Sharp well-posedness of the Ostrovsky, Stepanyams and Tsimring equation

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Abstract. In this paper, we study the Ostrovsky, Stepanyams and Tsimring equation. We show that the associated initial value problem is locally well-posed in Sobolev spaces $H^s(\mathbb{R})$ for $s > -3/2$. We also prove that our result is sharp in the sense that the flow map of this equation fails to be $C^2$ in $H^s(\mathbb{R})$ for $s < -3/2$.

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Key words: local well-posedness, OST equation, Sobolev spaces

1. Introduction

This paper is concerned with the well-posedness of the following initial value problem (IVP) for the Ostrovsky, Stepanyams and Tsimring (OST) equation:

$$
\begin{cases}
    u_t + u_{xxx} - \eta(\mathcal{H} u_x + \mathcal{H} u_{xxx}) + uu_x = 0, \quad x \in \mathbb{R}, \; t \geq 0, \\
    u(x,0) = u_0(x),
\end{cases}
$$

(1)

where $u = u(x,t)$ is a real-valued function, $\eta > 0$ and $\mathcal{H}$ denotes the usual Hilbert transformation given by

$$
\mathcal{H}\varphi(x) = \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{\varphi(x-y)}{y} \, dy,
$$

for $\varphi \in \mathcal{S}(\mathbb{R})$. Equation (1) was derived by Ostrovsky et al. in [18] to describe the radiational instability of long non-linear waves in a stratified flow caused by internal wave radiation from a shear layer.

We recall that the IVP for (1) is locally well-posed in Banach space $X$ if the solution uniquely exists in a certain time interval $[-T,T]$ (unique existence), the solution describes a continuous curve in $X$ in the interval $[-T,T]$ whenever initial data belong to $X$ (persistence), and the solution varies continuously depending upon the initial data (continuous dependence), i.e. continuity of application $u_0 \mapsto u(t)$ from $X$ to $C([-T,T];X)$.

Note that the OST equation is a modification of the well-known KdV equation

$$
u_t + u_{xxx} + uu_x = 0.$$
It is known that the KdV equation arises in modeling of one-dimensional long wavelength surface waves propagating in weakly nonlinear dispersive media [1, 4, 12, 22], as well as the evolution of weakly nonlinear ion acoustic waves in plasmas [21]. Different from the KdV equation which is of purely dispersive type, the OST equation is of the dispersive–dissipative type.

A model similar to (1) is the Korteweg-de Vries-Kuramoto-Sivashinsky (KdV-KS) equation

$$\begin{cases}
  u_t + u_{xxx} + \eta (u_{xx} + uu_{xxx}) + u_x^2 = 0, & x \in \mathbb{R}, \quad t \geq 0, \\
  u(x,0) = u_0(x).
\end{cases}$$

This equation arises as a model for long waves on a viscous fluid flowing down an inclined plane and describing drift waves in plasma [8, 20]. The IVP for (2) was studied by Biagioni et al. [3]. They proved that (2) is well-posed in $H^s(\mathbb{R})$ for $s \geq 1$, by using the properties of the semi-group associated with the linear problem. They also obtained a global solution in $H^s(\mathbb{R})$ for $s \geq 1$, making use of the conserved quantities for the Korteweg-de Vries equation. Recently, Carvajal and Panthee in [7], considered the derivative equation of (2) and obtained the local well-posedness of (2) in $H^s(\mathbb{R})$ for $s > -3/4$ (see also [6]).

The first work on the well-posedness of the IVP for (1) was carried out by Alvarez in [2]. He proved that (1) is locally well-posed in $H^s(\mathbb{R})$ for $s > 1/2$ and globally well-posed in $H^s(\mathbb{R})$ for $s \geq 1$. In [5], Carvajal improved these results. He proved that (1) is locally well-posed in $H^s(\mathbb{R})$, for $s \geq 0$, and globally well-posed in $L^2(\mathbb{R})$. Zhao and Cui in [23] used the ideas of Molinet and Ribaud in [15, 16, 17], employed the method of bilinear estimate in the Bourgain-type spaces and proved that (1) is locally well-posed in $H^s(\mathbb{R})$ for $s > -3/4$; which coincides with the sharp local well-posedness result for the KdV equation established by Kenig et al. in [14]. The authors in [24] improved their previous results by showing that the IVP for (1) is locally well-posed in $H^s(\mathbb{R})$ for $s > -1$.

