Erratum to “Total domination number of Cartesian products” [Math. Commun. 9(2004), 35–44]*

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Abstract. We correct a partial mistake for the total domination number of $\gamma_t(P_6 \square P_k)$ presented in the article “Total domination number of Cartesian products” [Math. Commun. 9(2004), 35–44].

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1. Introduction and preliminaries

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. We use the standard notations $N_G(v)$ for the open neighborhood $\{u : uv \in E(G)\}$ and $N_G[v]$ for the closed neighborhood $N_G(v) \cup \{v\}$ of a graph $G$. Throughout the article we only consider simple graphs.

The Cartesian product $G \square H$ of the graphs $G$ and $H$ is a graph with vertex set $V(G \square H) = V(G) \times V(H)$. Two vertices $(g, h)$ and $(g', h')$ are adjacent in $G \square H$ whenever $(gg' \in E(G)$ and $h = h')$ or $(g = g'$ and $hh' \in E(H))$. The Cartesian product is commutative and associative (see [5]). For a fixed $h \in V(H)$ we call $G^h = \{(g, h) \in V(G \times H) : g \in V(G)\}$ a $G$-layer in $G \times H$. An $H$-layer $^hH$ for a fixed $g \in V(G)$ is defined symmetrically. Any subgraph of $G \square H$ induced by $G^h$ or $^hH$ is isomorphic to $G$ or $H$, respectively. A Cartesian product graph is called a grid if both factors are isomorphic to paths. Since here we are only interested in grid graphs, more precisely in $P_6 \square P_k$, we use the following notation for vertices of $P_6 \square P_k$:

$$V(P_6 \square P_k) = \{(i, j) : i \in \{1, \ldots, 6\}, j \in \{1, \ldots, k\}\}.$$  

The domination number $\gamma(G)$ of a graph $G$ is one of the classical invariants in graph theory. It is given by the minimum cardinality of a set $S$ for which the union of
closed neighborhoods centered in the vertices of $S$ covers the whole vertex set of $G$. Such a set $S$ is called a dominating set of $G$. Hence, each vertex of $G$ is either in $S$ or adjacent to a vertex in $S$. In other words, we can say that vertices of $S$ control each vertex outside of $S$. A classical question in such a situation is: what controls the vertices of $S$? One possible solution to this dilemma is the total domination. A set $D \subseteq V(G)$ is a total dominating set of $G$ if every vertex of $G$ is adjacent to a vertex of $D$. (Hence, also vertices of $D$ are controlled by $D$.) The total domination number of a graph $G$ is the minimum cardinality of a total dominating set of $G$ and it is denoted by $\gamma_t(G)$. A total dominating set $D$ of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set. For more information about total domination in graphs we suggest the recent monograph [6].

Several graph products have been investigated in the last decades and a rich theory involving the structure and recognition of classes of these graphs has emerged, cf. [5]. Probably the most studied graph product is the Cartesian product, which is also the most problematic for domination related problems. We just mention the famous Vizing’s conjecture: $\gamma(G \square H) \geq \gamma(G)\gamma(H)$, which is probably the most challenging problem in the area of domination (see the latest survey on Vizing’s conjecture [1]).

Closely related to the problem of domination in grid graphs, recently solved in [4], also the total domination number of grid graphs attracted some attention in the past decade. For instance, in [3], the value of $\gamma_t(P_r \square P_t)$ was computed for $r \in \{1, 2, 3, 4\}$. This work was continued in [7] for $r \in \{5, 6\}$. Unfortunately, there is a partial mistake in the value of the total domination number of $P_6 \square P_t$ given in [7], which we correct in the next section.

