Math. Commun. **19**(2014), 43–55

The harmonic evolute of a surface in Minkowski 3-space

Željka Milin Šipuš[1,*] and Vladimir Volenec[1]

[1] *Department of Mathematics, Faculty of Science, University of Zagreb, Bijenička cesta 30, HR-10 000 Zagreb, Croatia*

Received December 20, 2012; accepted November 8, 2013

Abstract. In this paper we describe harmonic evolutes of surfaces in Minkowski 3-space. In particular, we study properties of harmonic evolutes of constant mean curvature surfaces and their relation to parallel surfaces. Furthermore, we study harmonic evolutes of surfaces of revolution.

AMS subject classifications: 53A35, 53B30

Key words: harmonic evolute, focal set, parallel surface, Minkowski 3-space

1. Introduction

The focal set (evolute) of a smooth surface in 3-dimensional Euclidean space is the locus of its centers of curvatures (the focal points), that is, the locus of points $p_i = p + r_i n$, $i = 1, 2$, where $r_i = 1/k_i$ are the radii of curvature, k_i the principal curvatures of a surface at a point p and n the unit normal vector at p. If p is not an umbilical point, two centers of curvatures are distinct points and they trace two distinct components of the focal set; at umbilical points, two components coincide. In particular, the focal set of a sphere degenerates to a point. Furthermore, if the Gaussian curvature of a surface vanishes at a point p with one principal curvature being equal to zero, the corresponding point of the focal set is a point at infinity. Generally, for a connected surface without umbilical points, two components of the focal set can both be surfaces (focal surfaces), one component can be a curve and the other a surface, or each component can be a curve.

The focal set coincides with a caustic of a smooth surface which is defined as the envelope of the normal rays to the surface. However, the focal set is not defined at points of a surface where the metric is degenerate, that is, at the points of the locus of degeneracy. The caustic provides an extension of the focal set to the locus of degeneracy within the framework of Lagrangian singularity theory.

In addition to the focal set, one can define a single surface, the so-called mean (middle) evolute of a surface. The mean evolute is the envelope of planes parallel to tangent planes and passing through midpoints of segments connecting the centers of curvatures of a given surface. There is also another type of a surface associated to a given one and obtained from its centers of curvatures, the so-called harmonic evolute of a surface. It is defined as the locus of points which are harmonic conjugates of p

*Corresponding author. *Email addresses:* `zeljka.milin-sipus@math.hr` (Ž. Milin Šipuš), `volenec@math.hr` (V. Volenec)

with respect to centers of curvature p_1, p_2 of S. These points are therefore centers of the so-called harmonic spheres, that is, spheres tangent to a surface and whose centers are exactly harmonic conjugates to points of tangency with respect to centers of curvatures. Harmonic evolutes of surfaces in Euclidean space have been studied in [1].

In this paper we study harmonic evolutes of surfaces in Minkowski 3-space within the framework of classical differential geometry. It is shown that they degenerate for minimal surfaces as well as for totally umbilical and totally quasi-umbilical surfaces. Special attention is paid to harmonic evolutes of constant mean curvature surfaces (cmc surfaces) with $H \neq 0$ which exhibit some interesting properties and their relation to parallel surfaces. Finally, as an example, we study harmonic evolutes of surfaces of revolution in Minkowski 3-space.

Cmc surfaces in Minkowski 3-space have been widely studied, see e.g. [7, 11]. In particular, the class of cmc surfaces of revolution is described as analogues of Delaunay cmc surfaces in Euclidean space, that is, as surfaces whose profile curve is the locus of a focus of a quadratic curve which is rolled along the axis of revolution ([3]). Cmc helicoidal mean surfaces whose generating curves are graphs of polynomials or Lorentzian circles are studied in [4].

There are not many results on surface evolutes in Minkowski 3-space. Caustics of surfaces in Minkowski 3-space within the framework of Lagrangian singularity theory are studied in [10]. Dupin-cyclides as surfaces whose evolutes degenerate to curves are studied in [9]. Some results on affine invariants of the focal surfaces of Minkowski minimal surfaces can be found in [5, 6].

2. Preliminaries

A Minkowski 3-space $\mathbb{R}^3_1$ is a real affine space whose underlying vector space $\mathbb{R}^3$ is endowed with a pseudo-scalar product, that is, with a non-degenerate indefinite symmetric bilinear form. If $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, we define this form by

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3$$

and denote the vector space by $\mathbb{R}^3_1$.

