On two Thomae-type transformations for hypergeometric series with integral parameter differences

Yong S. Kim¹, Arjun K. Rathie² and Richard B. Paris³,*

¹ Department of Mathematics Education, Wonkwang University, Iksan, Korea
² Department of Mathematics, Central University of Kerala, Kasaragod 671 123, Kerala, India
³ School of Computing, Engineering and Applied Mathematics, University of Abertay Dundee, Dundee DD1 1HG, UK

Received September 9, 2013; accepted January 15, 2014

Abstract. We obtain two new Thomae-type transformations for hypergeometric series with r pairs of numeratorial and denominatorial parameters differing by positive integers. This is achieved by application of the so-called Beta integral method developed by Krattenthaler and Rao [Symposium on Symmetries in Science (ed. B. Gruber), Kluwer (2004)] to two recently obtained Euler-type transformations. Some special cases are given.

AMS subject classifications: 33C15, 33C20
Key words: generalized hypergeometric series, Thomae transformations, generalized Euler-type transformations

1. Introduction

The generalized hypergeometric function \( pF_q(x) \) is defined for complex parameters and argument by the series

\[
pF_q \left[ \begin{array}{c} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{array} ; x \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k \cdots (a_p)_k}{(b_1)_k(b_2)_k \cdots (b_q)_k} \frac{x^k}{k!},
\]

(1)

When \( q \geq p \), this series converges for \( |x| < \infty \), but when \( q = p - 1 \), convergence occurs when \( |x| < 1 \) (unless the series terminates). In (1), the Pochhammer symbol or ascending factorial \((a)_n\) is given for integer \(n\) by

\[
(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & (n = 0) \\ a(a+1) \cdots (a+n-1) & (n \geq 1), \end{cases}
\]

where \(\Gamma\) is the gamma function. In what follows we shall adopt the convention of writing the finite sequence of parameters \((a_1, a_2, \ldots, a_p)\) simply by \((a_p)\) and the product of \(p\) Pochhammer symbols by

\[
((a_p))_k \equiv (a_1)_k \cdots (a_p)_k.
\]

*Corresponding author. Email addresses: yspkim@wonkwang.ac.kr (Y.S. Kim), akrathie@cukerala.edu.in (A.K. Rathie), r.paris@abertay.ac.uk (R.B. Paris)
where an empty product $p = 0$ is interpreted as unity.

Recent work has been carried out on the extension of various summations theorems, such as those of Gauss, Kummer, Bailey and Watson [1, 6, 7], and also of Euler-type transformations to higher-order hypergeometric functions with $r$ pairs of numeratorial and denominatorial parameters differing by positive integers [3, 4]. Our interest in this note is concerned with obtaining similar extensions of the two-term Thomae transformation [8, p. 52]

$$\binom{3}{2}_2 \left[ \begin{array}{ccc} a, b, c \\ d, e \end{array} ; 1 \right] = \frac{\Gamma(d)\Gamma(e)\Gamma(\sigma)}{\Gamma(a)\Gamma(b + \sigma)\Gamma(c + \sigma)} \binom{c - a, d - a, \sigma}{b + \sigma, c + \sigma} ; 1$$

for $\Re(\sigma) > 0, \Re(a) > 0$, where $\sigma = e + d - a - b - c$ is the parametric excess. Many other results of the above type, including three-term Thomae transformations, are given in [8, pp. 116-121]; see also [9].

The so-called Beta integral method introduced by Krathenthaler and Rao [2] generates new identities for hypergeometric series for some fixed value of the argument (usually 1) from known identities for hypergeometric series with a smaller number of parameters involving the argument $x, 1 - x$ or a combination of their powers. The basic idea of this method is to multiply the known hypergeometric identity by the factor $x^{d-1}(1-x)^{e-d-1}$, where $e$ and $d$ are suitable parameters, integrate term by term over $[0,1]$ making use of the beta integral representation

$$\int_0^1 t^{a-1}(1-t)^{b-1} dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} (\Re(a) > 0, \Re(b) > 0) \tag{2}$$

and finally to rewrite the result in terms of a new hypergeometric series. We apply this method to two Euler-type transformations obtained recently in [3, 4] to derive two two-term Thomae-type transformations for hypergeometric functions with $r$ pairs of numeratorial and denominatorial parameters differing by positive integers.

2. Extended Thomae-type transformations

Our starting point is the following Euler-type transformations for hypergeometric functions with $r$ pairs of numeratorial and denominatorial parameters differing by positive integers ($m_r$).

