Necessary and sufficient condition for
$L^1$-convergence of cosine trigonometric series with
$\delta$-quasimonotone coefficients

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Abstract. For a cosine trigonometric series with coefficients in the class $S_p(\delta)$, $1 < p \leq 2$, the necessary and sufficient condition for $L^1$-convergence is obtained.

Key words: $\delta$–quasi–monotone sequence, cosine trigonometric series, Fourier series, Dirichlet kernel, Abel’s transformation, H"{o}lder inequality, Hausdorff-Young inequality, $L^1$-convergence of Fourier series

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1. Introduction

Let $f$ be a $2\pi$-periodic and even function in $L^1(0, \pi)$, and let $\{a_k\}$ be the sequence of its Fourier coefficients. Denote by $J$ the class of sequences of Fourier coefficients of all such functions. It is well known that, in general, it does not follow from $\{a_n\} \in J$ that

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx$$

converges to $f$ in the $L^1$-norm, i.e. it does not follow that $\|S_n - f\| = o(1)$, $n \to \infty$.

There is a classical example for which $\|S_n - f\| = o(1)$, $n \to \infty$ is equivalent with $a_n \log n = o(1)$, $n \to \infty$.

Telyakovskii [8] introduced the following class $S$. A sequence $\{a_k\}$ belongs to the class $S$ if $a_k \to 0$ as $k \to \infty$ and there exists a monotonically decreasing sequence $\{A_k\}$ such that $\sum_{k=1}^{\infty} A_k < \infty$ and $|\Delta a_k| \leq A_k$, for all $k$. A sequence $\{a_k\}$ of positive numbers is said to be quasi-monotone if $a_k \to \infty$ as $k \to 0$ and $\Delta a_k \geq -\beta k^{1/2}$, for some $\beta > 0$.

A sequence $\{a_k\}$ is said to be $\delta$–quasi–monotone if $a_k \to 0$, $a_k > 0$, ultimately, and $\Delta a_k \geq -\delta_k$, where $\{\delta_k\}$ is a sequence of positive numbers.

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A sequence \( \{a_k\} \) is said to satisfy condition \( S' \), if \( a_k \to 0 \) as \( k \to \infty \) and there exists a sequence \( \{A_k\} \) such that \( \{A_k\} \) is quasi–monotone, \( \sum_{k=1}^{\infty} A_k < \infty \), \( |\Delta a_k| \leq A_k \), for all \( k \).

On the other hand, a sequence \( \{a_k\} \) is said to satisfy condition \( S(\delta) \), if \( a_k \to 0 \) as \( k \to \infty \) and there exists a sequence \( \{A_k\} \) such that \( \{A_k\} \) is \( \delta \)–quasi-monotone, \( \sum_{k=1}^{\infty} A_k < \infty \), \( \sum_{k=1}^{\infty} k\delta_k < \infty \) and \( |\Delta a_k| \leq A_k \), for all \( k \).

Now, we say that a null-sequence \( \{a_k\} \) belongs to the class \( S_p(\delta) \), \( 1 \leq p \leq 2 \) if there exists a sequence of numbers \( \{A_k\} \) such that:

(a) \( \{A_k\} \) is \( \delta \)–quasi–monotone and \( \sum_{k=1}^{\infty} k\delta_k < \infty \).

(b) \( \sum_{k=1}^{\infty} A_k < \infty \).

(c) \( \frac{1}{n} \sum_{k=1}^{n} \frac{|\Delta a_k|}{A_k} = O(1) \).

Thus, in view of the above definitions it is obvious that \( S' \subset S(\delta) \subset S_p(\delta) \).

2. Lemmas

For the proof of our theorem we require the following lemmas.