A vector x in Minkowski 3-space is called spacelike if $x \cdot x > 0$ or $x = 0$, timelike if $x \cdot x < 0$ and lightlike if $x \cdot x = 0$ and $x \neq 0$. A timelike vector is said to be positive (resp. negative) if $x_1 > 0$ (resp. $x_1 < 0$). The pseudo-norm of a vector x is defined as the real number $||x|| = \sqrt{|x \cdot x|} \geq 0$.

We denote by $S^2_1(p, r)$, $H^2(p, r)$, resp. $LC(p)$ the following quadrics in $\mathbb{R}^3_1$

$$\begin{aligned} S^2_1(p, r) &= \{q \in \mathbb{R}^3_1 : (q - p) \cdot (q - p) = r^2\},\\ H^2(p, r) &= \{q \in \mathbb{R}^3_1 : (q - p) \cdot (q - p) = -r^2\},\\ LC(p) &= \{q \in \mathbb{R}^3_1 : (q - p) \cdot (q - p) = 0\}.\end{aligned}$$

The set $S^2_1(p, r)$ is called a pseudo-sphere (de Sitter space) with center p and radius $r > 0$ and $H^2(p, r)$ a hyperbolic plane with center p and radius $r > 0$, and $LC(p)$ a light cone with the vertex p. We put $S^2_1 = S^2_1(0, 1)$, $H^2 = H^2(0, 1)$.

Let S be a smooth immersed surface in Minkowski 3-space and let $f : U \to \mathbb{R}^3_1$, be its local parametrization, where $U \subset \mathbb{R}^2$ is an open set. We denote by g or $\langle\ ,\ \rangle$ the induced metric on S (its first fundamental form), that is, the pull-back of the pseudo-scalar product $\langle\ ,\ \rangle$ of $\mathbb{R}^3_1$. In local coordinates we write $g_{ij} = \langle \frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j} \rangle$. For derivatives we use the notation $f_i = \frac{\partial f}{\partial u_i}$, $i = 1, 2$.

A surface S is called spacelike (resp. timelike, lightlike) if its first fundamental form g is positive definite (resp. indefinite, of rank 1). We exclude lightlike surfaces from our considerations. For spacelike (resp. timelike) surfaces we define locally the unit normal field

$$n = \frac{f_1 \times f_2}{||f_1 \times f_2||},$$

which is a timelike (resp. spacelike) field. Here $x \times y$ denotes the Lorentzian cross-product of vectors x, y which is defined by the condition $\langle x \times y, z \rangle = \det(x, y, z)$. The map $n : S \to S^2_1$ defined for timelike surfaces and the map $n : S \to H^2$ defined for spacelike surfaces is called the Gauss map of S. The Weigarten endomorphism (the shape operator) of S is defined by $L_p : T_pS \to T_pS$, $L_p v = -D_v n$. The second fundamental vector form $\mathbf{II}$ in p is the vector $\mathbf{II}_p$ orthogonal to T_pS which satisfies

$$\langle \mathbf{II}_p(v, w), n \rangle = \langle L_p v, w \rangle. \tag{1}$$

The shape operator L_p is a self-adjoint operator with respect to $\langle\ ,\ \rangle$, hence the following holds

$$\langle L_p v, w \rangle = \langle v, L_p w \rangle, \quad v, w \in T_pS.$$

For the scalar second fundamental II_p we have

$$\mathbf{II}_p(v, w) = II_p(v, w) n = \epsilon \langle L_p v, w \rangle n,$$

i.e. $II_p(v, w) = \epsilon \langle L_p(v), w \rangle$, where $\epsilon = \langle n, n \rangle \in \{-1, 1\}$. The eigenvalues k_1, k_2 of the shape operator L_p can be real or complex conjugate. When they are real, they are called the principal curvatures and the associated eigenvectors are the principal directions of a surface S in p. There are always two principal curvatures on the spacelike part of S and the shape operator is diagonalizable. In the case of a timelike surface, eigenvalues of L_p can be real or complex conjugate and L_p need not be diagonalizable.

In local coordinates we define the coefficients h_{ij} of $\mathbf{II}_p$ by

$$\mathbf{II}_p(f_i, f_j) = h_{ij} n$$

and therefore

$$h_{ij} = \epsilon \langle L_p(f_i), f_j \rangle = -\epsilon \langle n_i, f_j \rangle = \epsilon \langle f_{ij}, n \rangle.$$

We also use $E = g_{11}, F = g_{12} = g_{21}, G = g_{22}, L = h_{11}, M = h_{12} = h_{21}, N = g_{22}$.

Furthermore, the following formulas hold

$$L_p(f_i) = -n_i = \epsilon \sum_{k,s} h_{ik} g^{ks} f_s, \quad i = 1, 2, \tag{2}$$

where (g^{ij}) is the inverse matrix of (g_{ij}).