In this paper we shall prove that (1) is locally well-posed in $H^s(\mathbb{R})$ for $s > -3/2$. Indeed, we use purely dissipative methods as applied by Dix in [9] to study the IVP for the KdV-Burgers equation

$$\begin{cases}
  u_t + u_{xxx} + uu_x = u_{xx}, & x \in \mathbb{R}, \quad t \geq 0 \\
  u(x,0) = u_0(x).
\end{cases}$$

The main ingredient consists of applying a fixed-point theorem to the integral equation associated to (1) in time-weighted spaces.

Regarding the sharpness of our result, we establish that the flow map of the OST equation fails to be $C^2$ in $H^s(\mathbb{R})$ for $s < -3/2$. This means that a Picard iteration cannot be used to obtain a solution of (1).

Before presenting the precise statement of our main result, let us first introduce some definitions and notations.

Without loss of generality, later on we assume that $\eta = 1$. We shall denote by $\hat{\varphi}$ the Fourier transform of $\varphi$, defined as

$$\hat{\varphi}(\xi) = \int_{\mathbb{R}} \varphi(x)e^{-ix\xi} \, dx.$$
For $s \in \mathbb{R}$, by $H^s(\mathbb{R})$ we denote the nonhomogeneous Sobolev space defined by

$$H^s(\mathbb{R}) = \{ \varphi \in \mathcal{S}'(\mathbb{R}) : \| \varphi \|_{H^s(\mathbb{R})} < \infty \},$$

where

$$\| \varphi \|_{H^s(\mathbb{R})} = \left\| (1 + \xi^2)^{s/2} \hat{\varphi}(\xi) \right\|_{L^2(\mathbb{R})},$$

and $\mathcal{S}'(\mathbb{R})$ is the space of tempered distributions.

For any positive numbers $a$ and $b$, the notation $a \lesssim b$ means that there exists a positive constant $c$ such that $a \leq cb$; and we denote $a \sim b$ when, $a \lesssim b$ and $b \lesssim a$.

For $s \in \mathbb{R}$ and $u_0 \in H^s(\mathbb{R})$, consider the following linear problem associated to (1):

$$\begin{cases}
u_t + \nu_{xxx} - \mathcal{H}_x u - \mathcal{H}_x u_{xxx} = 0, & x \in \mathbb{R}, \ t \in \mathbb{R}, \\ u(x, 0) = u_0(x). &
\end{cases}(4)$$

The unique solution of (4) is given by the semigroup $\{U(t)\}_{t \geq 0}$ defined as follows:

$$u(t) = U(t)u_0 = \int_{\mathbb{R}} e^{t(i|\xi|^3 - |\xi|^3 + |\xi|)} \hat{u}_0(\xi) \, d\xi.$$ 

The main results of this paper read as follows:

**Theorem 1.** Let $s > -3/2$. Then for all $u_0 \in H^s(\mathbb{R})$, there exist

$$T = T \left( \| u_0 \|_{H^s(\mathbb{R})} \right) > 0,$$

a space

$$\mathcal{X}^s_T \hookrightarrow C \left( [0, T]; H^s(\mathbb{R}) \right)$$

and a unique solution $u(t)$ of (1) such that $u(0) = u_0$. Moreover, $u \in C((0,T); H^{\infty}(\mathbb{R}))$ and the map solution

$$\mathcal{F} : H^s(\mathbb{R}) \rightarrow \mathcal{X}^s_T \cap C \left( [0, T]; H^s(\mathbb{R}) \right), \ u_0 \mapsto u,$$

is smooth.

**Theorem 2.** Let $s < -3/2$, if there exists some $T > 0$ such that the Cauchy problem (1) is locally well-posed in $H^s(\mathbb{R})$, then the flow-map data solution

$$\mathcal{F} : H^s(\mathbb{R}) \rightarrow C([0, T]; H^s(\mathbb{R})), \ u_0 \mapsto u(t)$$

is not $C^2$ at zero.

The rest of this paper is as follows. In Section 2 we present the time-weighted space $\mathcal{X}^s_T$ and obtain some basic linear and bilinear estimates in this space. Section 3 is devoted to proving the local well-posedness in this space. We also establish that the flow map of the OST equation fails to be $C^2$ in $H^s(\mathbb{R})$ for $s < -3/2$. 

2. Linear and bilinear estimates

In this section, we introduce a suitable Banach space in order to derive appropriate linear and bilinear estimates.

