A characterization of continuous images of arcs by their images of weight $\leq \aleph_1$  

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Abstract. The main purpose of this paper is to characterize the continuous images of arcs by their images of the weight $\leq \aleph_1$. More precisely, we will show that a compact space $X$ is the continuous image of an arc if and only if every continuous image $Y = f(X)$ with $w(Y) \leq \aleph_1$ is a continuous image of an arc.

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1. Introduction

An arc is a continuum with precisely two nonseparating points. A space $X$ is said to be a continuous image of an arc if there exists an arc $L$ and a continuous surjection $f : L \to X$. Let $X$ be a non-degenerate locally connected continuum. A subset $Y$ of $X$ is said to be a cyclic element of $X$ if $Y$ is connected and maximal with respect to the property of containing no separating point of itself. A cyclic element of a locally connected continuum is again a locally connected continuum. Let

$$L_X = \{Y \subset X : Y \text{ is a non-degenerate cyclic element of } X\}.$$  

If $Y$ is a closed subset of $X$, we let $K(X \setminus Y)$ denote the family of all components of $X \setminus Y$. Let $X$ be a locally connected continuum. A subset $Y$ of $X$ is said to be a $T$-set if $Y$ is closed and $|Bd(J)| = 2$ for each $J \in K(X \setminus Y)$.

Theorem 1. [1, Theorem 1] A Hausdorff locally connected continuum $S$ is the continuous image of an arc if and only if each cyclic element of $S$ is the continuous image of an arc.

The following theorem is a part of Theorem 4.4 of [9].

Theorem 2. If $X$ is a locally connected continuum, then the following conditions are equivalent:

1. $X$ is a continuous image of an arc,

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2. \( X \) is a continuous image of an ordered compactum.

3. for each \( Y \in \mathbf{L}_X \) and any \( p, q, r \in Y \) there exists a metrizable \( T \)-set \( Z \) in \( Y \) such that \( p, q, r \in Z \).

4. For each \( Y \in \mathbf{L}_X \) and each closed metrizable subset \( M \) of \( Y \) there exists a metrizable \( T \)-set \( A \) in \( Y \) such that \( M \subseteq A \).

In this paper we shall use the notion of inverse systems \( \mathbf{X} = \{ X_a, p_{ab}, A \} \) and their limits in the usual sense [2, p. 135].

The notion of approximate inverse system \( \mathbf{X} = \{ X_a, p_{ab}, A \} \) will be used in the sense of S. Mardesić [6]. See also [8].

Let \( \tau \) be an infinite cardinal. We say that a partially ordered set \( A \) is \( \tau \)-directed if for each \( B \subseteq A \) with \( \text{card}(B) \leq \tau \) there is an \( a \in A \) such that \( a \geq b \) for each \( b \in B \).

Let \( \mathbf{X} = \{ X_a, p_{ab}, A \} \) be an approximate \( \tau \)-directed inverse system of compact spaces with surjective bonding mappings and with the limit \( X \).

Proof. Let \( B \) be a basis of \( Y \), \( \text{card}(B) = \tau \) and let \( \mathcal{V} \) be the collection of all finite subfamilies of \( B \). Clearly, \( \text{card}(\mathcal{V}) = \tau \).

We assume that \( \tau \) is an initial ordinal number. Hence, \( \mathcal{V} = \{ \mathcal{V}_\alpha : \alpha < \tau \} \). For each \( \mathcal{V}_\alpha \), \( f^{-1}(\mathcal{V}_\alpha) \) is a covering of \( \mathbf{X} \). There exists an \( a(\alpha) \in A \) such that for each \( b \geq a(\alpha) \) there is a cover \( \mathcal{V}_{ab} \) of \( X_b \) such that \( p_{b}(\mathcal{V}_{ab}) \) refines \( f^{-1}(\mathcal{V}_\alpha) \), i.e. \( p_{b}(\mathcal{V}_{ab}) \subset f^{-1}(\mathcal{V}_\alpha) \). From the \( \tau \)-directedness of \( A \) it follows that there is an \( a \in A \) such that \( a \geq a(\alpha) \), \( \alpha < \tau \). Let \( b \geq a \).