The Gaussian curvature of S is defined by ([8])

$$K = \epsilon \det L_p = \frac{\langle \mathbf{II}(X,X), \mathbf{II}(Y,Y)\rangle - \langle \mathbf{II}(X,Y), \mathbf{II}(Y,X)\rangle}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}, \tag{3}$$

or in local coordinates it is given by

$$K = \epsilon \frac{\det(h_{ij})}{\det(g_{ij})}. \tag{4}$$

The mean curvature vector field is defined by ([8])

$$\mathbf{H} = \frac{1}{2}\operatorname{tr}\mathbf{II} = \frac{1}{2}\left(\epsilon_1 \mathbf{II}(e_1,e_1) + \epsilon_2 \mathbf{II}(e_2,e_2)\right),$$

where (e_1, e_2) is an orthonormal frame on S at p and $\epsilon_i = \langle e_i, e_i\rangle$, $i = 1, 2$. If we put $\mathbf{H} = Hn$, then (1) implies

$$\begin{aligned} H\epsilon &= \frac{1}{2}\left(\epsilon_1\langle \mathbf{II}(e_1,e_1), n\rangle + \epsilon_2\langle \mathbf{II}(e_2,e_2), n\rangle\right) \\ &= \frac{1}{2}\left(\epsilon_1\langle L_p(e_1), e_1\rangle + \epsilon_2\langle L_p(e_2), e_2\rangle\right) \end{aligned} \tag{5}$$

and therefore

$$H = \frac{\epsilon}{2}\operatorname{tr}L_p. \tag{6}$$

In local coordinates we have

$$H = \frac{1}{2}\frac{h_{11}g_{22} - 2h_{12}g_{12} + h_{22}g_{11}}{\det(g_{ij})}. \tag{7}$$

Since the sign of H does not have a geometrical meaning, many authors define the mean curvature H without ϵ in (6).

When eigenvalues k_1, k_2 of the shape operator L_p are real, from (3), (6) it follows

$$\begin{aligned} &k_{1,2} = \epsilon H \pm \sqrt{H^2 - \epsilon K}, \\ &H^2 - \epsilon K \geq 0 \end{aligned} \tag{8}$$

Throughout the paper we deal with spacelike surfaces and only those timelike surfaces that satisfy the inequality $H^2 - K \geq 0$ ($\epsilon = 1$), that is, timelike surfaces having real principal curvatures.

3. The harmonic evolute in Minkowski 3-space

Having a local parametrization of a smooth surface S, $f : U \to \mathbb{R}^3_1$, the centers of curvature (i.e. the focal points) p_i of S can be obtained as $p_i(u_1, u_2) = f(u_1, u_2) + r_i(u_1, u_2)n(u_1, u_2)$, $i = 1, 2$, where $r_i = \frac{1}{\kappa_i}$ are the principal radii and $n = n(u_1, u_2)$ a unit surface normal in p. The harmonic conjugate point of a triple of collinear points p, p_1, p_2 is a point $\bar{p}$ such that the cross-ratio $(p_1, p_2; p, \bar{p}) = -1$. If points on a line determined by n are given by the coordinates $p = 0$, $p_1 = r_1$, $p_2 = r_2$, $\bar{p} = \lambda$, then we have

$$\frac{r_1}{r_2} \cdot \frac{r_2 - \lambda}{r_1 - \lambda} = -1$$

and therefore

$$\lambda = \frac{2r_1 r_2}{r_1 + r_2}.$$

By using (6), we have $\lambda = \frac{\epsilon}{H}$. Therefore, the harmonic evolute of a surface can be parametrized by

$$\bar{f}(u_1, u_2) = f(u_1, u_2) + \frac{\epsilon}{H(u_1, u_2)} n(u_1, u_2). \tag{9}$$

Notice that when S is a minimal surface, that is, S is a surface with $H = 0$, then its harmonic evolute $\bar{S}$ degenerates to a plane at infinity. Furthermore, in the case when $H \neq 0$, it is known that quadrics $S_1^2(p, r)$ and $H^2(p, r)$ are totally umbilical surfaces, since their shape operator (associated with an outward oriented normal vector) is a scalar operator, $L_p = -1/rI$, so their principal curvatures k_1, k_2 are real and identical. Therefore two centers of curvatures coincide for every point and the harmonic evolute of these surfaces degenerates to a point.