To prove Theorem 1, we will make the assumption $-3/2 < s < 0$, since the case $0 \leq s$ follows by similar arguments. Our strategy is to use a contraction argument on the integral equation associated to (1):

$$u(t) = \Phi(u(t)) := U(t)u_0 + \frac{1}{2} \int_0^t U(t-t')\partial_x(u^2(t')) \, dt'.$$

For $0 < T \leq T^* = \min\{1, 9|s|/2\}$, we define the Banach space

$$\mathcal{X}_{T}^s = \{u \in C\left([0,T];H^s(\mathbb{R})\right) : \|u\|_{\mathcal{X}_{T}^s} < \infty \},$$

where

$$\|u\|_{\mathcal{X}_{T}^s} = \sup_{t \in [0,T]} \left(\|u(t)\|_{H^s(\mathbb{R})} + t^{1/3}\|u(t)\|_{L^2(\mathbb{R})}\right).$$

We note that $T^* = 1$, if $s \leq -2/9$.

First we state the following lemma which is useful in establishing smoothness properties for the semigroup of (1). The proof is straightforward.

**Lemma 1.** For any $a > 0$ and $0 < t \leq 9a$, we have for all $\xi \in \mathbb{R}$,

$$\xi^{2a} e^{-t(|\xi|^3 - |\xi|)} \leq \rho^{2a} e^{-t(\rho^3 - \rho)} =: \psi(a,t),$$

where

$$\rho = \left(\frac{9a + \sqrt{81a^2 - t^2}}{3}\right)^{1/3} t^{-1/3} + \frac{t^{1/3}}{3 \left(9a + \sqrt{81a^2 - t^2}\right)^{1/3}}.$$

Moreover, if $a = 0$, then (6) holds for $\psi(0,t) = \exp\left(\frac{2t}{3\sqrt{3}}\right)$.

Now, we will turn our attention to estimate the linear part in $\mathcal{X}_{T}^s$.

**Proposition 1.** Let $0 < T \leq T^*$, $s < 0$ and $u_0 \in H^s(\mathbb{R})$, then

$$\sup_{t \in [0,T]} \|U(t)u_0\|_{H^s(\mathbb{R})} \leq e^{\frac{2T}{3\sqrt{3}}} \|u_0\|_{H^s(\mathbb{R})},$$

and

$$\sup_{t \in [0,T]} t^{1/3}\|U(t)u_0\|_{L^2(\mathbb{R})} \lesssim \Upsilon_s(T)\|u_0\|_{H^s(\mathbb{R})},$$

where

$$\Upsilon_s(t) = e^{\frac{2t}{3\sqrt{3}}} + t^{1/3}\psi(|s|/2,t)$$

is a continuous nondecreasing function on $[0,T^*]$ and $\psi$ is defined as in Lemma 1.
**Proof.** Inequality (7) follows immediately from Lemma 1. To prove inequality (8), we first observe from 0 < T ≤ 1 that

\[
I^{s/3} \leq \frac{(1 + t^{2/3} \xi^2)^{|s|/2}}{(1 + \xi^2)^{|s|/2}},
\]

for all t ∈ [0, T]. Hence, by using the Plancherel theorem and the definition of U(t), we deduce that

\[
I^{s/3} ||U(t)u_0||_{L^2(R)} \leq \left(1 + t^{2/3} \xi^2\right)^{|s|/2} e^{-t(\xi^2 - |\xi|)} \left(1 + \xi^2\right)^{|s|/2} u_0(\xi) \right)_{L^2(R)} \leq \left(\left|e^{-t(\xi^2 - |\xi|)}\right|_{L^\infty(R)} + \left|t^{2/3} \xi^2\right|^{s/2} e^{-t(\xi^2 - |\xi|)}\right)_{L^\infty(R)} \|u_0\|_{H^s(R)}.
\]

Lemma 1 implies the desired inequality in (8).

The next step is to derive the bilinear estimate.

**Proposition 2.** Let 0 < t ≤ T ≤ T* and s ∈ (-3/2, 0); then

\[
\left\|\int_0^t U(t-t')\partial_x(uv)(t') \, dt'\right\|_{\mathcal{X}^s_T} \lesssim e^{2\sqrt{T}/\sqrt{T^*}} T^{(2s+3)/6} \|u\|_{\mathcal{X}^s_T} \|v\|_{\mathcal{X}^s_T},
\]

for all u, v ∈ \mathcal{X}^s_T, where the constant of the above inequality depends only on s.