**Theorem 1.** Let $(m_r)$ be a sequence of positive integers with $m := m_1 + \cdots + m_r$. Then we have the two Euler-type transformations [3, 4] for $|\arg(1-x)| < \pi$

$$r+2 F_{r+1} \left[ \begin{array}{ccc} a, b, (f_r + m_r) \\ c, (f_r) \end{array} ; x \right] = (1-x)^{-a} m+2 F_{m+1} \left[ \begin{array}{ccc} a - b - m, (\xi_m + 1) \\ c, (\xi_m) \end{array} ; x \right] \tag{3}$$

provided $b \neq f_j (1 \leq j \leq r), (c - b - m)_m \neq 0$ and

$$r+2 F_{r+1} \left[ \begin{array}{ccc} a, b, (f_r + m_r) \\ c, (f_r) \end{array} ; x \right] = (1-x)^{c-a-b-m} m+2 F_{m+1} \left[ \begin{array}{ccc} c-a-m, c-b-m, (\eta_m + 1) \\ c, (\eta_m) \end{array} ; x \right] \tag{4}$$
provided \((c - a - m)_m \neq 0, (c - b - m)_m \neq 0\). The \((\xi_m)\) and \((\eta_m)\) are respectively the nonvanishing zeros of the associated parametric polynomials \(Q_m(t)\) and \(\hat{Q}_m(t)\) defined below.

The parametric polynomials \(Q_m(t)\) and \(\hat{Q}_m(t)\), both of degree \(m = m_1 + \cdots + m_r\), are given by

\[
Q_m(t) = \frac{1}{(\lambda)^m} \sum_{k=0}^{m} (b)_k C_{k;r}(t_k)(\lambda - t)^{m-k},
\]

where \(\lambda := b - a - m\), and

\[
\hat{Q}_m(t) = \sum_{k=0}^{m} \frac{(-1)^k C_{k;r}(a)(b)(t_k)}{(c - a - m)_k(c - b - m)_k} G_{m;k}(t),
\]

where

\[
G_{m;k}(t) := \binom{3}{2} \binom{-m + k; t + k; c - a - b - m}{c - a - m + k; c - b - m + k; 1}.
\]

The coefficients \(C_{k;r}\) are defined for \(0 \leq k \leq m\) by

\[
C_{k;r} = \frac{1}{\Lambda} \sum_{j=k}^{m} \sigma_j \mathcal{S}_j^{(k)}, \quad \Lambda = (f_1)_{m_1} \cdots (f_r)_{m_r},
\]

with \(C_{0;r} = 1, C_{m;r} = 1/\Lambda\). The \(\mathcal{S}_j^{(k)}\) denote the Stirling numbers of the second kind and the \(\sigma_j\) \(0 \leq j \leq m\) are generated by the relation

\[
(f_1 + x)_{m_1} \cdots (f_r + x)_{m_r} = \sum_{j=0}^{m} \sigma_j x^j.
\]

For \(0 \leq k \leq m\), the function \(G_{m;k}(t)\) is a polynomial in \(t\) of degree \(m - k\) and both \(Q_m(t)\) and \(\hat{Q}_m(t)\) are normalized so that \(Q_m(0) = \hat{Q}_m(0) = 1\).

**Remark 1.** In [5], an alternative representation for the coefficients \(C_{k;r}\) is given as the terminating hypergeometric series of unit argument

\[
C_{k;r} = \frac{(-1)^k}{k!} \binom{-k}{r; f_r + m_r; 1}.
\]

When \(r = 1\), with \(f_1 = f, m_1 = m\), Vandermonde’s summation theorem [8, p. 243] can be used to show that

\[
C_{k;1} = \binom{m}{k} \frac{1}{(f)_k}.
\]