However, contrary to the Euclidean case where the only surfaces satisfying $k_1 = k_2$ in every point are planes and spheres, in Minkowski 3-space, besides the above mentioned surfaces, there exist timelike surfaces with (real) identical principal curvatures and non-diagonalizable shape operator (so called totally quasi-umbilical surfaces). It is known they are all ruled surfaces with null-lines as rulings ([2]). For them, the following holds:

Proposition 1. *The harmonic evolute of a timelike surface with $H \neq 0$ and identical principal curvatures but a non-diagonalizable shape operator degenerates to a curve.*

Proof. A timelike surface with identical principal curvatures and a non-diagonalizable shape operator is a ruled surface and can be locally parametrized by

$$f(u_1, u_2) = c(u_1) + u_2 e(u_1),$$

where rulings are all null-lines ([2]). Its mean curvature $H \neq 0$ is a function of u_1 alone, $H = H(u_1)$, and the matrix of the shape operator L_p with respect to the basis (f_1, f_2) is a lower triangular matrix with equal entries on the diagonal. Hence f_2 is the eigenvector of L_p and therefore $\bar{f}_2 = 0$. Therefore in every point of the harmonic evolute $\bar{S}$ vectors $(\bar{f}_1, \bar{f}_2)$ span a one-dimensional space and $\bar{S}$ degenerates to a curve. □

Example 1. *Let S be a ruled helicoidal surface with a pitch h, $h > 0$, given by a local parametrization*

$$f(u_1, u_2) = (hu_1 + u_2, u_2 \cos u_1, u_2 \sin u_1).$$

It is a timelike surface with $K = \frac{1}{h^2}$, $H = \frac{1}{h}$, $k_1 = k_2 = \frac{1}{h}$. Its shape operator is not diagonalizable, and therefore the surface is not totally umbilical. Its harmonic evolute degenerates to a curve (a helix).

From now on, we exclude minimal surfaces, constant mean surfaces $S_1^2(p,r)$, $H^2(p,r)$ and timelike surfaces with identical principal curvatures and a non-diagonalizable shape operator from further investigations. With analogous exclusions, it is proved in [1] that the harmonic evolute of a surface in Euclidean space is a regular surface. Similarly, in Minkowski 3-space we have:

Theorem 1. *The harmonic evolute of a surface in Minkowski 3-space (which contains no umbilical or quasi-umbilical points) is a regular surface.*

Proof. The harmonic evolute of a surface (9) is regular if and only if $\bar{f}_i = f_i + \frac{\partial}{\partial u_i}(\frac{\epsilon}{H})n + \frac{\epsilon}{H}n_i$, $i = 1, 2$, are linearly independent, or equivalently, if and only if $\bar{f}_1 \times \bar{f}_2 \neq 0$. We can write $\bar{f}_1 \times \bar{f}_2 = a(f_1 \times f_2) + bf_1 + cf_2$, where $a, b, c : U \to \mathbb{R}$. Now (2), (3), (6) imply that the component a is equal to $-1 + \frac{\epsilon K}{H^2}$. When principal curvatures of a surface S exist (i.e. eigenvalues of L_p are real), then

$$1 - \frac{\epsilon K}{H^2} = 1 - \frac{4k_1k_2}{(k_1 + k_2)^2} = (\frac{k_1 - k_2}{k_1 + k_2})^2. \tag{10}$$

Therefore, if $k_1 \neq k_2$, then $\bar{f}_1 \times \bar{f}_2 \neq 0$, and the harmonic evolute $\bar{S}$ is a regular surface. □

We recall that a surface whose Weingarten function $D := \det(g_{ij}) = EG - F^2$ satisfies $D \neq 0$ is regular; $D > 0$ holds for spacelike surfaces and $D < 0$ for timelike surfaces. If $D = 0$, a surface is singular or lightlike.

Proposition 2. *The coefficients of the first fundamental form of the harmonic evolute of a parametrized surface $f = f(u_1, u_2)$ in Minkowski 3-space are given by*

$$\begin{aligned} \bar{E} &= E(1 - \tfrac{\epsilon K}{H^2}) + \epsilon(\tfrac{H_1}{H^2})^2, \\ \bar{F} &= F(1 - \tfrac{\epsilon K}{H^2}) + \epsilon\tfrac{H_1H_2}{H^4}, \\ \bar{G} &= G(1 - \tfrac{\epsilon K}{H^2}) + \epsilon(\tfrac{H_2}{H^2})^2, \end{aligned} \tag{11}$$

where H_i stands for $\frac{\partial H}{\partial u_i}$, $i = 1, 2$.