**Proof.** Let 0 ≤ t ≤ T. We have (1 + \xi^2)^{s/2} ≤ |\xi|^s, since s < 0. So by using the Minkowski inequality and the definition of U(t), we obtain that

\[
\left\|\int_0^t U(t-t')\partial_x(uv)(t') \, dt'\right\|_{H^s(R)} \leq \int_0^t \|\xi(1 + \xi^2)^{s/2} e^{(t-t')(\xi^2 - |\xi|^2)} (u(t')v(t'))\|^s(\xi) \|_{L^2(R)} \, dt' \leq \int_0^t \|\xi^{1+s} e^{(t-t')(\xi^2 - |\xi|^2)}\|^s(\xi) \|u(t') * v(t')(\xi)\|_{L^\infty(R)} \, dt'.
\]

The Young inequality implies that

\[
\left\|u(t') * v(t')(\xi)\right\|_{L^\infty(R)} \leq \frac{\|u\|_{\mathcal{X}^s_T} \|v\|_{\mathcal{X}^s_T}}{|t'|^{2|s|/3}}.
\]

Therefore, by changing the variable, we obtain

\[
\left\|\int_0^t U(t-t')\partial_x(uv)(t') \, dt'\right\|_{H^s(R)} \leq \left(\int_0^t \|\xi^{1+s} e^{(t-t')(\xi^2 - |\xi|^2)}\|^s(\xi) \left|\frac{1}{|t - t'|^{2|s|/3}}\right| \, dt'\right) \|u\|_{\mathcal{X}^s_T} \|v\|_{\mathcal{X}^s_T}.
\]
To estimate the integral on the right-hand side of (12), we use a change of the variable to deduce that
\[
\left\|\xi^{1+s} e^{t'(|\xi|-|t'|^2)}\right\|_{L^2(\mathbb{R})} \leq |t'|^{-(2s+3)/6} \left\| e^{(t'\xi^2/3 - |\xi|^3/2)} \right\|_{L^\infty(\mathbb{R})} \left\| \xi^{1+s} e^{-|\xi|^3/2} \right\|_{L^2(\mathbb{R})}
\]
where in the last inequality we used the following inequality
\[
e^{(t'\xi^2/3 - |\xi|^3/2)} \leq \frac{1}{\sqrt{27}}, \quad \forall \xi \in \mathbb{R}.
\]
Therefore, we get from (12), (13) and a change of the variable that
\[
\left\| \int_0^t U(t - t')\partial_u(uv)(t')dt' \right\|_{H^s(\mathbb{R})} \leq e^{2\sqrt{27}/3 - (2s+3)/6} \left\| t'\left| |t'|^{-(2s+3)/6} |1 - t'|^{2s/3} dt' \right\|_2 \left\| u \right\| \| v \| \| \mathcal{X} \|^2,
\]
for all $0 \leq t \leq T$. On the other hand, a similar argument allows us to deduce for all $0 \leq t \leq T$ that
\[
\left\| t^{s/3} \int_0^t U(t - t')\partial_u(uv)(t')dt' \right\|_{L^2(\mathbb{R})} \leq t^{s/3} \int_0^t \left\| \xi e^{(t'\xi^2/3 - |\xi|^3/2)} \right\|_{L^2(\mathbb{R})} \left\| u(t') \ast v(t')(\xi) \right\|_{L^\infty(\mathbb{R})} \| \mathcal{X} \|_2 \| v \| \| \mathcal{X} \|
\]
\[
\leq t^{s/3} \left( \int_0^t \left\| \xi e^{(t'\xi^2/3 - |\xi|^3/2)} \right\|_{L^2(\mathbb{R})} \left| t - t' \right|^{2s/3} \| \mathcal{X} \|_2 \| v \| \| \mathcal{X} \|
\]
\[
\leq e^{2\sqrt{27}/3} T^{-(2s+3)/6} \left( \int_0^t \left| t'\right|^{-1/2} |1 - t'|^{-2s/3} dt' \right) \left\| u \right\| \| v \| \| \mathcal{X} \|^2.
\]

This completes the proof. □

Remark 1. If we consider $s' > s > -3/2$, then after modifying the space $\mathcal{X}^{s'}_T$ by
\[
\mathcal{X}^{s'}_T = \{ u \in \mathcal{X}^{s'}_T; \| u \|_{\mathcal{X}^{s'}_T} < \infty \}
\]
with
\[
\| u \|_{\mathcal{X}^{s'}_T} = \| u \|_{\mathcal{X}^{s'}_T} + \sup_{t \in [0,T]} t^{s'/3} \left\| (1 - \partial_x^2)^{(s'-s)/2} u(t) \right\|_{L^2(\mathbb{R})}
\]
and using
\[
(1 + \xi^2)^{s'/2} \leq (1 + \xi^2)^{s/2} (1 + \xi^2)^{(s'-s)/2} + (1 + \xi^2)^{s/2} (1 + (\xi - \xi_1)^2)^{(s'-s)/2}
\]
and Proposition 2 we can deduce that for \( s > s' > -3/2, \) we have (see (10))
\[
\left\| \int_0^t U(t - t') \partial_x (uv)(t') \, dt' \right\|_{\mathcal{X}^{s'}} \lesssim e^{\sqrt{T}/\sqrt{2} T^\theta(s)} \left( \|u\|_{\mathcal{X}_T^{s'}} \|v\|_{\mathcal{X}_T^{s'}} + \|v\|_{\mathcal{X}_T^{s'}} \|u\|_{\mathcal{X}_T^{s'}} \right).
\]