We claim that \( f(p_{b}^{-1}(x)) \) for \( x \in X_b \) is degenerate. Suppose that there exists a pair \( u, v \) of distinct points of \( Y \) such that \( u, v \in f(p_{b}^{-1}(x)) \). Then there exists a pair \( x, y \) of distinct points of \( p_{b}^{-1}(x) \) such that \( f(x) = u \) and \( f(y) = v \). Let \( U, V \) be a pair of disjoint open sets of \( Y \) such that \( u \in U \) and \( v \in V \). Consider the covering \( \{ U, V, Y \setminus \{ u, v \} \} \).

There exists a covering \( \mathcal{V}_{ab} \in \mathcal{V} \) such that \( \mathcal{V}_{ab} \subset \{ U, V, X \setminus \{ u, v \} \} \). We infer that there is a covering \( \mathcal{V}_{ab} \) of \( X_b \) such that \( p_{b}^{-1}(\mathcal{V}_{ab}) \subset f^{-1}(\mathcal{V}_\alpha) \). It follows that \( p_{b}(x) \neq p_{b}(y) \) since \( x \) and \( y \) lie in disjoint members of the covering \( f^{-1}(\mathcal{V}_\alpha) \). This is impossible since \( x, y \in p_{b}^{-1}(x) \). Thus, \( f(p_{b}^{-1}(x)) \) is degenerate. Now we define \( g_b : X_b \to Y \) by \( g_b(x) = f(p_{b}^{-1}(x)) \). It is clear that \( g_b \) is continuous.

The following theorem is Theorem 1.7 from [5].

Theorem 3. Let \( \mathbf{X} = \{ X_a, p_{ab}, A \} \) be a \( \sigma \)-directed inverse system of compact metrizable spaces and surjective bonding mappings. Then \( X = \lim \mathbf{X} \) is metrizable if and only if there exists an \( a \in A \) such that \( p_b : X_b \to X_b \) is a homeomorphism for each \( b \geq a \).
2. The main theorems

We first establish the following theorem.

Theorem 4. Let $X$ be a compact Hausdorff space. The following are equivalent:

a) $X$ is a continuous image of an arc,

b) If $f : X \to Y$ is a continuous surjection and $w(Y) \leq \aleph_1$, then $Y$ is a continuous image of an arc.

Proof. a) $\Rightarrow$ b) Obvious.

b) $\Rightarrow$ a) If $w(X) \leq \aleph_1$, then, by b) $X$ is a continuous image of an arc since there exists the identity $i : X \to X$ and $w(X) \leq \aleph_1$. Let $w(X) > \aleph_1$. The proof consists of several steps.

(i) There exists an $\aleph_1$-directed inverse system $X = \{X_\alpha, P_{ab}, A\}$ such that $w(X_\alpha) \leq \aleph_1$ and $X$ is homeomorphic to $\lim X$.

By [2, Theorem 2.3.23] the space $X$ is embeddable in $I^{w(X)}$. We identify the cardinal $w(X)$ with an initial ordinal number $\Omega$, i.e. with the set of all ordinal numbers of the cardinality $< w(X)$. Consider the set $A = \{\alpha, \text{card}(\alpha) = \aleph_1\}$ of all subsets of $\Omega$ of the cardinality $\aleph_1$ ordered by inclusion. It is obvious that $A$ is $\aleph_1$-directed. For each $\alpha$ we have the cube $I^\alpha$. It is clear that every $I^\alpha$ is a proper subspace of $I^{w(X)}$ since $w(X) > \aleph_1$. If $\alpha$ is a subset of $\beta$, let $P_{\alpha, \beta}$ be the natural projection of $I^\beta$ onto $I^\alpha$. Arguing as in [2, 2.5.3. Example] we infer that $I = \{I^\alpha, P_{\alpha, \beta}, A\}$ is an inverse system with limit homeomorphic to $I^{w(X)}$. Let $P_\alpha : I^{w(X)} \to I^\alpha$, $\alpha \in A$, be the natural projection. For every $\alpha \in A$ put $X_\alpha = P_\alpha(X)$. Every $X_\alpha$ has the weight $\leq \aleph_1$ and is a closed subspace of $I^\alpha$ since $X$ is a closed subset of $I^{w(X)}$. Let $p_\alpha$ be the restriction of $P_\alpha$ on $X$. We have the inverse system $X = \{X_\alpha, P_{\alpha, \beta}, A\}$ whose limit is homeomorphic to $X$. Clearly, $X$ is $\aleph_1$-directed since $A$ is $\aleph_1$-directed.

