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On the Isoptic Hypersurfaces in the n-Dimensional Euclidean Space

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ABSTRACT

The theory of the isoptic curves is widely studied in the Euclidean plane $\mathbf{E}^2$ (see [1] and [13] and the references given there). The analogous question was investigated by the authors in the hyperbolic $\mathbf{H}^2$ and elliptic $\mathcal{E}^2$ planes (see [3], [4]), but in the higher dimensional spaces there is no result according to this topic.

In this paper we give a natural extension of the notion of the isoptic curves to the n-dimensional Euclidean space $\mathbf{E}^n$ $(n \geq 3)$ which are called isoptic hypersurfaces. We develope an algorithm to determine the isoptic hypersurface $\mathcal{H}_{\mathcal{D}}$ of an arbitrary $(n-1)$ dimensional compact parametric domain $\mathcal{D}$ lying in a hyperplane in the Euclidean n-space. We will determine the equation of the isoptic hypersurfaces of rectangles $\mathcal{D} \subset \mathbf{E}^2$ and visualize them with Wolfram Mathematica. Moreover, we will show some possible applications of the isoptic hypersurfaces.

Key words: isoptic curves, hypersurfaces, differential geometry, elliptic geometry

MSC2010: 53A05, 51N20, 68A05

O izooptičkim hiperplohama u n-dimenzionalnom euklidskom prostoru

SAŽETAK

Teorija o izooptičkim krivuljama dosta se proučava u euklidskoj ravnini $\mathbf{E}^2$ (vidi [1] i [13] te u referencama koje se tamo mogu naći). Autori su proučavali analogno pitanje u hiperboličkoj $\mathbf{H}^2$ i eliptičkoj ravnini $\mathcal{E}^2$ (vidi [3], [4]), međutim u višedimenzionalnim prostorima nema rezultata vezanih za ovu temu.

U ovom članku dajemo prirodno proširenje pojma izooptičkih krivulja na n-dimenzionalni euklidski prostor $\mathbf{E}^n$ $(n \geq 3)$ koje zovemo izooptičke hiperplohe. Razvijamo algoritam kojim određujemo izooptičke hiperplohe $\mathcal{H}_{\mathcal{D}}$ proizvoljne $(n-1)$-dimenzionalne kompaktne parametarske domene $\mathcal{D}$ koja leži u hiperravnini u n-dimenzionalnom euklidskom prostoru.

Odredit ćemo jednadžbu izooptičkih hiperploha pravokutnika $\mathcal{D} \subset \mathbf{E}^2$ i vizualizirati ih koristeći program Wolfram Mathematica. Štoviše, pokazat ćemo neke moguće primjene izoptičkih hiperploha.

Ključne riječi: izooptičke krivulje, hiperplohe, diferencijalna geometrija, eliptička geometrija

1 Introduction

Definition 1 *Let X be one of the constant curvature plane geometries $\mathbf{E}^2$, $\mathbf{H}^2$, $\mathcal{E}^2$. The isoptic curve $\mathcal{C}^\alpha$ of an arbitrary given plane curve $\mathcal{C}$ of X is the locus of points P where $\mathcal{C}$ is seen under a given fixed angle α $(0 < \alpha < \pi)$.*

An isoptic curve formed from the locus of two tangents meeting at right angle $(\alpha = \frac{\pi}{2})$ are called orthoptic curve. The name isoptic curve was suggested by C. Taylor in his work [12] in 1884.

In the Euclidean plane $\mathbf{E}^2$ the easiest case if $\mathcal{C}$ is a line segment then the set of all points (locus) for which a line segment can be seen under angle α contains two arcs in both half-plane of the line segment, each are with central angle 2α. In the special case $\alpha = \frac{\pi}{2}$, we get exactly the so-called Thales circle (without the endpoints of the given segment) with center the middle of the line segment.

Further curves appearing as isoptic curves are well studied in the Euclidean plane geometry $\mathbf{E}^2$, see e.g. [8],[13]. In [1] and [2] can be seen the isoptic curves of the closed, strictly convex curves, using their support function. The papers [14] and [15] deal with curves having a circle or an ellipse for an isoptic curve. Isoptic curves of conic sections have been studied in [6], [8] and [11]. A lot of papers concentrate on the properties of the isoptics e.g. [9], [7], [10] and the references given there.

In the hyperbolic and elliptic planes $\mathbf{H}^2$ and $\mathcal{E}^2$ the isoptic curves of segments and proper conic sections are determined by the authors ([3], [4], [5]).