**Remark 2.** We should note that Proposition 2 holds for \( s \geq 0. \) Indeed since \( H^s(\mathbb{R}) \) is an algebra for \( s > 1/2, \) then bilinear estimate (9) holds easily. When \( s \in [0, 1/2], \) we have
\[
\left\| \int_0^t U(t - t') \partial_x (uv)(t') \, dt' \right\|_{H^s(\mathbb{R})} \lesssim \left\| \int_0^t \mathcal{V}(t - t') \ast \partial_x (uv)(t') \, dt' \right\|_{H^s(\mathbb{R})},
\]
where
\[
\mathcal{V}(t) = \int_{\mathbb{R}} e^{i\xi t} e^{i(\xi^2 - |\xi|^2)} d\xi.
\]
Observe that for any \( 1 \leq p \leq \infty \) and \( \nu \geq 0, \) we have for some \( K > 0 \) that
\[
\left\| D^{\nu} \mathcal{V}(t) \right\|_{L^p(\mathbb{R})} \leq e^{Kt} t^{-\frac{1}{2}(\nu + \frac{p}{2})} \lesssim t^{-\frac{1}{2}(\nu + \frac{p}{2})},
\]
for \( 0 \leq t \leq T \leq 1, \) where \( D^{\nu} \mathcal{V} = |\xi|^\nu \mathcal{V}. \) Then by using the fractional Leibnitz rule, we get from (15), (16) and the Sobolev embedding that
\[
\left\| \int_0^t U(t - t') \partial_x (uv)(t') \, dt' \right\|_{H^s(\mathbb{R})} \lesssim \int_0^t \left\| \partial_x \mathcal{V}(t - t') \right\|_{L^{z/(2z + 1)}(\mathbb{R})} \left\| (D)^{s} (uv)(t') \right\|_{L^{1/(1-s)}(\mathbb{R})} dt'
\]
\[
\lesssim \int_0^t (t - t')^{s/3 - 1/2} \left\| u(t') \right\|_{L^{z/(1-s)}(\mathbb{R})} \left\| v(t') \right\|_{H^s(\mathbb{R})} dt'
\]
\[
\lesssim T^{\theta(s)} \left\| u \right\|_{\mathcal{X}_T^s} \left\| v \right\|_{\mathcal{X}_T^s},
\]
where \( \langle \cdot \rangle = 1 + |\cdot| \) and \( \theta(s) > 0 \) for any \( s > 0. \)

Next, we derive a regularity property which will be helpful in the regularity property in Theorem 1.

**Proposition 3.** Let \( 0 \leq t \leq T \leq T^*, \) \( s \in (-3/2, 0) \) and \( \kappa \in [0, s + 3/2); \) then
\[
\mathcal{V} : t \mapsto \int_0^t U(t - t') \partial_x (u^2(t')) \, dt',
\]
is in \( C([0, T]; H^{s + \kappa}(\mathbb{R})), \) for all \( u \in \mathcal{X}_T^s. \)

**Proof.** Let \( t_0, t_1 \in [0, T] \) be fixed such that \( t_0 < t_1. \) Then by the Minkowski inequality, we have
\[
\left\| \mathcal{V}(t_1) - \mathcal{V}(t_0) \right\|_{H^{s + \kappa}(\mathbb{R})} \leq \mathcal{V}_1(t_0, t_1) + \mathcal{V}_2(t_0, t_1),
\]
where
\[
\mathcal{V}_1(t_0, t_1) := \int_{t_0}^{t_1} \left\| \partial_x \right\|_{L^{z/(2z + 1)}(\mathbb{R})} \left\| (D)^{s} (uv)(t') \right\|_{L^{1/(1-s)}(\mathbb{R})} dt'
\]
\[\text{and}\]
\[
\mathcal{V}_2(t_0, t_1) := \int_{t_0}^{t_1} (t - t')^{s/3 - 1/2} \left\| u(t') \right\|_{L^{z/(1-s)}(\mathbb{R})} \left\| v(t') \right\|_{H^s(\mathbb{R})} dt'.
\]
This completes the proof.