(ii) The space $X$ is a locally connected continuum.

By b) each $X_\alpha$ is a continuous image of an arc since $w(X_\alpha) \leq \aleph_1$. This means that each $X_\alpha$ is locally connected. Hence, $X$ is a locally connected continuum since $X$ is $\aleph_1$-directed and thus also $\sigma$-directed [3, Theorem 3].

(iii) There exists an $\aleph_1$-directed inverse system $Y = \{Y_\alpha, q_{\alpha, \beta}, A\}$ of continuous images of arcs such that $q_{\alpha, \beta}$ are monotone and $X$ is homeomorphic to $\lim Y$.

Let $X = \{X_\alpha, P_{\alpha, \beta}, A\}$ be as in (i) and let $p_\alpha$ be the natural projection of $X$ onto $X_\alpha \in X$. Applying the monotone-light factorization [13] to $p_\alpha$, we get the compact spaces $Y_\alpha$, monotone surjection $m_\alpha : X \to Y_\alpha$ and the light surjection $l_\alpha : Y_\alpha \to X_\alpha$ such that $p_\alpha = l_\alpha \circ m_\alpha$. By [7, Lemma 8] there exists a monotone surjection $q_{\alpha, \beta} : Y_\beta \to Y_\alpha$ such that $q_{\alpha, \beta} \circ m_\beta = m_\alpha$, $\alpha \leq \beta$. It follows that $Y = \{Y_\alpha, q_{\alpha, \beta}, A\}$ is an inverse system such that $X$ is homeomorphic to $\lim Y$. Every $Y_\alpha$ is locally connected since $X$ is locally connected. Moreover, by [7, Theorem 1] it follows that $w(Y_\alpha) = w(X_\alpha) \leq \aleph_1$. By b) we infer that every $Y_\alpha$ is a continuous image of an arc. The proof of (iii) is completed.

In the following step we shall represent every cyclic element of $X$ as the limit of some inverse system of cyclic elements of $Y_\alpha$, $\alpha \in A$.

(iv) For each nondegenerate cyclic element $W$ of $X$ there exists an $\aleph_1$-directed inverse system $W = \{W_\alpha, P_{ab}, A^*\}$ such that $W_\alpha$ is a nondegenerate cyclic element of some $X_\alpha$, $P_{ab}$ are monotone and $A^*$ is a cofinal subset of $A$. 
By (iii) $X$ is the limit of $Y = \{Y_\alpha, q_\alpha, A\}$. Let $q_\alpha : X \to Y_\alpha$ be the natural projection. Every $q_\alpha(W)$ is a locally connected continuum because it is the image of the locally connected continuum $W$ [12, p. 70, Lemma 1.5]. Moreover, every $q_\alpha(W)$ is the image of an ordered compactum since every $Y_\alpha$ is the continuous image of an arc. By Theorem 2, it follows that every $q_\alpha(W)$ is the continuous image of an arc. It easily follows that $W = \lim q_\alpha(W)$, $q_\alpha|q_\beta(W)$, $A$. Define $r_\alpha = q_{\alpha\beta}|q_\beta(W)$. As in the proof of Theorem 5.1 of [9] we infer that there exists an $\alpha_0$ in $A$ and a non-degenerate cyclic element $W_{\alpha_0}$ of $q_{\alpha_0}(W)$. Let $A^* = \{\alpha : \alpha \geq \alpha_0\}$. For each $\alpha \geq \alpha_0$ there exists a non-degenerate cyclic element $W_\alpha$ of $q_\alpha(W)$ such that $r_{\alpha\alpha}(W_\alpha) \supseteq W_{\alpha_0}$ (Lemma 1.5) since the restrictions $q_\alpha|W \to q_\alpha(W)$ are monotone ([9, Lemma 2.2]). Let $\rho_\alpha : q_\alpha(W) \to W_\alpha$ be the canonical retraction [9, p. 5]. We define $P_{\alpha\beta} = \rho_\alpha \circ r_{\alpha\beta}$ for each pair $\alpha, \beta$ such that $\alpha_0 \leq \alpha \leq \beta$. As in the proof of Theorem 5.1 of [9, p. 25] it follows that $\{W_\alpha, P_{\alpha\beta}, A^*\}$ is an $\aleph_1$-directed inverse system with monotone bonding mappings $P_{\alpha\beta}$ whose limit is $W$. The proof of (iv) is complete.

