_3$.

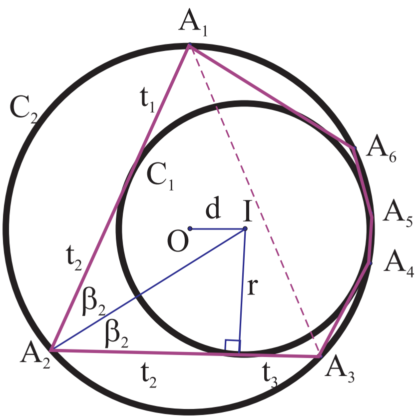



*Figure 2.2*

Since

$$\begin{aligned}\cos 2\beta_2 &= \cos^2\beta_2 - \sin^2\beta_2 = 2\cos^2\beta_2 - 1\\ &= \frac{2}{1+\tan^2\beta_2} - 1 = \frac{1-\tan^2\beta_2}{1+\tan^2\beta_2}\end{aligned}$$

and $\tan\beta_2 = \frac{r}{t_2}$, we have

$$\cos 2\beta_2 = \frac{t_2^2 - r^2}{t_2^2 + r^2},$$

$$\frac{|A_1A_3|^2}{(t_1+t_3)^2} = \frac{\left[(t_1+t_2)^2 + (t_2+t_3)^2 - 2(t_1+t_2)(t_2+t_3)\cdot\frac{t_2^2-r^2}{t_2^2+r^2}\right]}{(t_1+t_3)^2}. \tag{2.47}$$

Now, we shall show that holds (2.46), that is

$$\frac{\left[(t_1+t_2)^2 + (t_2+t_3)^2 - 2(t_1+t_2)(t_2+t_3)\cdot\frac{t_2^2-r^2}{t_2^2+r^2}\right]}{(t_1+t_3)^2} = \left(\frac{4t^2Rr}{r^4-t^4}\right)^2. \tag{2.48}$$

So, using the expressions for $t_3$ and $t_2$ given by (2.22) and (2.29), after rationalization we get

$$\phi_6 \cdot \left(\phi_6 + 16r^2\left[(R+d)^2 - r^2\right]\left[(R-d)^2 - r^2\right]\left[R^2 - d^2 + r^2\right]\right) = 0 \tag{2.49}$$

where

$$\phi_6 = 3(R^2-d^2)^4 - 4r^2(R^2+d^2)(R^2-d^2)^2 - 16R^2r^4d^2. \tag{2.50}$$

But $\phi_6 = 0$ is Fuss' relation for bicentric Hexagons given by (1.1). This proves theorem since all others factors in (2.49) are different from zero. Namely,

$$\begin{gathered}(R+d)^2 - r^2 = t_M^2 > 0, \quad (R-d)^2 - r^2 = t_m^2 > 0\\ \text{and } R^2 - d^2 + r^2 > 0 \text{ because } R > d.\end{gathered} \tag{2.51}$$

In the same way it can be shown that for each $i = 2,3,4,5,6$ analogously holds

$$\frac{|A_iA_{i+2}|^2}{(t_i+t_{i+2})^2} = \frac{4t^2Rr}{r^4-t^4}, \quad i = 2,\ldots,6. \tag{2.52}$$

REMARK 2. It may be interesting that Theorem 5 can also be proved in the following way. For short we shall prove that $2\hat{\beta}_1 = 2\overline{\beta}_1$, where

$$2\hat{\beta}_1 = \text{ measure of } \measuredangle BAC, \quad 2\overline{\beta}_1 = \text{ measure of } \measuredangle A_3A_1A_5.$$

In the proof we shall use relation $t_4 = \frac{t^2}{t_1}$ given by (2.29) and relation $r_T = \frac{t^2}{r}$ given by (2.18). (See Figure 4.)