Proof. From (9) it follows $\bar{f}_i = f_i + \frac{\partial}{\partial u_i}(\frac{\epsilon}{H})n + \frac{\epsilon}{H}n_i$ which then implies

$$\bar{E} = E + 2\frac{\epsilon}{H}\langle f_1, n_1\rangle + (\frac{\partial}{\partial u_1}(\frac{\epsilon}{H}))^2\langle n, n\rangle + (\frac{\epsilon}{H})^2\langle n_1, n_1\rangle.$$

Now (2) implies

$$\bar{E} = E + \epsilon(\frac{H_1}{H^2})^2 - E\frac{1}{H^2}\frac{LN - M^2}{EG - F^2} = E(1 - \frac{\epsilon K}{H^2}) + \epsilon(\frac{H_1}{H^2})^2.$$

The coefficients $\bar{F}$, $\bar{G}$ are calculated analogously. □

The Weingarten function $\bar{D}$ of the harmonic evolute $\bar{S}$ can be calculated by using the previous proposition:

Corollary 1. *The Weingarten function $\bar{D}$ of the harmonic evolute of a surface satisfies*

$$\bar{D} = D(1 - \frac{\epsilon K}{H^2})^2 + \epsilon \frac{EH_2^2 - 2FH_1H_2 + GH_1^2}{H^4}(1 - \frac{\epsilon K}{H^2}). \tag{12}$$

Notice that equation (12) can be written in the following way. We recall that the gradient of a function h on S is defined by

$$\operatorname{grad} h = \sum_{i,j} g^{ij} \frac{\partial h}{\partial x_i} \partial_j, \quad (g^{ij}) = (g_{ij})^{-1}.$$

Then for the mean curvature H of S with respect to the induced metric we have

$$\operatorname{grad} H = \frac{1}{D}\left((GH_1 - FH_2)f_1 + (-FH_1 + EH_2)f_2\right) \in T_pS,$$

where f_1, f_2 span the tangent plane T_pS. Simple calculation shows

$$\langle \operatorname{grad} H, \operatorname{grad} H \rangle = \frac{1}{D}(EH_2^2 - 2FH_1H_2 + GH_1^2).$$

Therefore, from (10) and (12) now it follows:

Corollary 2. *The Weingarten function $\bar{D}$ of the harmonic evolute of a surface is given by*

$$\bar{D} = (\frac{k_1 - k_2}{k_1 + k_2})^2 \left(D(\frac{k_1 - k_2}{k_1 + k_2})^2 + \frac{\epsilon}{H^4}\langle \operatorname{grad} H, \operatorname{grad} H \rangle \right). \tag{13}$$

Equation (13) allows us to discuss the casual character of the harmonic evolute of a surface with $k_1 \neq k_2$. If a surface S is spacelike, then $\operatorname{grad} H \in T_pS$ can only be spacelike, whereas for a timelike surface S, $\operatorname{grad} H$ can either be spacelike, timelike or lightlike. Therefore, the harmonic evolute of a spacelike surface ($\epsilon = -1$) can either be a spacelike, a timelike or a lightlike surface, depending on a mutual relationship between $D(\frac{k_1-k_2}{k_1+k_2})^2$ and $\frac{1}{H^4}\langle \operatorname{grad} H, \operatorname{grad} H \rangle$. For a timelike surface ($\epsilon = 1$, $D < 0$), we have the following:

Proposition 3. *The harmonic evolute of a timelike surface with non-identical principal curvatures for which* $\operatorname{grad} H$ *is timelike or lightlike is a timelike surface.*

For a cmc surface, $\operatorname{grad} H$ is the zero-vector. Therefore:

Corollary 3. *The harmonic evolute of a spacelike (timelike) surface of constant mean curvature is a spacelike (timelike) surface.*

Some further properties of the harmonic evolute are the following:

Proposition 4. *If the local coordinates (u_1, u_2) on S and the corresponding local coordinates on $\bar{S}$ given by equation (9) are both orthogonal coordinate systems, then the mean curvature of S is constant along one family of its parametric curves.*

Proof. First notice that a surface is parametrized by orthogonal coordinates if and only if $F = 0$. It holds both on spacelike as well on timelike surfaces, where on the latter ones the orthogonality of parametric curves in tangent plane with induced metric of index one, implies that one family of parametric curves is timelike and the other spacelike. Therefore their angle is defined by $\sinh\varphi = \frac{F}{EG} = 0$. Now, the assumption implies $F = \bar{F} = 0$, which with (11) yields $H_1 = 0$ or $H_2 = 0$. □

Proposition 5. *The map from S to $\bar{S}$ defined in coordinate representation by equation (9) is a conformal equivalence if and only if S is a surface of constant mean curvature.*

Proof. A map $\psi : S \to \bar{S}$ is a conformal equivalence of surfaces $S, \bar{S}$ if it is a C^∞-diffeomorphism satisfying $\psi^*(g_{\bar{S}}) = hg_S$, where $h \in C^\infty(S)$ is either $h > 0$ or $h < 0$. When $S, \bar{S}$ are locally parametrized by the same parameters u_1, u_2, a map $\psi : S \to \bar{S}$ is a conformal equivalence if and only if the induced coefficients of the first fundamental forms satisfy

$$\bar{E} = hE, \ \ \bar{F} = hF, \ \ \bar{G} = hG. \tag{14}$$

Note that a conformal equivalence preserves the casual character of a surface.