In the higher dimensions by our best knowledge there are no results in this topic thus in this paper we give a natural extension of the Definition 1 in the n-dimensional Euclidean space $\mathbf{E}^n$. Moreover, we develope a procedure to determine the isoptic hypersurface $\mathcal{H}_{\mathcal{D}}$ of an arbitrary $(n-1)$ dimensional compact parametric domain $\mathcal{D}$ lying in a hyperplane in the Euclidean space. We will determine the equation of the isoptic hypersurfaces (see Theorem 1) of rectangles $\mathcal{D} \subset \mathbf{E}^2$ and visualize them with Wolfram Mathematica (see Fig. 2-3). Moreover, we will show some possible applications of the isoptic hypersurfaces.

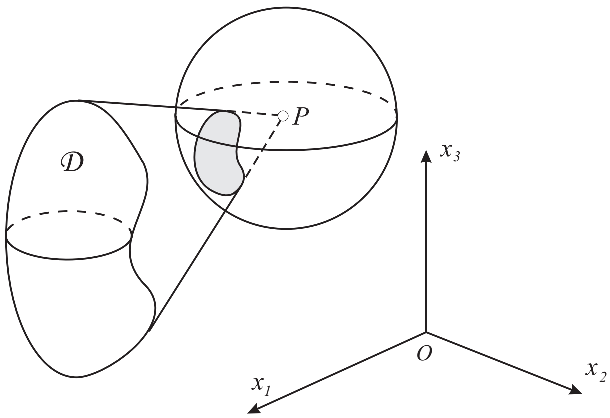

Figure 1: *Projection of a compact domain $\mathcal{D}$ to unit sphere in $\mathbf{E}^3$*

2 Isoptic hypersurface of a compact domain lying in a hyperplane of $\mathbf{E}^n$

In Definition 1 we have considered that, the angle can be measured by the arc length on the unit circle around the point. From this statement, Definition 1 can be extended to the n-dimensional Euclidean space $\mathbf{E}^n$.

Definition 2 *The isoptic hypersurface $\mathcal{H}_{\mathcal{D}}^{\alpha}$ in $\mathbf{E}^n$ $(n \geq 3)$ of an arbitrary d dimensional compact parametric domain $\mathcal{D}(2 \leq d \leq n)$ is the locus of points P where the measure of the projection of $\mathcal{D}$ onto the unit $(n-1)$-sphere around P is a given fixed value α $\left(0 < \alpha < \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}\right)$ $\left[\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dt\right]$ (see Fig. 1).*

We consider a compact parametric $(n-1)$ $(n \geq 3)$-dimensional domain $\mathcal{D}$ lying in a hyperplane of $\mathbf{E}^n$. We can suppose the next form of parametrization:
$\phi(x,y)$ plane surface

$$\widetilde{\boldsymbol{\phi}}(u_1,u_2,\dots,u_{n-1}) = \begin{pmatrix} \widetilde{f}_1(u_1,u_2,\dots,u_{n-1}) \\ \widetilde{f}_2(u_1,u_2,\dots,u_{n-1}) \\ \vdots \\ \widetilde{f}_{n-1}(u_1,u_2,\dots,u_{n-1}) \\ 0 \end{pmatrix}, \tag{2.1}$$

where $u_i \in [a_i, b_i]$, $(a_i, b_i \in \mathbb{R})$, $(i = 1,\dots,n-1)$.

For the point $P(x_1,x_2,\dots,x_n) = P(\mathbf{x})$ the inequality $x_n > 0$ will be assumed. Projecting the surface onto the unit sphere with centre P, we have the following parametrization:

$$\boldsymbol{\phi}(u_1,u_2,\dots,u_{n-1}) =$$
$$= \begin{pmatrix} f_1(u_1,u_2,\dots,u_{n-1}) \\ f_2(u_1,u_2,\dots,u_{n-1}) \\ \vdots \\ f_n(u_1,u_2,\dots,u_{n-1}) \end{pmatrix} = \begin{pmatrix} f_1(\mathbf{u}) \\ f_2(\mathbf{u}) \\ \vdots \\ f_n(\mathbf{u}) \end{pmatrix}. \tag{2.2}$$