Let $\beta_i =$ measure of $\measuredangle IA_iA_{i+1}$, $i = 1,\ldots,6$. First we can write

$$\begin{aligned}\cos 2\hat{\beta}_1 &= \frac{1-\tan^2\hat{\beta}_1}{1+\tan^2\hat{\beta}_1} = \frac{1-\left(\frac{r_T}{t_1}\right)^2}{1+\left(\frac{r_T}{t_1}\right)^2} = \frac{t_1^2 - r_T^2}{t_1^2 + r_T^2}\\ &= \frac{t_1^2 - \left(\frac{t^2}{r}\right)^2}{t_1^2 - \left(\frac{t^2}{r}\right)^2} = \frac{r^2t_1^2 - t^4}{r^2t_1^2 + t^4}.\end{aligned} \tag{2.53}$$

Since $A_1A_3A_4A_5$ is a chordal quadrilateral (see Figure 2.3), it holds

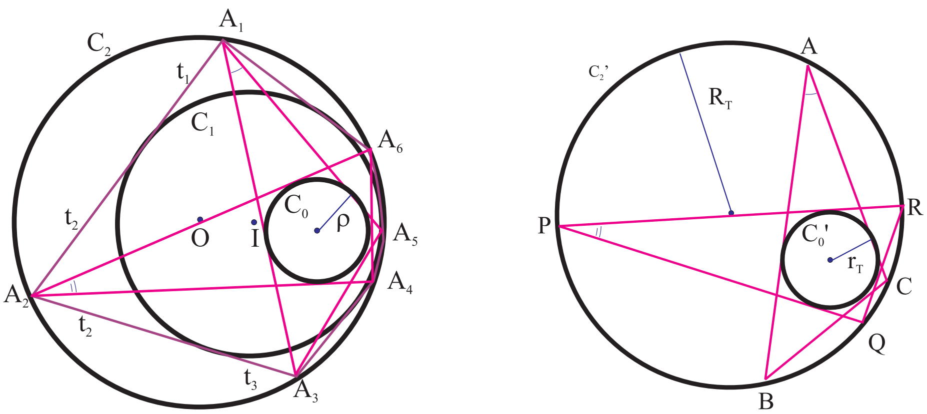
$$\begin{aligned} 2\bar{\beta}_1 + 2\beta_4 &= 180^\circ, \\ \cos 2\bar{\beta}_1 &= -\cos 2\beta_4. \end{aligned} \tag{2.54}$$

Also we have

$$\cos 2\beta_4 = \frac{t_4^2 - r^2}{t_4^2 + r^2}$$

or, since $t_4 = \frac{t^2}{t_1}$,

$$\cos 2\beta_4 = \frac{t^4 - r^2 t_1^2}{t^4 + r^2 t_1^2}. \tag{2.55}$$




*(a)* *(b)*

*Figure 2.3*

From (2.53) and (2.55) we see that $\cos 2\hat{\beta}_1 = -\cos 2\beta_4$. Since holds (2.54), it follows that $2\hat{\beta}_1 = 2\bar{\beta}_1$.

In the same way it can be proved for other pairs of corresponding angles.

COROLLARY 5.1. *It holds*

$$\frac{|A_1A_3|}{t_1+t_3} = \frac{|A_3A_5|}{t_3+t_5} = \frac{|A_5A_1|}{t_5+t_1} = \frac{|A_2A_4|}{t_2+t_4} = \frac{|A_4A_6|}{t_4+t_6} = \frac{|A_6A_2|}{t_6+t_2}.$$

*Proof.* If instead of (2.46) we write

$$\frac{|A_iA_{i+2}|}{t_i+t_{i+2}} = \frac{4t^2Rr}{r^4-t^4},$$

all essential remains the same.

COROLLARY 5.2. *Let ABC and PQR be triangles such that*

$$|AB| = t_1 + t_3, \qquad |BC| = t_3 + t_5, \qquad |CA| = t_5 + t_1$$
$$|PQ| = t_2 + t_4, \qquad |QR| = t_4 + t_6, \qquad |RP| = t_6 + t_2$$

*where*

$$t_i + t_{i+1} = |A_iA_{i+1}|, \quad i = 1, \dots, 6.$$

*Then*

$$A_1A_3A_5 \sim ABC, \quad A_2A_4A_6 \sim PQR.$$

*If $\Pi$ is the plane which contain Figure 2.3, then there is a similarity $f : \Pi \to \Pi$ such that*

$$f(A_1) = A,\ f(A_3) = B,\ f(A_5) = C,\ f(A_2) = P,\ f(A_4) = Q,\ f(A_6) = R.$$

*The coefficient of similarity is* $\frac{4t^2Rr}{r^4-t^4}$.