Thus, we deduce from the Lebesgue dominated convergence theorem that

\[ \lim_{t_1 \to t_0} V_1(t_0, t_1) = 0. \]

On the other hand, we have

\[ V_2(t_0, t_1) \leq \left( \int_0^{t_0} \left\| g(t_0, t_1, t', \xi) \right\|_{L^2(\mathbb{R})} |t'|^{-2|\xi|^3} \, dt' \right) \| u \|_{X_T^2}^2, \]

where

\[ g(t_0, t_1, t', \xi) = \left| \xi \right|^{\kappa+1} \left[ e^{i(t_1-t')(|\xi|-|\xi'|)} e^{i(t_1-t')\xi^3} \right] \]
\[ - \left| \xi \right|^{\kappa+1} \left[ e^{i(t_0-t')(|\xi|-|\xi'|)} e^{i(t_0-t')\xi^3} \right]. \]

It is clear that \( g(t_0, t_1, t', \xi) \) tends to zero pointwise for almost every \( \xi \in \mathbb{R} \) and \( t' \in [0, t_0] \) when \( |t_1 - t_0| \to 0 \). Hence

\[ |g(t_0, t_1, t', \xi)| \lesssim \chi_{\{|\xi| \leq 1\}}(\xi) e^{2\sqrt{T}/\sqrt{T'}} + |\xi|^{\kappa+1} e^{(t_0-t')(|\xi|-|\xi'|)}. \]

Thus, we deduce from the Lebesgue dominated convergence theorem that

\[ \lim_{t_1 \to t_0} V_2(t_0, t_1) = 0. \]

This completes the proof. \( \Box \)
3. Local existence and ill-posedness

All the elements are now in place to mount a proof of the local well-posedness result in Theorem 1.

Proof of Theorem 1. Let $s > -3/2$ and $u_0 \in H^s(\mathbb{R})$. We are going to show that the operator $\Phi$ defined in (5) is a contraction in some closed ball of $X_T^s$. By Propositions 1 and 2, there exist two positive constant $C = C(s)$ and $\theta = \theta(s)$ such that

$$
\|\Phi(u)\|_{X_T^s} \leq C \left( \|u_0\|_{H^s(\mathbb{R})} + T^\theta \|u\|_{X_T^s}^2 \right),
$$

(17)

and

$$
\|\Phi(u) - \Phi(v)\|_{X_T^s} \leq CT^\theta \|u - v\|_{X_T^s} \|u + v\|_{X_T^s},
$$

(18)

for all $u, v \in X_T^s$ and $0 < T \leq T^*$. Now we define

$$
X_T^s(b) = \{ u \in X_T^s : \|u\|_{X_T^s} \leq b \} \quad \text{with} \quad b = 2C\|u_0\|_{H^s(\mathbb{R})}
$$

and we choose

$$
0 < T < \min \left\{ 1, (2Cb)^{-1/\theta} \right\}.
$$

Estimates (17) and (18) imply that $\Phi$ is a contraction on the Banach space $X_T^s(b)$; so that we deduce by the fixed point theorem, the existence of a unique solution $u$ of the integral equation (5) in $X_T^s(b)$ with the initial data $u(0) = u_0$. Note that Proposition 3 assures that $\Phi(u) \in C([0,T]; H^s(\mathbb{R}))$.

The uniqueness of the solution of (5) on the whole space $X_T^s$ and the smoothness of the flow map solution follow by standard arguments (see for example [13]).

Note that a similar contraction argument shows that the existence result holds for any $s' > s > -3/2$, in the time interval $[0,T]$ with $T = T(\|u_0\|_{H^s(\mathbb{R})})$ (see Remark 1). Finally, we know that the map $t \mapsto U(t)u_0$ is continuous in the time interval $[0,T]$ with respect to the topology of $H^\infty(\mathbb{R})$. Since our solution $u$ belongs to $X_T^s$, we deduce from Proposition 3 that there exists $\kappa > 0$ such that the map $V$ belongs to $C([0,T]; H^{s+\kappa}(\mathbb{R}))$, so that

$$
u \in C((0,T]; H^{s+\kappa}(\mathbb{R})).
$$

Therefore, by a standard bootstrapping argument, using the uniqueness result and the fact that the time interval of the existence of the solutions depends only on the $H^s(\mathbb{R})$-norm of the initial data, we deduce that