**Lemma 1. (Hausdorff–Young, see [3])** Let the sequence of complex numbers \( \{c_n\} \in l^p \), \( 1 < p \leq 2 \). Then \( \{c_n\} \) is the sequence of Fourier coefficients of some \( \varphi \in L^q \left( \frac{1}{p} + \frac{1}{q} = 1 \right) \), and

\[
\left( \frac{1}{2\pi} \int_0^{2\pi} |\varphi(x)|^q \, dx \right)^{1/q} \leq \left( \sum_{n=-\infty}^{\infty} |c_n|^p \right)^{1/p}.
\]

**Lemma 2. (see [1],[11] case \( v=1 \))** If \( \{a_n\} \) is a \( \delta \)–quasi-monotone sequence with \( \sum_{n=1}^{\infty} n\delta_n < \infty \), then the convergence of \( \sum_{n=1}^{\infty} a_n \) implies that \( na_n = o(1) \), \( n \to \infty \).

**Lemma 3. (see [11])** Let \( \{a_n\} \) be a \( \delta \)–quasi-monotone sequence with \( \sum_{n=1}^{\infty} n\delta_n < \infty \).

If \( \sum_{n=1}^{\infty} a_n < \infty \), then \( \sum_{n=1}^{\infty} (n + 1) |\Delta a_n| < \infty \).

**Lemma 4.** Let the coefficients of the series (1) satisfy the condition \( S_p(\delta) \), \( 1 < p \leq 2 \). Then the following relations hold.
\begin{enumerate}
\item \[ a\int_{0}^{\pi} \left| \sum_{j=0}^{k} \frac{\Delta a_{j}}{A_{j}} D_{j}(x) \right| dx = O_{p}(k), \text{ where } O_{p} \text{ depends on } p. \]
\item \[ A_{n} \int_{0}^{\pi} \left| \sum_{j=0}^{n} \frac{\Delta a_{j}}{A_{j}} D_{j}(x) \right| dx = o(1), \ n \to \infty. \]
\end{enumerate}

\textbf{Proof.} a) We have
\[ \int_{0}^{\pi} \left| \sum_{j=0}^{k} \frac{\Delta a_{j}}{A_{j}} D_{j}(x) \right| dx = \int_{0}^{\pi/k} + \int_{\pi/k}^{\pi} = I_{k} + J_{k}. \]
Recalling the uniform estimate of the Dirichlet kernel we have:
\[ I_{k} \leq A \sum_{j=0}^{k} \left| \frac{\Delta a_{j}}{A_{j}} \right| \leq A k \left( \frac{1}{k} \sum_{j=0}^{k} \frac{|\Delta a_{j}|^{p}}{A_{j}^{p}} \right)^{1/p} \]
where \( A \) is an absolute constant.
To estimate the second integral:
\[ J_{k} = \int_{\pi/k}^{\pi} \left| \sum_{j=0}^{k} \frac{\Delta a_{j}}{A_{j}} D_{j}(x) \right| dx = \int_{\pi/k}^{\pi} \frac{1}{2 \sin \frac{x}{2}} \left| \sum_{j=0}^{k} \frac{\Delta a_{j}}{A_{j}} \sin \left( j + \frac{1}{2} \right) x \right| dx. \]
We shall first apply the Hölder inequality, where \( \frac{1}{p} + \frac{1}{q} = 1 \),
\[ J_{k} \leq \left[ \int_{\pi/k}^{\pi} \left( \frac{1}{2 \sin \frac{x}{2}} \right)^{p} dx \right]^{1/p} \left[ \int_{0}^{\pi} \left| \sum_{j=0}^{k} \frac{\Delta a_{j}}{A_{j}} \sin \left( j + \frac{1}{2} \right) x \right|^{q} dx \right]^{1/q}. \]
Since
\[ \int_{\pi/k}^{\pi} \left( \frac{dx}{\sin \frac{x}{2}} \right)^{p} \leq \pi^{p} \int_{\pi/k}^{\pi} \frac{dx}{x^{p}} \leq \frac{\pi}{p-1} k^{p-1}, \]
it follows that
\[ J_{k} \leq \frac{1}{2} \left( \frac{\pi}{p-1} \right)^{1/p} k^{(p-1)/p} \left[ \int_{0}^{\pi} \left| \sum_{j=0}^{k} \frac{\Delta a_{j}}{A_{j}} \sin \left( j + \frac{1}{2} \right) x \right|^{q} dx \right]^{1/q}. \]
Then using the Hausdorff-Young inequality we get:
\[ \left[ \int_{0}^{\pi} \left| \sum_{j=0}^{k} \frac{\Delta a_{j}}{A_{j}} \sin \left( j + \frac{1}{2} \right) x \right|^{q} dx \right]^{1/q} \leq \left[ \int_{0}^{\pi} \left| \sum_{j=0}^{k} \frac{\Delta a_{j}}{A_{j}} \sin \left( j + \frac{1}{2} \right) x \right| dx \right] \left( \sum_{j=0}^{k} \frac{|\Delta a_{j}|^{p}}{A_{j}^{p}} \right)^{1/p}. \]
Finally,
\[ J_k \leq B_p \left( \frac{1}{k} \sum_{j=0}^{k} \frac{|\Delta a_j|^p}{A_j^p} \right)^{1/p}, \]
where \( B_p \) is an absolute constant dependent on \( p \). Thus
\[ \pi \int_0^\pi \left| \sum_{j=0}^{n} \frac{\Delta a_j}{A_j} D_j(x) \right| dx = O_p(nA_n) = o(1), \ n \to \infty. \]