COROLLARY 5.3. *It holds*

$$\textit{area of } ABC \cdot \textit{area of } PQR = t^2r^2. \tag{2.56}$$

*Proof.* Since

$$\text{area of } ABC = (t_1 + t_3 + t_5)\frac{t^2}{r},$$

$$\text{area of } PQR = (t_2 + t_4 + t_6)\frac{t^2}{r},$$

and by Theorem 3 in [3] it holds

$$\sum_{i=1}^{6} t_it_{i+1} = \frac{r^4 - 3t^4}{t^2},$$

we can write

$$(t_1 + t_3 + t_5)(t_2 + t_4 + t_6)\left(\frac{t^2}{r}\right)^2 = \left(\frac{r^4 - 3t^4}{t^2} + 3t^2\right)\left(\frac{t^2}{r}\right)^2 = t^2r^2. \tag{2.57}$$

$$\text{area of } A_1A_3A_5 \cdot \text{ area of } A_2A_4A_6 = \left(\frac{4t^2Rr}{r^4 - t^4}\right)^4 t^2r^2. \tag{2.58}$$

COROLLARY 5.4. *It holds*

$$\textit{perimeter of } A_1A_3A_5 \cdot \textit{ perimeter of } A_2A_4A_6 = \left(\frac{8t^2Rr}{r^4 - t^4}\right)^2 \frac{r^4}{t^2}. \tag{2.59}$$

*Proof.* By Theorem 5 it holds

$$\begin{aligned}|A_1A_3|+|A_3A_5|+|A_5A_1| &= \frac{4t^2Rr}{r^4-t^4}\left[(t_1+t_3)+(t_3+t_5)+(t_5+t_1)\right]\\ &= \frac{8t^2Rr}{r^4-t^4}(t_1+t_3+t_5),\\ |A_2A_4|+|A_4A_6|+|A_6A_2| &= \frac{8t^2Rr}{r^4-t^4}(t_2+t_4+t_6).\end{aligned}$$

Thus

$$(|A_1A_3|+|A_3A_5|+|A_5A_1|)\cdot(|A_2A_4|+|A_4A_6|+|A_6A_2|) = \left(\frac{8t^2Rr}{r^4-t^4}\right)^2\frac{r^4}{t^2},$$

since from (2.57) we have $(t_1+t_3+t_5)(t_2+t_4+t_6) = \frac{r^4}{t^2}$.

COROLLARY 5.5. *Let $h_1$, $h_2$, $h_3$ be altitudes of the triangle $A_1A_3A_5$, and let $\bar{h}_1$, $\bar{h}_2$, $\bar{h}_3$ be altitudes of the triangle $A_2A_4A_6$. Then*

$$H(h_1,h_2,h_3)H(\bar{h}_1,\bar{h}_2,\bar{h}_3) = 144\left(\frac{t^4R}{r^4-t^4}\right)^2. \tag{2.60}$$

*where $H(x_1,x_2,x_3)$ denotes harmonic mean of $x_1$, $x_2$, $x_3$.*

*Proof.* Let areas of triangles$A_1A_3A_5$ and $A_2A_4A_6$ be denoted respectively by $J_1$ and $J_2$. From

$$\begin{aligned}|A_1A_3| &= \frac{2J_1}{h_1}, & |A_3A_5| &= \frac{2J_1}{h_2}, & |A_5A_1| &= \frac{2J_1}{h_3}\\ |A_2A_4| &= \frac{2J_2}{\bar{h}_1}, & |A_4A_6| &= \frac{2J_2}{\bar{h}_2}, & |A_6A_2| &= \frac{2J_2}{\bar{h}_3}\end{aligned}$$

it follows

$$\frac{9J_1J_2}{(|A_1A_3|+|A_3A_5|+|A_5A_1|)\cdot(|A_2A_4|+|A_4A_6|+|A_6A_2|)} = H(h_1,h_2,h_3)H(\bar{h}_1,\bar{h}_2,\bar{h}_3),$$

which can be written as (2.60) since hold (2.58) and (2.59).