If S cmc surface, then (11) implies (14), where $h = 1 - \frac{\epsilon K}{H^2}$, and (10) implies $h > 0$. Conversely, if (14) is satisfied, then

$$\begin{aligned} E(1 - \tfrac{\epsilon K}{H^2} - h) + \epsilon(\tfrac{H_1}{H^2})^2 &= 0, \\ F(1 - \tfrac{\epsilon K}{H^2} - h) + \epsilon\tfrac{H_1 H_2}{H^4} &= 0, \\ G(1 - \tfrac{\epsilon K}{H^2} - h) + \epsilon(\tfrac{H_2}{H^2})^2 &= 0, \end{aligned} \tag{15}$$

implies

$$(EG - F^2)(1 - \frac{\epsilon K}{H^2} - h)^2 = (\frac{H_1}{H^2})^2(\frac{H_2}{H^2})^2 - (\frac{H_1 H_2}{H^4})^2.$$

Since the right-hand side of the previous equation equals 0 and $EG - F^2 \neq 0$, we have $h = 1 - \frac{\epsilon K}{H^2}$ and therefore (15) implies $H_1 = H_2 = 0$. □

4. Harmonic evolutes as parallel surfaces

In the following, we need some results on parallel surfaces in Minkowski 3-space. A surface is called parallel to a surface $f = f(u_1, u_2)$ (at the distance a) if it admits a parametrization

$$\widetilde{f}(u_1, u_2) = f(u_1, u_2) + an(u_1, u_2), \tag{16}$$

where $a \in \mathbb{R}$ is a constant.

Proposition 6. *Surfaces with constant mean curvature are the only surfaces parallel to their harmonic evolutes.*

Proof. Immediately from the definition of a parallel surface. A harmonic evolute is a parallel surface with a directed distance equal to $\frac{\epsilon}{H}$. □

Since for parallel surfaces we have $\widetilde{n} = \pm n$, where n is a unit normal field of a surface and $\widetilde{n}$ of a parallel surface, parallel surfaces are surfaces of the same casual character. Assuming $\widetilde{n} = n$, the shape operators of parallel surfaces $S, \widetilde{S}$ are related by $\widetilde{L} = L(I - aL)^{-1}$, where I is the identity operator. Therefore curvatures of parallel surfaces in Minkowski space are given by ([11])

$$\widetilde{K} = \frac{K}{1 - 2a\epsilon H + a^2\epsilon K}, \quad \widetilde{H} = \frac{H - aK}{1 - 2a\epsilon H + a^2\epsilon K}. \tag{17}$$

Direct application of formulas (17) implies the following two analogues of the Bonnet's theorem in Minkowski space $\mathbb{R}^3_1$:

Theorem 2. *Let S be a surface of constant Gaussian curvature $K = \epsilon/a^2$. Then there are two surfaces parallel to S which have the constant mean curvature $\widetilde{H} = \pm\epsilon/2a$. (Directed) distances of these surfaces to S are $\mp a$, respectively.*

Theorem 3. *Let S be a surface with constant mean curvature H. Then the parallel surface at the distance $a = \frac{\epsilon}{H}$ has constant mean curvature $\widetilde{H} = -H$.*

Furthermore, for parallel surfaces the following theorem holds:

Theorem 4. *The only pair of parallel surfaces for which the map defined in coordinate representation by equation (9) is a conformal equivalence is the pair of constant mean curvature surfaces from the Theorem 3.*

Proof. Let S and $\widetilde{S}$ be a pair of parallel surfaces at a distance a given locally by (16). Since $\widetilde{f_i} = (I - aL)f_i$, $i = 1, 2$, where L is the shape operator of S, by using (2) it can be shown that the coefficients of the first fundamental form of $\widetilde{S}$ satisfy

$$\begin{aligned}
\widetilde{E} &= E - 2a\epsilon L + \tfrac{a^2}{D}(GL^2 - 2FLM + EM^2),\\
\widetilde{F} &= F - 2a\epsilon M + \tfrac{a^2}{D}(GLM + EMN - F(M^2 + LN)),\\
\widetilde{G} &= G - 2a\epsilon N + \tfrac{a^2}{D}(GM^2 - 2FMN + EN^2).
\end{aligned} \tag{18}$$

Therefore, if S is a cmc surface and $\widetilde{S}$ a parallel surface at the distance $a = \frac{\epsilon}{H}$, then simple calculation using (4), (7) yields $\widetilde{E} = hE$, $\widetilde{F} = hF$, $\widetilde{G} = hG$, where $h = 1 - \frac{\epsilon K}{H^2} > 0$. This implies that (9) is a conformal equivalence between $S, \widetilde{S}$.