b) Applying first the relation a) of this lemma, then Lemma 2 yields
\[ \pi \int_0^\pi \left| \sum_{j=0}^{n} \frac{\Delta a_j}{A_j} D_j(x) \right| dx = O_p(nA_n) = o(1), \ n \to \infty. \]

\[ \square \]

3. Main result

**Theorem 1.** Let \( \{a_k\} \in S_p(\delta), \ 1 < p \leq 2 \). Then (11) is a Fourier series of some \( f \in L^1(0, \pi) \) and \( \|S_n - f\| = o(1), \ n \to \infty \) if and only if \( a_n \log n = o(1), \ n \to \infty \).

**Proof.** By summation by parts, we have:
\[
\begin{align*}
\sum_{k=1}^{n} |\Delta a_k| &= \sum_{k=1}^{n} A_k |\Delta a_k| A_k \leq \sum_{k=1}^{n-1} \Delta A_k \sum_{j=1}^{k} \frac{|\Delta a_j|^p}{A_j^p} + A_n \sum_{j=1}^{n} \frac{|\Delta a_j|^p}{A_j^p} \\
&\leq \sum_{k=1}^{n-1} k |\Delta A_k| \left( \frac{1}{k} \sum_{j=1}^{k} \frac{|\Delta a_j|^p}{A_j^p} \right)^{1/p} + nA_n \left( \frac{1}{n} \sum_{j=1}^{n} \frac{|\Delta a_j|^p}{A_j^p} \right)^{1/p} \\
&= O(1) \left[ \sum_{k=1}^{n-1} k |\Delta A_k| + nA_n \right] \leq O(1) \left[ \sum_{k=1}^{\infty} (k+1) |\Delta A_k| + nA_n \right].
\end{align*}
\]

Application of Lemma 2 and Lemma 3 yields, \( \sum_{n=1}^{\infty} |\Delta a_n| < \infty \), i.e. \( S_n(x) \) converges to \( f(x) \), for \( x \neq 0 \).