COROLLARY 5.6. *It holds*

$$|A_1A_3|\cdot|A_2A_4|\cdot|A_3A_5|\cdot|A_4A_6|\cdot|A_5A_1|\cdot|A_6A_2| = \left(\frac{4t^2Rr}{r^4-t^4}\right)^6 16t^2R_T^2r^2. \tag{2.61}$$

*In other words, the product of side-lengths of the triangles $A_1A_3A_5$ and $A_2A_4A_6$ is a constant.*

*Proof.* Since

$$|A_1A_3| = \frac{4t^2Rr}{r^4-t^4}(t_1+t_3),\dots,|A_6A_2| = \frac{4t^2Rr}{r^4-t^4}(t_6+t_2),$$

$$\begin{aligned}\frac{(t_1+t_3)(t_3+t_5)(t_5+t_1)}{4R_T} &= (t_1+t_3+t_5)\frac{t^2}{r} = \text{ area of } ABC,\\ \frac{(t_2+t_4)(t_4+t_6)(t_6+t_2)}{4R_T} &= (t_2+t_4+t_6)\frac{t^2}{r} = \text{ area of } PQR,\end{aligned} \tag{2.62}$$

and holds (2.57), we get (2.61).

THEOREM 6. *Let $ABC$ and $PQR$ be triangles as it is said in Theorem 2 and Theorem 3. Then there is bicentric hexagon $B_1 \dots B_6$ whose vertices are $A$, $P$, $B$, $Q$, $C$, $R$. Corresponding values $R_H$, $r_H$, $d_H$ are given by*

$$R_H = R_T \tag{2.63}$$

$$r_H = \frac{(4R_Tr_T+r_T^2)^2-\hat{t}^4}{2\hat{t}^2\left(\sqrt{(R_T+d_T)^2+4R_Tr_T}+\sqrt{(R_T-d_T)^2+4R_Tr_T}\right)}\,, \tag{2.64}$$

$$d_H = \frac{\left[(4R_Tr_T+r_T)^2-\hat{t}^4\right]\left(\sqrt{(R_T+d_T)^2+4R_Tr_T}-\sqrt{(R_T-d_T)^2+4R_Tr_T}\right)}{4\hat{t}^2\sqrt{4R_Tr_T+r_T}\left(\sqrt{(R_T+d_T)^2+4R_Tr_T}+\sqrt{(R_T-d_T)^2+4R_Tr_T}\right)}\,. \tag{2.65}$$

*Proof.* The above relations can be written as

$$R_H = k\cdot R, \quad r_H = k\cdot r, \quad d_H = k\cdot d, \tag{2.66}$$

where $R$, $r$, $d$ are given by (2.7)-(2.9) and $k$ is given by

$$k = \frac{r^4-\hat{t}^4}{4Rr\hat{t}^2} \tag{2.67}$$

Thus, there is similarity with ratio $k$ which maps hexagon $A_1 \dots A_6$ onto hexagon $B_1 \dots B_6$.

Of course, tangent lengths of bicentric hexagon $B_1 \dots B_6$ are given by

$$k\cdot t_i,\ i = 1,\dots,6$$

where $t_1$, $t_3$, $t_5$ and $t_2$, $t_4$, $t_6$ are tangent lengths of triangles $ABC$ and $PQR$ respectively.

In this connection let us remark that

$$r_H^2 = (kt_1)(kt_3)+(kt_3)(kt_5)+(kt_5)(kt_1) = k^2(t_1t_3+t_3t_5+t_5t_1) = k^2r^2,$$

from which follows $r_H = kr$.