We first apply the Beta integral method [2] to the result in (4) to obtain a new hypergeometric identity. Multiplying both sides by \(x^{d-1}(1 - x)^{e-d-1}\), where \(e, d\) are arbitrary parameters satisfying \(\Re(e - d) > 0, \Re(d) > 0\), we integrate over the
interval \([0, 1]\). The left-hand side yields

\[
\int_0^1 x^{d-1}(1-x)^{c-d-1} \binom{r+2}{r}F_{r+1} \left[ a, b, (f_r + m_r) \left| c, (f_r) \right. ; x \right] dx
\]

\[
= \sum_{k=0}^{\infty} \frac{(a)_k(b)_k ((f_r + m_r))_k}{(c)_k k! (f_r)_k} \int_0^1 x^{d+k-1}(1-x)^{c-d-1} dx
\]

\[
= \sum_{k=0}^{\infty} \frac{(a)_k(b)_k ((f_r + m_r))_k}{(c)_k k! (f_r)_k} \Gamma(d+k)\Gamma(c-d) \Gamma(e-k) \Gamma(e)
\]

\[
\frac{\Gamma(d)\Gamma(e-d)}{\Gamma(e)} \binom{r+3}{r}F_{r+2} \left[ a, b, d, (f_r + m_r) \left| c, e \left| (f_r) ; 1 \right. \right. \right]
\]

\[
(10)
\]

upon evaluation of the integral by (2) and use of the definition (1) when it is supposed that \(\Re(s) > 0\), where \(s\) is the parametric excess given by

\[
s := c + e - a - b - d - m.
\]

(11)

Proceeding in a similar manner with the right-hand side of (4), we obtain

\[
\int_0^1 x^{d-1}(1-x)^{s-1} \binom{m+2}{m}F_{m+1} \left[ c-a-m, c-b-m, (\eta_m+1) \left| \eta_m \right. ; x \right] dx
\]

\[
= \sum_{k=0}^{\infty} \frac{(c-a-m)_k(c-b-m)_k (\eta_m+1)_k}{(c)_k k! (\eta_m)_k} \int_0^1 x^{d+k-1}(1-x)^{s-1} dx
\]

\[
= \frac{\Gamma(d)\Gamma(s)}{\Gamma(c+e-a-b-m)} \binom{m+3}{m}F_{m+2} \left[ c-a-m, c-b-m, d, (\eta_m+1) \left| \eta_m \right. ; 1 \right]
\]

(12)

Then by (10) and (12) we obtain the two-term Thomae-type hypergeometric identity given in the following theorem, where the restriction \(\Re(d) > 0\) can be removed by appeal to analytic continuation:

**Theorem 2.** Let \((m_r)\) be a sequence of positive integers with \(m := m_1 + \cdots + m_r\). Then

\[
\binom{r+3}{r}F_{r+2} \left[ a, b, d, (f_r + m_r) \left| c, e \left| (f_r) ; 1 \right. \right. \right]
\]

\[
= \frac{\Gamma(c)\Gamma(s)}{\Gamma(c-d)\Gamma(s+d)} \binom{m+3}{m}F_{m+2} \left[ c-a-m, c-b-m, d, (\eta_m+1) \left| \eta_m \right. ; 1 \right]
\]

(13)

provided \((c-a-m)_m \neq 0, (c-b-m)_m \neq 0, \Re(c-d) > 0\) and \(\Re(s) > 0\), where \(s\) is defined by (11).

The same procedure can be applied to (3) when the parameter \(a = -n\) (to ensure convergence of the resulting integral at \(x = 1\)), where \(n\) is a non-negative integer, to
yield the right-hand side of (3) given by
\[
\int_0^1 x^{d-1}(1-x)^{e-d+n-1} \binom{n}{m} F_{m+1} \left[ \begin{array}{c} -n, c-b-m, (\xi m + 1) \\ c, (\xi m) \end{array} ; \frac{x}{x-1} \right] \, dx \\
= \sum_{k=0}^{n} \frac{(-1)^k (n)_k (c-b-m)_k}{(c)_k k!} \frac{(\xi m + 1)_k}{((\xi m))_k} \int_0^1 x^{d+k-1}(1-x)^{e-d+n-k-1} \, dx \\
= \frac{\Gamma(d)\Gamma(e-d+n)}{\Gamma(e+n)} \sum_{k=0}^{n} \frac{(-1)^k (n)_k (c-b-m)_k (\xi m + 1)_k}{(c)_k (1-e+d-n)_k k!} \frac{((\xi m))_k}{((\xi m))_k} \\
= \frac{\Gamma(d)\Gamma(e-d+n)}{\Gamma(e+n)} m+3 F_{m+2} \left[ \begin{array}{c} -n, c-b-m, d, (\xi m + 1) \\ c, 1-e+a+d, (\xi m) \end{array} ; 1 \right] \\
\tag{14}
\]
provided \( R(e-d) > 0 \), \( R(d) > 0 \). From (10) and (14), and appeal to analytic continuation to remove the restriction \( R(d) > 0 \), we then obtain the finite Thomae-type transformation