Using Abel’s transformation, we obtain:
\[ f(x) = \sum_{k=0}^{\infty} \Delta a_k D_k(x), \]
by the fact that \( \lim_{n \to \infty} a_n D_n(x) = 0 \), if \( x \neq 0 \), where \( D_n(x) \) is the Dirichlet kernel.
Then, \[
\|S_n - f\| = \left\| \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - f(x) \right\|
\]
\[
= \left\| \frac{a_0}{2} - \frac{a_{n+1}}{2} + \sum_{k=1}^{n} (a_k - a_{n+1}) \cos kx - f(x) + \frac{a_{n+1}}{2} + \sum_{k=1}^{n} a_{n+1} \cos kx \right\|
\]
\[
= \left\| \frac{1}{2} \sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta a_j \cos kx - f(x) + a_{n+1} D_n(x) \right\|
\]
\[
= \|g_n(x) - f(x) + a_{n+1} D_n(x)\|,
\]
where \(g_n(x) = \frac{1}{2} \sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta a_j \cos kx\) are the Rees-Stanojević sums (see [2],[6],[7]).

We have:
\[
g_n(x) = \frac{\Delta a_0}{2} + \sum_{k=1}^{n} \left( \frac{1}{2} \Delta a_k + \sum_{j=k}^{n} \Delta a_j \cos kx \right)
\]
\[
= \frac{\Delta a_0}{2} + \sum_{k=1}^{n} \frac{1}{2} \Delta a_k + \sum_{k=1}^{n} a_k \cos kx - a_{n+1} D_n(x) + \frac{1}{2} a_{n+1}.
\]

Using Abel’s transformation, we obtain:
\[
g_n(x) = \frac{\Delta a_0}{2} + \sum_{k=1}^{n} \frac{1}{2} \Delta a_k + \sum_{k=1}^{n-1} \Delta a_k \left( D_k(x) - \frac{1}{2} \right) + a_n \left( D_n(x) - \frac{1}{2} \right)
- a_{n+1} D_n(x) + \frac{1}{2} a_{n+1}
\]
\[
= \Delta a_0 D_0(x) + \sum_{k=1}^{n-1} \Delta a_k D_k(x) + a_n D_n(x) - a_{n+1} D_n(x)
\]
\[
= \sum_{k=0}^{n} \Delta a_k D_k(x).
\]

Since \(\sum_{n=1}^{\infty} |\Delta a_n| < \infty\), the series \(\sum_{k=0}^{\infty} \Delta a_k D_k(x)\) converges. Hence \(\lim_{n \to \infty} g_n(x)\) exists for \(x \neq 0\).

Then,
\[
||f(x) - g_n(x)|| = \left\| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right\| = \frac{1}{\pi} \int_{0}^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| \, dx.
\]

Application of Abel’s transformation and Lemma 4.b) yields
\[
\int_{0}^{\pi} \left| \sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x) \right| \, dx \leq \sum_{k=n+1}^{\infty} |\Delta A_k| \int_{0}^{\pi} \left| \sum_{j=0}^{k} \frac{\Delta a_j}{A_j} D_j(x) \right| \, dx + o(1), \ n \to \infty.
\]

Then, by Lemma 4.a) and Lemma 3, we have:
\[
\int_{0}^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| \, dx = O_p \left( \sum_{k=n+1}^{\infty} |\Delta A_k| (k + 1) \right) + o(1) = o(1), \ n \to \infty.
\]
Thus \(|f(x) - g_n(x)|| = o(1), \ n \to \infty\).

"If": Let \(|S_n - f|| = o(1), \ n \to \infty\), then by the formulae:

\[ S_n(x) = g_n(x) + a_{n+1}D_n(x), \]

we get:

\[ \|a_{n+1}D_n(x)\| = \|S_n - g_n\| \leq \|S_n - f\| + \|f - g_n\| = a(1) + o(1), \ n \to \infty. \]

Since \(|D_n(x)|| = O(\log n)|, we have, \(a_n \log n = o(1), \ n \to \infty\).

"Only if": Let \(a_n \log n = o(1), \ n \to \infty\). Then,

\[ \|S_n - f\| \leq \|g_n - f\| + \|a_{n+1}D_n(x)\| = a(1) + a_{n+1}O(\log n) = o(1), \ n \to \infty. \]

References