$$
u \in C((0,T]; H^\infty(\mathbb{R})).
$$

Remark 3. A standard argument similar to [3], one can observe that if $u_0 \in H^s(\mathbb{R})$, for $s \geq 0$, the corresponding local solution of (1) extends globally in time. More precisely, since the solution $u$ of (1) is in $C((0,T]; H^\infty(\mathbb{R}))$, one only needs to prove an a priori estimate for $u$. So $u$ solves the Cauchy problem (1) in the classical sense. Recall that $T = T(\|u_0\|_{H^s(\mathbb{R})})$. This allows us to take the $L^2$-scalar product of (1) with $u$, integrate by parts and use the properties of the Hilbert transform (see for
example \cite{10, 11}, the Gagliardo-Nirenberg inequality and the Young inequality to obtain
\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2(\mathbb{R})}^2 = \|D^{1/2}u\|_{L^2(\mathbb{R})}^2 - \|D^{3/2}u\|_{L^2(\mathbb{R})}^2 \\
\leq C \|D^{3/2}u\|_{L^2(\mathbb{R})}^2 \|u\|_{L^2(\mathbb{R})}^{4/3} - \|D^{3/2}u\|_{L^2(\mathbb{R})}^2 \leq C \|u\|_{L^2(\mathbb{R})}^2,
\]
where \( C > 0 \) is independent of \( t \). Then by the Gronwall inequality, it yields
\[
\|u(t)\|_{L^2(\mathbb{R})} \leq \|u_0\|_{L^2(\mathbb{R})} e^{CT}, \quad \text{for all } t \in [0, T].
\]

Next, we are going to show that our well-posedness result is sharp. We will first prove that we cannot solve the Cauchy problem (1) in \( H^s(\mathbb{R}) \) using the fixed point theorem when \( s < -3/2 \). Then we show that this fact implies Theorem 2.

**Remark 4.** With a slight modification, the proofs of Theorems 2 and 3 (below) are very similar to Pastrán’s results in his thesis \cite{19}. The author should mention that he proved Theorems 2 and 3 independent of Pastrán’s thesis in \cite{19}, and for the sake of completeness of this paper, the author gives the proofs in details here.

**Theorem 3.** Let \( s < -3/2 \) and \( T > 0 \). Then, there does not exist any space \( X_T^s \) such that \( X_T^s \) is continuously embedded in \( C([0, T]; H^s(\mathbb{R})) \), i.e.
\[
\|u\|_{L^\infty(\mathbb{R})} \lesssim \|u\|_{X_T^s}, \quad \forall u \in X_T^s
\]
and such that
\[
\|U(t)u_0\|_{X_T^s} \lesssim \|u_0\|_{H^s(\mathbb{R})}, \quad \forall u_0 \in H^s(\mathbb{R})
\]
and
\[
\left\| \int_0^t U(t - t') (uv)(t') \, dt' \right\|_{X_T^s} \lesssim \|u\|_{X_T^s} \|v\|_{X_T^s},
\]
for all \( u, v \in X_T^s \).

**Proof.** Suppose that there exists a space \( X_T^s \) as in Theorem 3. Take \( u_0 \in H^s(\mathbb{R}) \), \( u(t) = U(t)u_0 \), and fix \( 0 < t < T \). Then by using relations (19), (20) and (21), we see that
\[
\left\| \int_0^t U(t - t') \partial_x (U(t')u_0^2) \, dt' \right\|_{H^s(\mathbb{R})} \lesssim \|u_0\|_{H^s(\mathbb{R})}^2.
\]
(22)
We will show that (22) fails for an appropriate choice of \( u_0 \), which would lead to a contradiction. Define \( u_0 \) by
\[
\tilde{u}_0(\xi) = N^{-s} \gamma^{-1/2} (\chi_{I_1}(\xi) + \chi_{I_2}(\xi)),
\]
where \( N \gg 1, \gamma = N^{1-\epsilon_0} \) (\( 0 < \epsilon_0 \ll 1 \) fixed) and
\[
I_1 = [N, N + 2\gamma], \quad I_2 = [-N - 2\gamma, -N].
\]
It is easy to see that
\[
\|u_0\|_{H^s(\mathbb{R})} \sim 1.
\]
Then, we use the definition of $U(t)$ and Fubini’s theorem to get
\[
\left| h(t, \xi) (\xi) \right| = \left| \int_0^t U(t - t') \partial_x \left( (U(t') u_0)^2 \right) dt' \right| (\xi)
\]
\[
= \left| \int_0^t i \xi e^{i \xi t} (\xi^2 - (|\xi|^2 - |\xi|^2)) \overline{U(t')} \overline{u_0} \ast U(t') u_0 (\xi) dt' \right|
\]
\[
= \left| \int_{\mathbb{R}} i \xi \mathcal{F}_0 (\xi_1) \mathcal{F}_0 (\xi_2) f(t, \xi, \xi_1) d\xi_1 \right|
\]
\[
\geq \frac{1}{N^{2s}} \int_{\mathcal{M}} \xi f(t, \xi, \xi_1) d\xi_1,
\]
where
\[
f(t, \xi, \xi_1) = \frac{e^{-t(|\xi|^2 - |\xi_1|^2 + |\xi|^2)} e^{i t (\xi_1^2 + \xi_2^2 - t^2)} - e^{-t(|\xi|^2 - |\xi|^2)}}{\omega(\xi, \xi_1)},
\]
\[
\xi_2 = \xi - \xi_1,
\]
\[
\omega(\xi, \xi_1) = |\xi_1| - |\xi|^2 - |\xi_2| + |\xi_2| + |\xi| + 3i \xi_1 \xi_2.
\]
and
\[
\mathcal{M} = \{ \xi_1 : \xi_1 \in \mathbb{I}_1, \xi_2 \in \mathbb{I}_2 \}.
\]
When $\xi_1 \in \mathbb{I}_1$ and $\xi_2 \in \mathbb{I}_2$, we deduce that $\xi \in [2N, 2N + 4\gamma]$ and $\omega(\xi, \xi_1) \lesssim N^3$.
Now we choose a sequence of times $t_N = N^{-\frac{3}{4}}$, so that $e^{-t(|\xi|^2 - |\xi_1|^2)} t_N \sim e^{-N^{3+\epsilon}} \sim e^{-N^{-\epsilon}} > C > 0$. Hence
\[
\left| \frac{e^{-t(|\xi|^2 - |\xi_1|^2 + |\xi|^2) - |\xi|^2 + |\xi|^2)} e^{i t (\xi_1^2 + \xi_2^2 - t^2)} - 1}{\omega(\xi, \xi_1)} \right| = \frac{1}{N^{3+\epsilon_0}} + O \left( \frac{1}{N^{3+2\epsilon_0}} \right).
\]
Therefore,
\[
\| h(t, \xi) \|_{H^s(\mathbb{R})} \gtrsim N^{s-3/2-3\epsilon_0/2}.
\]
Hence, we obtain that
\[
N^{-s-3/2-3\epsilon_0/2} \lesssim 1, \quad \forall \ N \gg 1;
\]
which contradicts the assumption $s < -3/2$.