Here, if $i \neq n$ we have

$$f_i(\mathbf{u}) = \frac{\widetilde{f}_i(u_1,\dots,u_{n-1}) - x_1}{\sqrt{(\widetilde{f}_1(u_1,\dots,u_{n-1}) - x_1)^2 + \cdots + (\widetilde{f}_{n-1}(u_1,\dots,u_{n-1}) - x_{n-1})^2 + (x_n)^2}},$$

else $(i = n)$

$$f_n(\mathbf{u}) = \frac{-x_n}{\sqrt{(\widetilde{f}_1(u_1,\dots,u_{n-1}) - x_1)^2 + \cdots + (\widetilde{f}_{n-1}(u_1,\dots,u_{n-1}) - x_{n-1})^2 + (x_n)^2}}.$$

Now, it is well known, that the measure of the $n-1$-surface can be calculated using the forumla below:

$$S(x_1,x_2,\dots,x_n) =$$
$$= \int_{a_1}^{b_1}\int_{a_2}^{b_2}\dots\int_{a_{n-1}}^{b_{n-1}} \sqrt{\det G}\ \ \mathrm{d}u_{n-1}\,\mathrm{d}u_{n-2}\dots\mathrm{d}u_1 \tag{2.3}$$

by successive integration, where

$$G = J^T J =$$
$$= \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_{n-1}} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_{n-1}}{\partial u_1} & \frac{\partial f_{n-1}}{\partial u_2} & \cdots & \frac{\partial f_{n-1}}{\partial u_{n-1}} \end{pmatrix}^T \cdot$$
$$\cdot \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_{n-1}} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_{n-1}}{\partial u_1} & \frac{\partial f_{n-1}}{\partial u_2} & \cdots & \frac{\partial f_{n-1}}{\partial u_{n-1}} \end{pmatrix}.$$

The isoptic hypersurface $\mathcal{H}_{\mathcal{D}}^{\alpha}$ by the Definition 2 is the following:

$$\mathcal{H}_{\mathcal{D}}^{\alpha} = \{\mathbf{x} \in \mathbf{E}^n | \alpha = S(x_1,x_2,\dots,x_n)\}$$

In the general case, the isoptic hypersurface can be determined only by numerical computations. In the next section we show an explicite application of our algorithm.

3 Isoptic surface of the rectangle

Now, let suppose that $n=3$ and $\mathcal{D} \subset \mathbf{E}^2$ is a rectangle lying in the $[x,y]$ plane in a given Cartesian coordinate system. Moreover, we can assume, that it is centered, so the parametrization is the following:

$$\widetilde{\phi}(x,y) = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}, \tag{3.1}$$

where $x \in [-a,a]$ and $y \in [-b,b]$ $(a,b \in \mathbf{R})$. And the parametrization of the projection from $P(x_0,y_0,z_0)$ can be seen below:

$$\phi(x,y) = \begin{pmatrix} \frac{x-x_0}{\sqrt{(x-x_0)^2+(y-y_0)^2+z_0^2}} \\ \frac{y-y_0}{\sqrt{(x-x_0)^2+(y-y_0)^2+z_0^2}} \\ \frac{-z_0}{\sqrt{(x-x_0)^2+(y-y_0)^2+z_0^2}} \end{pmatrix}. \tag{3.2}$$

Remark 1 *It is clear, that the computations is similar if $\mathcal{D}$ is a normal domain concerning to x or y on the plane. The difference is appered only on the boundaries of the integrals.*

Now, we need the partial derivatives, to calculate the surface area:

$$\phi_x(x,y) = \begin{pmatrix} \frac{(y-y_0)^2+z_0^2}{\left((x-x_0)^2+(y-y_0)^2+z_0^2\right)^{3/2}} \\ \frac{-(x-x_0)(y-y_0)}{\left((x-x_0)^2+(y-y_0)^2+z_0^2\right)^{3/2}} \\ \frac{z_0(x-x_0)}{\left((x-x_0)^2+(y-y_0)^2+z_0^2\right)^{3/2}} \end{pmatrix},$$

$$\phi_y(x,y) = \begin{pmatrix} \frac{-(x-x_0)(y-y_0)}{\left((x-x_0)^2+(y-y_0)^2+z_0^2\right)^{3/2}} \\ \frac{(x-x_0)^2+z_0^2}{\left((x-x_0)^2+(y-y_0)^2+z_0^2\right)^{3/2}} \\ \frac{z_0(y-y_0)}{\left((x-x_0)^2+(y-y_0)^2+z_0^2\right)^{3/2}} \end{pmatrix}$$

/medskip
Taking the cross product of the vectors above, we obtain:

$$\phi_x(x,y) \times \phi_y(x,y) = \begin{pmatrix} \frac{z_0(x_0-x)}{\left((x-x_0)^2+(y-y_0)^2+z_0^2\right)^2} \\ \frac{z_0(y_0-y)}{\left((x-x_0)^2+(y-y_0)^2+z_0^2\right)^2} \\ \frac{z_0^2}{\left((x-x_0)^2+(y-y_0)^2+z_0^2\right)^2} \end{pmatrix}$$