**Theorem 3.** Let \((m_r)\) be a sequence of positive integers with \( m := m_1 + \cdots + m_r \).
Then, for non-negative integer \( n \)
\[
r+3 F_{r+2} \left[ \begin{array}{c} -n, b, d, (f_r + m_r) \\ c, e, (f_r) \end{array} ; 1 \right] = \frac{(e-d)_n}{(e)_n} m+3 F_{m+2} \left[ \begin{array}{c} -n, c-b-m, d, (\xi m + 1) \\ c, 1-e+a+d, (\xi m) \end{array} ; 1 \right] \\
\tag{15}
\]
provided \( b \neq f_j (1 \leq j \leq r) \), \( c-b-m)_m \neq 0 \) and \( R(e-d) > 0 \).

**3. Examples**

When \( r = 0 \) (with \( m = 0 \)), from (13) and (15) we recover the known results [9]
\[
3 F_2 \left[ \begin{array}{c} a, b, d \\ c, e \end{array} ; 1 \right] = \frac{\Gamma(e)\Gamma(c+e-a-b-d)}{\Gamma(e-d)\Gamma(c+e-a-b)} 3 F_2 \left[ \begin{array}{c} c-a, c-b, d \\ c+e-a-b \end{array} ; 1 \right]
\]
for \( R(e-d) > 0 \), \( R(e+c-a-b-d) > 0 \) and
\[
3 F_2 \left[ \begin{array}{c} -n, b, d \\ c, e \end{array} ; 1 \right] = \frac{(e-d)_n}{(e)_n} 3 F_2 \left[ \begin{array}{c} -n, c-b, d \\ c, 1-e+d-n \end{array} ; 1 \right]
\]
for \( R(e-d) > 0 \) with \( n \) a non-negative integer.

In the particular case \( r = 1 \), \( m_1 = m = 1 \), \( f_1 = f \), we have the parametric polynomial from (5)
\[
Q_1(t) = 1 + \frac{(b-f)t}{(c-b-1)f}
\]
with the nonvanishing zero \( \xi_1 = \xi \) (provided \( b \neq f \), \( c-b-1 \neq 0 \)) given by
\[
\xi = \frac{(c-b-1)f}{f-b},
\tag{16}
\]
and from (6)

\[ Q_1(t) = 1 - \frac{(c-a-b-1)f + ab)\ell}{(c-a-1)(c-b-1)f} \]

with the nonvanishing zero \( \eta_1 = \eta \) (provided \( c-a-1 \neq 0, c-b-1 \neq 0 \)) given by

\[ \eta = \frac{(c-a-1)(c-b-1)f}{ab + (c-a-b-1)f} . \tag{17} \]

Then from (13) and (15) we have the transformations

\[ _4F_3 \left[ \begin{array}{c} a, b, d, f+1 \\ c, e, f \end{array} \right] = \frac{\Gamma(c)\Gamma(s)}{\Gamma(e-d)\Gamma(s+d)} _4F_3 \left[ \begin{array}{c} c-a-1, c-b-1, d, \eta + 1 \\ c, s+d, \eta \end{array} \right] \]

provided \( c-a-1 \neq 0, c-b-1 \neq 0, \Re(e-d) > 0 \) and \( \Re(s) > 0 \), where \( s \) is defined by (11) with \( m = 1 \), and

\[ _4F_3 \left[ \begin{array}{c} -n, b, d, f+1 \\ c, e, f \end{array} \right] = \frac{(d-c)n}{(c)n} _4F_3 \left[ \begin{array}{c} -n, c-b-1, d, \xi + 1 \\ c, -c+e+d-n, \xi \end{array} \right] \]

for non-negative integer \( n \) and \( \Re(e-d) > 0 \).

In the case \( r = 1, m_1 = 2 \), \( f_1 = f \), we have \( C_{0,1} = 1, C_{1,1} = 2/f \) and \( C_{2,1} = 1/(f)2 \) by (9). From (5) and (6) we obtain after a little algebra the quadratic parametric polynomials \( Q_2(t) \) (with zeros \( \xi_1 \) and \( \xi_2 \)) and \( \hat{Q}_2(t) \) (with zeros \( \eta_1 \) and \( \eta_2 \)) given by

\[ Q_2(t) = 1 - \frac{2(f-b)t}{(c-b-2)f} + \frac{(f-b)t(t+1)}{(c-b-2)(f)2} \]

and

\[ \hat{Q}_2(t) = 1 - \frac{2Bt}{(c-a-2)(c-b-2)} + \frac{Ct(1+t)}{(c-a-2)(c-b-2)^2} \]

where

\[ B := \sigma' + \frac{ab}{f}, \quad C := \sigma' \sigma' + 1 + \frac{2ab\sigma'}{f} + \frac{a_2(b)2}{(f)2}, \quad \sigma' := c-a-b-2. \]