Conversely, let S, $\widetilde{S}$ be parallel surfaces that are conformally equivalent with a function h. Then from (14) and (18), it follows

$$(1 - h)^2 = a^2 K \left(a^2 K - 4aH + 4\epsilon\right). \tag{19}$$

Now, the only possible solution of equations (18), (19) is given by $a = \frac{\epsilon}{H} = const.$, $h = 1 - \frac{\epsilon K}{H^2}$, and therefore the statement follows. □

Since the harmonic evolute of a cmc surface is a parallel surface at the directed distance $\frac{\epsilon}{H}$, Theorem 3 implies that the harmonic evolute of a surface is exactly the surface from the Bonnet's theorem.

Finally, we can notice that it is of interest also to investigate higher harmonic evolutes of a surface (that is, the harmonic evolute of a harmonic evolute, etc.). For the cmc surfaces we can conclude the following:

Theorem 5. *Let S be a surface with constant mean curvature. Then its harmonic evolute is a surface with constant mean curvature having the surface S as the harmonic evolute (that is, the second harmonic evolute of S).*

Proof. Theorem 3 implies that the harmonic evolute $\bar{S}$ is a parallel surface of S at the distance $\frac{\epsilon}{H}$ and has constant mean curvature $\bar{H} = -H$. $\square$

5. Harmonic evolutes of surfaces of revolution

According to the types of rotation, there are three types of surfaces of revolution in Minkowski 3-space. By adequate choice of coordinates, the axis of revolution can be considered as the x-axis (a timelike axis), z-axis (a spacelike axis) or the line spanned by $(1,1,0)$ (a lightlike axis). A Lorentzian rotation with respect to the x-axis (resp. z-axis) is given by a matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{pmatrix}, \qquad \begin{pmatrix} \cosh\varphi & \sinh\varphi & 0 \\ \sinh\varphi & \cosh\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A rotation with a lightlike axis (e.g. the diagonal $(1,1,0)$ in xy-plane) is given by a matrix

$$\begin{pmatrix} 1+\frac{\varphi^2}{2} & -\frac{\varphi^2}{2} & \varphi \\ \frac{\varphi^2}{2} & 1-\frac{\varphi^2}{2} & \varphi \\ \varphi & -\varphi & 1 \end{pmatrix}.$$

By rotating a spacelike (timelike, lightlike) curve, a spacelike (timelike, lightlike) surface of revolution is obtained. As we consider non-lightlike surfaces only, we take spacelike and timelike curves as profile curves.

Cylinders are surfaces of revolution with constant mean curvature. They are represented by the Lorentz circular cylinder, $S^1_1 \times \mathbb{R} = \{(x,y,z) | -x^2+y^2 = R^2\}$, the circular cylinder of index 1, $\mathbb{R}^1_1 \times S^1 = \{(x,y,z) | y^2+z^2 = R^2\}$, which are timelike surfaces, and the hyperbolic cylinder $H^1 \times \mathbb{R} = \{(x,y,z) | -x^2+y^2 = -R^2\}$, which is a spacelike surface. We have:

Proposition 7. *The harmonic evolute of a cylinder in Minkowski 3-space is the very same cylinder.*

Proof. Cylinders $S^1_1 \times \mathbb{R}$ and $H^1 \times \mathbb{R}$ can be obtained as the sets $q^{-1}(\epsilon R^2)$, where $q(x,y,z) = -x^2+y^2$, whereas for the circular cylinder $\mathbb{R}^1_1 \times S^1$ we have $\mathbb{R}^1_1 \times S^1 = q^{-1}(R^2)$, where $q(x,y,z) = y^2+z^2$. If we take the normal field for $S^1_1 \times \mathbb{R}$, $H^1 \times \mathbb{R}$ as $n = \frac{\operatorname{grad} q}{||\operatorname{grad} q||} = \frac{1}{R}(x,y,0)$, where $\operatorname{grad} q$ is determined with respect to pseudo-metric, then the shape operators (in the basis for T_pS consisting of a tangent vector of a (pseudo)-circle and a tangent vector of a ruling) are

$$L_p = \begin{pmatrix} -\frac{1}{R} & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore $H = -\frac{\epsilon}{2R}$. Now, a point of the harmonic evolute obtained from a point $P(x, y, z)$ of a cylinder is

$$(x, y, z) + \frac{\epsilon}{H} n = (x, y, z) - 2R\frac{1}{R}(x, y, 0) = (-x, -y, z),$$

which is a point of a cylinder symmetrical to a starting point with respect to the axis of cylinder. Similarly, for $\mathbb{R}^1_1 \times S^1 = q^{-1}(R^2)$, $q(x, y, z) = y^2 + z^2$. □