Now we can substitute $\left|\phi_x(x,y) \times \phi_y(x,y)\right|$ into formula (2.3) to get the spatial angle:

$$S(x_0,y_0,z_0) =$$

$$\int_{-a}^{+a}\int_{-b}^{+b} \frac{|z_0|}{\left((x-x_0)^2+(y-y_0)^2+z_0^2\right)^{\frac{3}{2}}} \mathrm{d}y\,\mathrm{d}x = \tag{3.3}$$

$$\arctan\left(\frac{(a-x_0)(b-y_0)}{z_0\sqrt{(a-x_0)^2+(b-y_0)^2+z_0^2}}\right) +$$
$$\arctan\left(\frac{(a+x_0)(b-y_0)}{z_0\sqrt{(a+x_0)^2+(b-y_0)^2+z_0^2}}\right) +$$
$$\arctan\left(\frac{(a-x_0)(b+y_0)}{z_0\sqrt{(a-x_0)^2+(b+y_0)^2+z_0^2}}\right) +$$
$$\arctan\left(\frac{(a+x_0)(b+y_0)}{z_0\sqrt{(a+x_0)^2+(b+y_0)^2+z_0^2}}\right).$$

Remark 2 *It is easy to see, if $a \to \infty$ and $b \to \infty$, then the angle tendst to 2π for every z_0. This implies some kind of elliptic properties. The normalised cross pruduct of the two partial derivatives can be interpreted as a weight function on this elliptic plane. Now, if we have a domain on the plane, we can integrate this function over the domain, to obtain the angle. But the symbolic integral for a given domain almost never works, so in this case, it is suggested also, to use numerical approach.*

Using the results abowe, we can claim the following theorem:

Theorem 1 *Let us given a rectangle $\mathcal{D} \subset \mathbf{E}^2$ lying in the $[x,y]$ plane in a given Cartesian coordinate system. Moreover, we can assume, that it is centered at the origin with sides $(2a,2b)$. Then the isoptic surface for a given spatial angle α $(0 < \alpha < 2\pi)$ is determined by the following equation:*

$$\alpha = \arctan\left(\frac{(a-x)(b-y)}{z\sqrt{(a-x)^2+(b-y)^2+z^2}}\right) +$$
$$\arctan\left(\frac{(a+x)(b-y)}{z\sqrt{(a-x)^2+(b-y)^2+z^2}}\right) +$$
$$\arctan\left(\frac{(a-x)(b+y)}{z\sqrt{(a-x)^2+(b-y)^2+z^2}}\right) +$$
$$\arctan\left(\frac{(a+x)(b+y)}{z\sqrt{(a-x)^2+(b-y)^2+z^2}}\right).$$

In the following figures, there can be seen the isoptic surface of the rectangle:

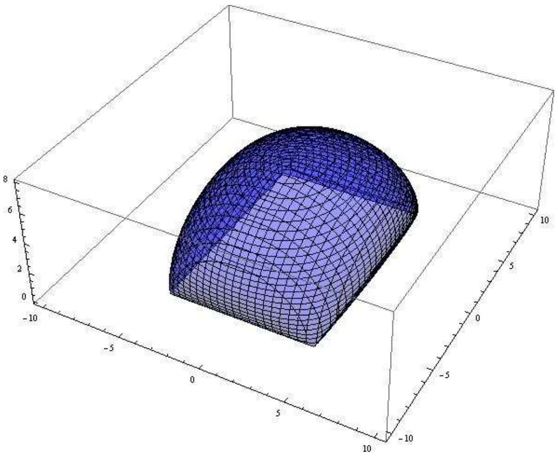

Figure 2: $2a = 9$, $2b = 13$, $\alpha = \frac{\pi}{2}$

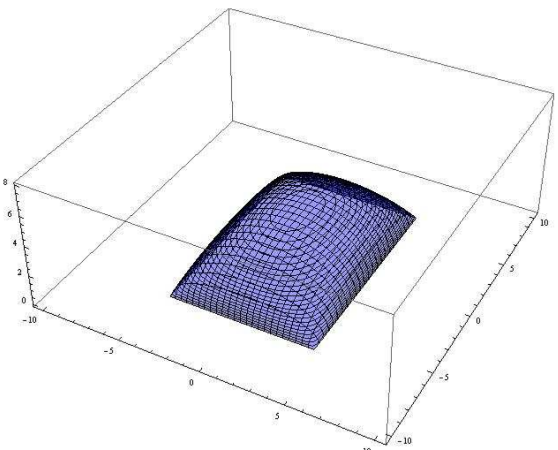

Figure 3: $2a = 9$, $2b = 13$, $\alpha = \pi$

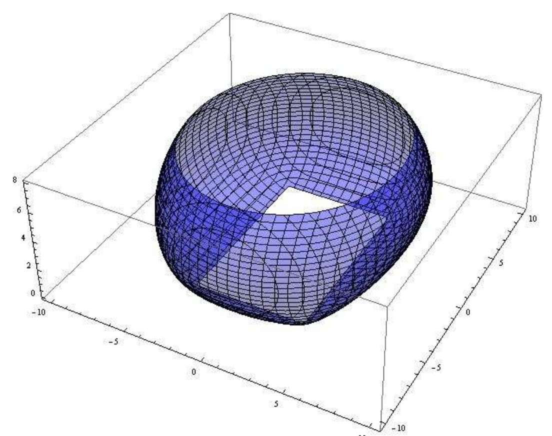

Figure 4: $2a = 7$, $2b = 11$, $\alpha = \frac{\pi}{6}$

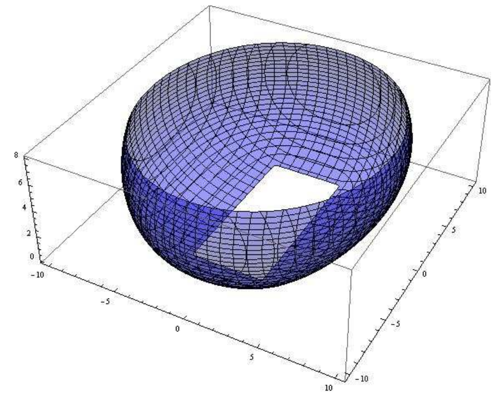

Figure 5: $2a = 11$, $2b = 5$, $\alpha = \frac{\pi}{12}$

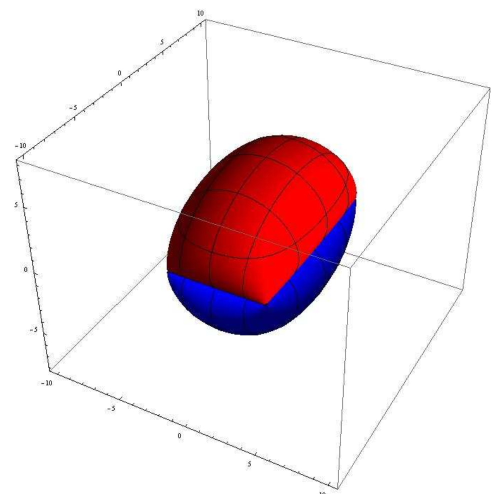

Figure 6: $2a = 7$, $2b = 13$, $\alpha = \frac{\pi}{2}$ *(both half-spaces)*

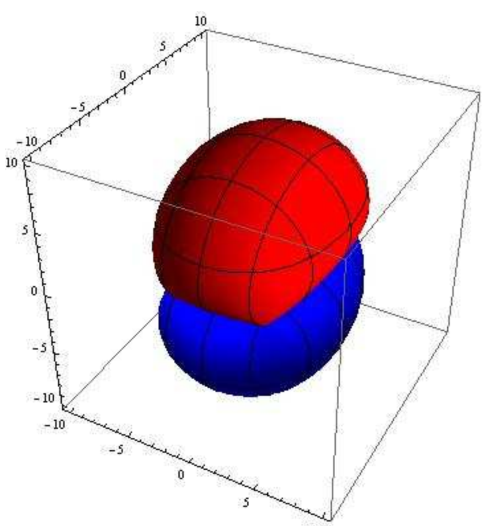

Figure 7: $2a = 7$, $2b = 13$, $\alpha = \frac{\pi}{4}$ *(both half-spaces)*

Remark 3 *The figures show us, that this topic has severeal applications, for example designing stadiums, theaters or cinemas. It can be interesting, if we have a stadium, which has the property, that from every seat on the grandstand, the field can be seen under a same angle.*

Designing a lecture hall, it is important, that the screen or the blackboard is clearly visible from every seat. In this case, the isoptic lecture hall is not feasible, but it can be optimized.

Figure 8: *MetLife Stadium:*
`http://www.bonjovi.pl/forum/topics58/25-27072013-east-rutherford-vt3278.htm`

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