A proof of Theorem 2 is now in sight.

**Proof of Theorem 2.** Let $s < -3/2$, suppose that there exists $T > 0$ such that the Cauchy problem (1) is locally well-posed in $H^s(\mathbb{R})$ in the time interval $[0, T]$ and that the flow map solution $\mathcal{F} : H^s(\mathbb{R}) \rightarrow C([0, T]; H^s(\mathbb{R}))$ is $C^2$ at the origin. When $u_0 \in H^s(\mathbb{R})$, we will denote $u_{u_0}(t) = \mathcal{F}(u_0)(t)$ the solution of equation (1) with initial datum $u_0$. This means that $u_{u_0}$ is a solution of the integral equation
\[
u_{u_0}(t) = \mathcal{F}(u_0)(t) = U(t) u_0 - \frac{1}{2} \int_0^t U(t - t') \partial_x (u_{u_0}^2) (t') \ dt'.
\]
By computing the Fréchet derivative of $\mathcal{F}$ at $\varphi$ in the direction $u_0$, we obtain that

$$d_\varphi \mathcal{F}(u_0)(t) = U(t)u_0 - \int_0^t U(t - t') \mathcal{B}[u_\varphi(t'), d_\varphi \mathcal{F}(u_0)(t')] \, dt',$$

where $\mathcal{B}[\varphi, \psi] = (\varphi \psi)_x$. Since the Cauchy problem (1) is supposed to be well-posed, we know by using the uniqueness that $\mathcal{F}(0)(t) = u_0(t) = 0$ and then we deduce from (23) that

$$d_0 \mathcal{F}(u_0)(t) = U(t)u_0.$$  \hspace{1cm} (24)

Using (23), we compute the second Fréchet derivative at the origin in the direction $(u_0, \psi)$ and using (24), we deduce that

$$d_0^2 \mathcal{F}(u_0, \psi)(t) = - \int_0^t U(t - t') \mathcal{B}[U(t')\psi, U(t')u_0] \, dt'.$$

The assumption of $C^2$ regularity of $\mathcal{F}$ at the origin would imply that

$$d_0^2 \mathcal{F} \in \mathcal{L}(H^s(\mathbb{R}) \times H^s(\mathbb{R}), H^s(\mathbb{R})), \hspace{1cm}$$

which would lead to the following inequality

$$\|d_0^2 \mathcal{F}(u_0, \psi)(t)\|_{H^s(\mathbb{R})} \lesssim \|u_0\|_{H^s(\mathbb{R})} \|\psi\|_{H^s(\mathbb{R})},$$

for all $u_0, \psi \in H^s(\mathbb{R})$. But (25) is equivalent to (22) which has been shown to fail in the proof of Theorem 3. \hspace{1cm} \checkmark

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References

The OST equation