Proposition 8. *The harmonic evolute of a surface of revolution is a coaxial surface of revolution.*

Proof. A surface of revolution is a surface invariant by a 1-parameter group of Lorentzian rotations. By the construction of the harmonic evolute, since a Lorentzian cross product is invariant under Lorentzian rotations, and thus such is the unit normal of a surface of revolution, the harmonic evolute is a surface invariant by the same 1-parameter group of Lorentzian rotations, i.e. it is a surface of revolution coaxial with the initial surface of revolution. □

Example 2. *In what follows we give the local parametrizations of the harmonic evolutes of surfaces of revolution. If S is a surfaces of revolution obtained by rotating with respect to the x-axis (a timelike axis) a profile curve $c(u) = (g(u), 0, h(u))$ in xz-plane, where g, h are smooth functions, $h > 0$, and c is, without loss of generality, parametrized by the arc-length $-g'^2 + h'^2 = \pm 1$, then*

$$f(u, v) = (g(u), h(u)\sin v, h(u)\cos v)$$

and

$$\bar{f}(u, v) = (\bar{g}(u), \bar{h}(u)\sin v, \bar{h}(u)\cos v),$$

where

$$\bar{g}(u) = g(u) + \frac{\epsilon}{H(u)}h'(u), \quad \bar{h}(u) = h(u) + \frac{\epsilon}{H(u)}g'(u),$$

$H = H(u)$ is the mean curvature of a surface

$$H(u) = -\epsilon\frac{g''(u)h(u) + g'(u)h'(u)}{2h(u)h'(u)}.$$

The harmonic evolute of a surface of revolution with respect to the z-axis (a spacelike axis)

$$f(u, v) = (g(u)\cosh v, g(u)\sinh v, h(u))$$

obtained by rotating a profile curve $c(u) = (g(u), 0, h(u))$ in the xz-plane, where g, h are smooth functions, $g > 0$, and which is, without loss of generality, parametrized by the arc-length $-g'^2 + h'^2 = \pm 1$, is parametrized by

$$\bar{f}(u, v) = (\bar{g}(u)\cosh v, \bar{g}(u)\sinh v, \bar{h}(u)),$$

where

$$\bar{g}(u) = g(u) + \frac{\epsilon}{H(u)}h'(u), \quad \bar{h}(u) = h(u) + \frac{\epsilon}{H(u)}g'(u), \quad H(u) = -\epsilon\frac{g''g + h'^2}{2gh'}.$$

The harmonic evolute of a surface of revolution with respect to a lightlike axis (e.g. the axis $x = y$, $z = 0$)

$$f(u,v) = ((1+\frac{v^2}{2})g(u) - \frac{v^2}{2}h(u), \frac{v^2}{2}g(u) + (1-\frac{v^2}{2})h(u), v(g(u)-h(u))$$

obtained by rotating a profile curve $c(u) = (g(u), h(u), 0)$ in the xy-plane by isotropic rotations, where g, h are smooth functions, $g - h > 0$, and which is, without loss of generality, parametrized by arc-length $-g'^2 + h'^2 = \pm 1$, can be parametrized by

$$\bar{f}(u,v) = ((1+\frac{v^2}{2})\bar{g}(u) - \frac{v^2}{2}\bar{h}(u), \frac{v^2}{2}\bar{g}(u) + (1-\frac{v^2}{2})\bar{h}(u), v(\bar{g}(u)-\bar{h}(u)),$$

where

$$\begin{aligned} \bar{g}(u) &= g(u) - \frac{\epsilon}{H(u)}h'(u), \quad \bar{h}(u) = h(u) - \frac{\epsilon}{H(u)}g'(u), \\ H(u) &= -\epsilon\frac{(g'(u) - h'(u) + \epsilon(g(u)-h(u))(g''(u)h'(u) - g'(u)h''(u)))}{2(g(u)-h(u))}. \end{aligned}$$

Among surfaces of revolution, we particularly distinguish the class of cmc-surfaces, so-called Delaunay surfaces. From the aforementioned, we have the following proposition:

Proposition 9. *Harmonic evolutes of Delaunay surfaces in Minkowski 3-space are Delaunay surfaces.*

Acknowledgement

The authors thank the anonymous referees for valuable suggestions that improved the final version of the paper.

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