Also let us remark that

$$|A_1A_2| = t_1 + t_2, \quad |AP| = kt_1 + kt_2 = k(t_1 + t_2)$$
$$|A_2A_3| = t_2 + t_3, \quad |PB| = kt_2 + kt_3 = k(t_2 + t_3)$$

and so on. Thus

$$|A_1A_2| : |AP| = |A_2A_3| : |PB| = \ldots = |A_6A_1| : |RA| = k.$$

(See Figure 2.5 and 2.6.)

COROLLARY 6.1. *It holds*

$$\frac{R_T}{R} = k. \tag{2.68}$$

*Proof.* Follows from (2.63) and (2.66), that is, from $R_H = R_T$ and $R_H = kR$.

COROLLARY 6.2. *Let $A_1 \ldots A_6$ be any given bicentric hexagon. Then*

$$\frac{t_1+t_3}{|A_1A_3|} = \frac{t_3+t_5}{|A_3A_5|} = \frac{t_5+t_1}{|A_5A_1|} = \frac{t_2+t_4}{|A_2A_4|} = \frac{t_4+t_6}{|A_4A_6|} = \frac{t_6+t_2}{|A_6A_2|} \tag{2.69}$$

*where $t_i + t_{i+1} = |A_iA_{i+1}|$, $i = 1, \ldots, 6$.*

*Proof.* For every given bicentric hexagon $A_1 \ldots A_6$ there is bicentric hexagon $APBQCR$ as it is described in given theorems. According to Theorem 6 it holds

$$\frac{t_i + t_{i+2}}{|A_iA_{i+2}|} = \frac{R_T}{R}, \; i = 1, \ldots, 6. \tag{2.70}$$

If similarity with ratio $\frac{r^4-t^4}{4Rrt^2}$ is denoted by $s$, then

$$s(A_1) = A, \quad s(A_3) = B, \quad s(A_2) = P, \quad s(A_4) = Q \text{ and so on.}$$

COROLLARY 6.3. *Triangles $A_1A_2A_3$ and $A_2A_4A_6$ have not only the same circumcircle but also the same incircle.*

COROLLARY 6.4. *If $\rho =$ radius of incircle of $A_1A_3A_5$, then $\rho = kr_T$.*

COROLLARY 6.5. *area of $A_1A_3A_5 = \frac{1}{k^2} \cdot$ area of $ABC$.*

COROLLARY 6.6. *It holds*

$$\textit{area of } A_1A_3A_5 \cdot \textit{area of } A_2A_4A_6 = \frac{r^2t^2}{k^4}.$$

COROLLARY 6.7. *If triangle $ABC$ is given, then triangle $PQR$ such that $APBQCR$ is a bicentric hexagon, can be constructed as shown in Figure 2.4. Point $S$ can be constructed using Figure 2.4.a, where $EFG$ and $KLM$ are axial symmetric triangles.*

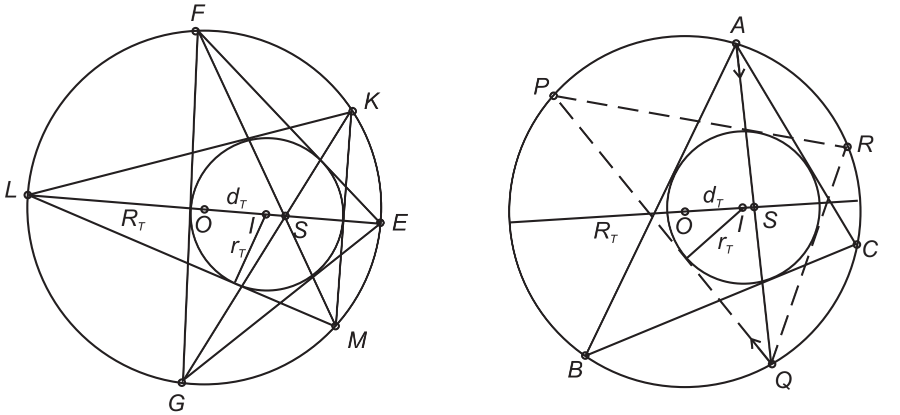



*Figure 2.4*

As it is well-known, it holds: If $A_1 \dots A_{2n}$ is bicentric $2n$-gon then for each $i = 1, 2, \dots, n$, the chord $A_i A_{i+n}$ passes through $S$.