(v) Every non-degenerate cyclic element $W$ of $X$ is a continuous image of an arc.

Let $x, y$ and $z$ be points of $W$. By (3) of Theorem 2, it suffices to prove that there exists a metrizable $T$-set in $W$ which contains $x, y$ and $z$. It remains to prove that $N$ is metrizable. By virtue of $\dim N \geq \aleph_1$. There exists a countable dense subset $Z = \{z_n : n \in \mathbb{N}\}$ of $N$. For each $n$, there is an $M_n \subseteq M$ such that $w(M_n) \subseteq N$. We infer that $w(N) \subseteq N_0$ since $M_n$ is a compact metric subspace of $X$. This contradicts the assumption $w(N) \geq \aleph_1$.

**Claim 1.** Let $\mathcal{M} = \{M_\mu : \mu \in \mathcal{M}\}$ be a family of compact metric subspaces $M_\mu$ of a space $M$ partially ordered by inclusion $\subseteq$. If it is $\aleph_1$-directed, then $N = \bigcup\{M_\mu : \mu \in \mathcal{M}\}$ is a compact metrizable subspace of $M$.

Suppose that $w(N) \geq \aleph_1$. By virtue of [4] (or [10, Theorem 1.1]), for $\lambda = \aleph_1$, there exists a subspace $N_{\aleph_1}$ of $N$ such that $\dim(N_{\aleph_1}) \leq \aleph_1$ and $w(N_{\aleph_1}) \geq \aleph_1$. For each $x \in N_{\aleph_1}$, there exists an $M_{\mu_0}(x) \in \mathcal{M}$ such that $x \in M_{\mu_0}(x)$. The family $\mathcal{M}_0 = \{M_\mu(x) : x \in N_{\aleph_1}\}$ has the cardinality $\leq \aleph_1$. By the $\aleph_1$-directedness of $\mathcal{M}$ there exists an $M_\mu \in \mathcal{M}$ such that $M_\mu \supseteq M_{\mu_0}(x)$ for each $x \in N_{\aleph_1}$. This means that $N_{\aleph_1} \subseteq M_\mu$. We infer that $w(N_{\aleph_1}) \leq \aleph_0$ since $M_\mu$ is a compact metric subspace of $X$. This contradicts the assumption $w(N_{\aleph_1}) \geq \aleph_1$. Hence, $w(N) \leq \aleph_0$. There exists a countable dense subset $Z = \{z_n : n \in \mathbb{N}\}$ of $N$. For each $z_n$ there is an $M_{\mu_0}(n) \subseteq \mathcal{M}$ such that $z_n \in M_{\mu_0}(n)$. It is clear that $L = \bigcup\{M_{\mu}(n) : n \in \mathbb{N}\}$ is dense in $N$. By virtue of the $\aleph_1$-directedness of $\mathcal{M}$ there exists an $M_\mu \in \mathcal{M}$ such that $M_\mu \supseteq M_{\mu_0}(n)$ for each $n$. We infer that $M_\mu \supseteq L$ and, consequently, $M_\mu$ is dense in $N$. From the compactness of $M_\mu$ it follows that $N = M_\mu$. Hence, $N$ is a compact metrizable subspace of $M$. The proof of Claim 1 is complete.