For example, if \( a = \frac{1}{2}, b = \frac{5}{2}, c = \frac{3}{2} \) and \( f = \frac{1}{2} \) we have

\[ Q_2(t) = 1 - \frac{5}{3}t + \frac{5}{7}t(1+t), \quad \hat{Q}_2(t) = 1 + \frac{10}{9}t - \frac{10}{9}t(1+t), \]

whence \( \xi_1 = \frac{1}{2}, \xi_2 = \frac{9}{2} \) and \( \eta_1 = \frac{1}{2}, \eta_2 = -\frac{27}{54} \). The transformations in (13) and (15) then yield

\[ _4F_3 \left[ \begin{array}{c} \frac{1}{2}, \frac{5}{2}, d, \frac{5}{2} \\ \frac{3}{2}, c, \frac{1}{2} \end{array} \right] = \frac{\Gamma(e)\Gamma(e-d-\frac{13}{4})}{\Gamma(e-d)\Gamma(e-\frac{13}{4})} _4F_3 \left[ \begin{array}{c} -\frac{3}{4}, -3, d, \frac{7}{54} \\ \frac{1}{2}, \frac{5}{2}, -\frac{27}{54} \end{array} \right] \tag{18} \]

provided \( \Re(e-d) > \frac{13}{4} \), and

\[ _4F_3 \left[ \begin{array}{c} -n, \frac{5}{2}, d, \frac{5}{2} \\ \frac{3}{2}, c, \frac{1}{2} \end{array} \right] = \frac{(e-d)n}{(c)n} _4F_3 \left[ \begin{array}{c} -n, -3, d, \frac{11}{2} \\ 1- c + e+ n, \frac{1}{2}, \frac{9}{2} \end{array} \right] \tag{19} \]
for non-negative integer \( n \). We remark that a contraction of the order of the hypergeometric functions on the right-hand sides of (18) and (19) has been possible since \( c = \xi_1 + 1 = \eta_1 + 1 = \frac{3}{2} \). In addition, both series on the right-hand sides terminate: the first with summation index \( k = 3 \) and the second with index \( k = \min\{n, 3\} \). A final point to mention is that for real parameters \( a, b, c \) and \( f \) it is possible (when \( m \geq 2 \)) to have complex zeros.

4. Concluding remarks

We have employed the Beta Integral method of Krattenthaler and Rao [2] applied to two recently obtained Euler-type transformations for hypergeometric functions with \( r \) pairs of numeratorial and denominatorial parameters differing by positive integers \( (m_r) \). By this, we have established two Thomae-type transformations given in Theorems 2 and 3.

In order to write the hypergeometric series in (13) and (15) we require the zeros \( (\eta_m) \) and \( (\xi_m) \) of the parametric polynomials \( \tilde{Q}_m(t) \) and \( Q_m(t) \), respectively. However, to evaluate the series on the right-hand sides of (13) and (15), it is not necessary to evaluate these zeros. This observation can be understood by reference to the hypergeometric series

\[
F \equiv m+2F_{m+1} \left[ \begin{array}{c}
\alpha, \beta, \xi_m + 1 \\
\gamma, (\xi_m) \end{array} ; 1 \right] = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k k!} \left( 1 + \frac{k}{\xi} \right) \cdots \left( 1 + \frac{k}{\xi_m} \right)
\]

upon use of the fact that \((a+1)_k/(a)_k = 1+(k/a)\). Since the parametric polynomial \( Q_m(t) \) in (5) can be written as \( Q_m(t) = \prod_{r=1}^{m} \{1 - (t/\xi_r)\} \), it follows that

\[
F = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k k!} Q_m(-k).
\]

Consequently, it is sufficient to know only the parametric polynomial \( Q_m(t) \). A similar remark applies to the series involving the zeros \( (\eta_m) \) with the parametric polynomial \( Q_m(-k) \) replaced by \( \tilde{Q}_m(-k) \).

Acknowledgement:

Y. S. Kim acknowledges the support of the Wonkwang University Research Fund (2014).

References