REMARK 3. It may be interesting that

$$Rd = R_T d_T. \tag{2.71}$$

This follows from (2.4) and (2.5), that is, from $t_m^2 - t_M^2 = \hat{t}_m^2 - \hat{t}_M^2$.

EXAMPLE 1. Let $\hat{C}_1$ and $\hat{C}_2$ be given circles, one inside of the other, such that holds Euler's relation

$$R_T^2 - d_T^2 = 2R_T r_T,$$

where $R_T = 3.208333333$, $r_T = 0.666666666$, $d_T = 2.45267711$. Then

$$\hat{t}_m = 0{,}355769427, \quad \hat{t}_M = 5.6216185, \quad \hat{t}^2 = 2. \tag{2.72}$$

Then there is a triangle $ABC$ whose incircle is $\hat{C}_1$ and circumcircle $\hat{C}_2$ such that

$$t_1 + t_3 = |AB|, \quad t_3 + t_5 = |BC|, \quad t_5 + t_1 = |CA|,$$

where

$$t_1 = 5, \quad t_3 = 1.176380598, \quad t_5 = 0.504842108. \tag{2.73}$$

The tangent lengths of triangle $PQR$ which is connected with triangle $ABC$ as it is said in Theorem 3, are given by

$$t_2 = \frac{\hat{t}^2}{t_5} = 3.961634659, \quad t_4 = \frac{\hat{t}^2}{t_1} = 0.4, \quad t_6 = \frac{\hat{t}^2}{t_3} = 1.700130047.$$

The lengths $R_H$, $r_H$, $d_H$, according to relations (2.66), are given by

$$R_H = R_T, \quad r_H = 2.049390153, \quad d_H = 1.144582651.$$

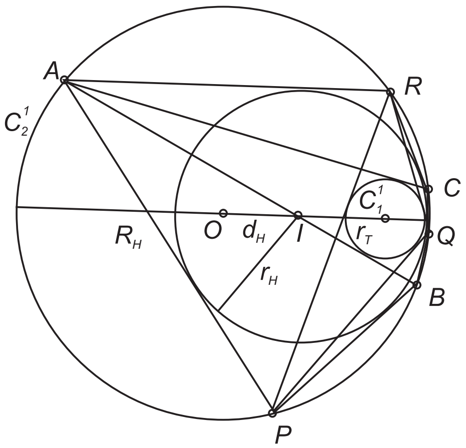



*Figure 2.5*

The bicentric hexagon $APBQCR$ is sketched in Figure 2.5.

Concerning bicentric hexagon $A_1 \ldots A_6$ for which hold relations (2.7), (2.8) and (2.9), we have

$$R = 4.696519101, \quad d = 1.67549744, \quad r = 3.$$

Corresponding bicentric hexagon is sketched in Figure 2.6.

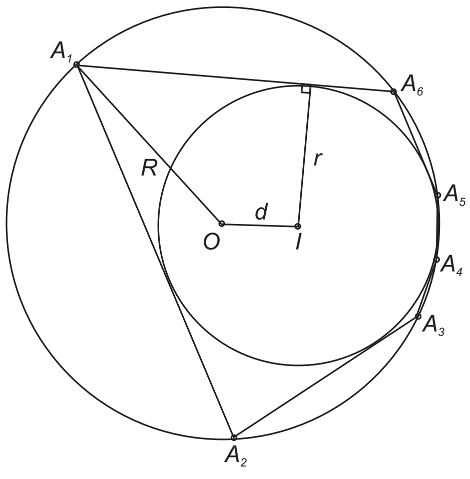



*Figure 2.6*

Similarity between this hexagon and hexagon shown in Figure 2.5 is observable.

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