It is obvious that the collection $\mathcal{N} = \{N_{\alpha}, p_{\alpha\beta}, A^*\}$ is an inverse system. Every $N_{\alpha}$ is a $T$-set in $W_{\alpha}$ [9, Theorem 3.1]. By [9, Theorem 3.13] $N = \lim \mathcal{N}$ is a $T$-set in $W$ which contains $x, y$ and $z$. It remains to prove that $N$ is metrizable. This is established by the following Claim 2.

**Claim 2.** Let $\mathcal{Z} = \{Z_{\alpha}, p_{\alpha\beta}, A\}$ be an $\aleph_1$-directed inverse system of compact metric spaces $Z_{\alpha}$ and surjective bonding mappings. Then $Z = \lim \mathcal{Z}$ is a compact metrizable space.
By virtue of Theorem 3, it suffices to prove that there exists an \( a \in A \) such that \( p_{ab} : Z_b \rightarrow Z_a \) is a homeomorphism for each \( b \geq a \). Suppose that this is not true, i.e. that for each \( a \in A \) there exists a \( b \geq a \) such that \( p_{ab} : Z_b \rightarrow Z_a \) is not a homeomorphism. Let \( a_1 \) be any element of \( A \). By assumption, there exists an \( a_2 \in A \) such that \( a_2 \geq a_1 \) and \( p_{a_1a_2} : Z_{a_2} \rightarrow Z_{a_1} \) is not a homeomorphism. Suppose that for each ordinal number \( \alpha < \beta < \omega_1 \) the element \( a_\alpha \) is defined. Let us define \( a_\beta \). If there exists \( \beta -1 \), then we define \( a_\beta \) so that \( p_{a_\beta -1a_\beta} : Z_{a_\beta} \rightarrow Z_{a_\beta -1} \) is not a homeomorphism. If \( \beta \) is a countable limit ordinal, then there exists an \( a_\beta \) such that \( a_\beta \geq a_\alpha \) for each \( \alpha < \beta \) since \( Z \) is \( \aleph_1 \)-directed. Now, we have the transfinite sequence \( \Omega = \{ a_\alpha : \alpha < \omega_1 \} \) and a well-ordered inverse system \( Z_\Omega = \{ W_\alpha, p_{\alpha\beta}, \Omega \} \). Let \( Y = \lim Z_\Omega \). We shall prove that \( Y \) is metrizable. By virtue of the \( \aleph_1 \)-directedness of \( A \) there exists an \( a \in A \) such that \( a \geq a_\alpha \) for each \( \alpha < \omega_1 \). It is clear that there exists a mapping \( q : X_a \rightarrow Y \) induced by the mappings \( P_{a_\alpha} \). This means that \( Y \) is metrizable since \( X_a \) is metrizable. By Theorem 3, there exists an \( a_0 \) such that \( p_{a_0a_\gamma} : Z_{a_\gamma} \rightarrow Z_{a_0} \) is a homeomorphism, \( a_0 < \beta < \gamma < \omega_1 \). This contradicts the construction of \( \Omega = \{ a_\alpha : \alpha < \omega_1 \} \) and the well-ordered inverse system \( Z_\Omega = \{ W_\alpha, P_{a_\beta}, \Omega \} \). Hence, \( Z = \lim Z_\Omega \) is a compact metrizable space.

Finally, the proof of (vi) is complete.

(vi) \( X \) is the continuous image of an arc. This follows from (v) and Theorem 1. \( \square \)

Corollary 1. Let \( X \) be a locally connected continuum. The following are equivalent:

a) \( X \) is a continuous image of an arc,

b) If \( f : Z \rightarrow Y \) is a continuous surjection, where \( Z \) is a cyclic element of \( X \) and \( w(Y) \leq \aleph_1 \), then \( Y \) is a continuous image of an arc.

Proof. a) \( \Rightarrow \) b) If \( X \) is a continuous image of an arc, then every cyclic element \( Z \) of \( X \) is a continuous image of an arc (Theorem 1). We infer that \( Y \) is a continuous image of an arc since there exists a surjection \( f : Z \rightarrow Y \).

b) \( \Rightarrow \) a) Let \( Z \) be any cyclic element of \( X \). Applying Theorem 4, for \( Z \), we infer that \( Z \) is a continuous image of an arc. By Theorem 1, we infer that \( X \) is a continuous image of an arc since every cyclic element \( Z \) of \( X \) is a continuous image of an arc.

A space \( X \) is said to be rim-finite (rim-countable) if it has a basis \( B \) such that \( \text{card}(\text{Bd}(U)) \leq \aleph_0 \) \((\text{card}(\text{Bd}(U)) \leq \aleph_0)\) for each \( U \in B \). Equivalently, a space \( X \) is rim-finite (rim-countable) if and only if for each pair \( F,G \) of disjoint closed subsets of \( X \) there exist a finite (countable) subset of \( X \) which separates \( F \) and \( G \). This follows from the fact that if \( \{ A_\alpha \} \) is a locally finite family of subsets of \( X \), then \([2, \text{p. 46}]\)

\[ \text{Bd}(\bigcup A_\alpha) \subseteq \bigcup \text{Bd}(A_\alpha). \]

Every rim-finite continuum is a continuous image of an arc [11]. Hence, every rim-finite continuum is locally connected and hereditarily locally connected.

Lemma 2. Let \( f : X \rightarrow Y \) be a monotone surjection. If \( X \) is rim-finite (rim-countable), then \( Y \) is rim-finite (rim-countable).

Theorem 5. [9, Theorem 9.9] Let \( X = \{ X_a, p_{ab}, A \} \) be a \( \sigma \)-directed inverse system of rim-finite continua. Then \( X = \lim X \) is rim-finite.
Now we are ready to prove the following characterization theorem for rim-finite continua.

**Theorem 6.** Let $X$ be a continuum. The following are equivalent:

a) $X$ is rim-finite;

b) If $f : X \to Y$ is a monotone surjection and if $Y$ is metrizable, then $Y$ is rim-finite.

**Proof.** a)⇒b). Apply Lemma 2.

b)⇒a). By virtue of [9, Theorem 9.5] there exists a $\sigma$-directed monotone inverse system $X = \{X_\alpha, p_{\alpha\beta}, A\}$ such that $w(X_\alpha) \leq \aleph_0$ and $X$ is homeomorphic to $\lim X$. From b) it follows that each $X_\alpha$ is rim-finite. By Theorem 5, we infer that $X$ is rim-finite. 

3. Applications

In this section some applications of Theorems 4. and 6. are given.

**Theorem 7.** Let $X = \{X_\alpha, p_{\alpha\beta}, A\}$ be an approximate $\aleph_1$-directed inverse system of continuous images of arcs. Then $X = \lim X$ is the continuous image of an arc.

**Proof.** By Theorem 4, it suffices to prove that if $f : X \to Y$ is a continuous surjection and $w(Y) \leq \aleph_1$, then $Y$ is a continuous image of an arc. Using Lemma 1, for $\tau = \aleph_1$, we will find an $a \in A$ and a continuous surjection $g_a : X_a \to Y$ such that $f = g_a p_a$. Hence, $Y$ is a continuous image of an arc since $X_a$ is a continuous image of an arc. By Theorem 4, we infer that $X$ is a continuous image of an arc.

**Corollary 2.** Let $X = \{X_\alpha, p_{\alpha\beta}, A\}$ be an $\aleph_1$-directed inverse system of continuous images of arcs. Then $X = \lim X$ is the continuous image of an arc.

**Remark 1.** Let us observe that mappings $p_{\alpha\beta}$ in Theorem 7. and Corollary 2. are not necessarily monotone.

**Theorem 8.** Let $X = \{X_\alpha, p_{\alpha\beta}, A\}$ be an approximate $\sigma$-directed inverse system of rim-finite continua. Then $X = \lim X$ is a rim-finite continuum.

**Proof.** By Theorem 6, it suffices to prove that if $f : X \to Y$ is a monotone surjection onto a metrizable space $Y$, then $Y$ is rim-finite. By Lemma 1, there exists an $a \in A$ and a continuous mapping $g_a : X_a \to Y$ such that $f = g_a p_a$. It follows that $g_a$ is monotone since $f$ is monotone. From Lemma 2, it follows that $Y$ is rim-finite. Hence, $X$ is rim-finite (Theorem 6.).

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References


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