2. The grid $P_6 \square P_k$

The following formula appeared in [7] for $k \geq 6$:

$$\gamma_t(P_6 \square P_k) = \left\lfloor \frac{12k + 21}{7} \right\rfloor.$$  \hspace{1cm} (1)

However, this formula is not correct in some cases. As we will show below, this result is not correct when $k \equiv x \pmod{7}$ for $x \in \{0, 4, 5, 6\}$. The mistake is due to the facts that, on one hand, not all optimal patterns have been considered in [7] and, on the other hand, the number stated in Equation 1 is incorrect (for $x = 4$). To do so, we need to introduce some terminology. A graph $G$ is an efficient open domination graph if there exists a set $D$, called an efficient open dominating set, for which

$$\bigcup_{v \in D} N_G(v) = V(G) \text{ and } N_G(u) \cap N_G(v) = \emptyset$$

for every pair $u$ and $v$ of distinct vertices of $D$ (see [2]). The following result from [8] is useful to prove our results. (We also state the proof to make the present work self contained.)

**Lemma 1** (see [8]). If $G$ is an efficient open domination graph with an efficient open dominating set $D$, then $\gamma_t(G) = |D|$.
Proof. If D is an efficient open dominating set of G, then D is also a total dominating set of G and \( \gamma_t(G) \leq |D| \) follows. On the other hand, every vertex of D has at least one neighbor in every \( \gamma_t(G) \)-set \( D' \), since
\[
\bigcup_{v \in D'} N_G(v) = V(G).
\]
Moreover, these neighbors must be different, since
\[
\bigcup_{v \in D} N_G(v)
\]
forms a partition of \( V(G) \). Hence \( \gamma_t(G) \geq |D| \) and the equality follows.

Theorem 1. Let \( k \geq 6 \). Then
\[
\gamma_t(P_6 \Box P_k) = \begin{cases} 
\frac{12k+14}{7}, & \text{if } k \equiv 0 \pmod{7}, \\
\frac{12k+16}{7}, & \text{if } k \equiv 1 \pmod{7}, \\
\frac{12k+18}{7}, & \text{if } k \equiv 2 \pmod{7}, \\
\frac{12k+20}{7}, & \text{if } k \equiv 3 \pmod{7}, \\
\frac{12k+22}{7}, & \text{if } k \equiv 4 \pmod{7}, \\
\frac{12k+24}{7}, & \text{if } k \equiv 5 \pmod{7}, \\
\frac{12k+12}{7}, & \text{if } k \equiv 6 \pmod{7}.
\end{cases}
\]

Proof. First we try to find a total dominating set D for \( G = P_6 \Box P_k \) where every vertex is totally dominated exactly once. Notice that if every vertex of G is totally dominated by D exactly once, then G is an efficient open domination graph. Thus, by Lemma 1 we have that \( \gamma_t(G) = |D| \). We have only three options, up to the symmetry, to totally dominate each vertex exactly once in the first layer \( P_1 \) of G, see Figure 1. Moreover, each of these three possibilities expands to the whole G in a unique way (the pattern is forced by the starting position in the first layer \( P_1 \)), again see Figure 1. Double dotted lines in each graph of Figure 1 show the positions in which the pattern can stop to obtain a total dominating set for G where every vertex is totally dominated exactly once. This is done when \( k \equiv x \pmod{7} \) for \( x \in \{1, 4, 6\} \). Hence, if \( x \in \{1, 4, 6\} \), then D is a \( \gamma_t(G) \)-set and we only need to know the cardinality of the set D. If we split G into consecutive blocks isomorphic to \( P_6 \Box P_7 \) and the remainder \( P_6 \Box P_x \), \( x \in \{1, 4, 6\} \), then it is easy to see that each block contains twelve vertices of D. In the remainder \( P_6 \Box P_7 \) we get additional four vertices (see Figure 1 b)), in \( P_6 \Box P_3 \) additional eight vertices (see Figure 1 a)) and in \( P_6 \Box P_5 \) additional twelve vertices (see Figure 1 b)). For \( k \equiv 1 \pmod{7} \) we have
\[
k = 7n + 1 \quad \text{and} \quad \gamma_t(P_6 \Box P_k) = 12n + 4 = \frac{12k + 16}{7},
\]
by Lemma 1. By doing a similar computation we obtain that \( \gamma_t(P_6 \Box P_k) = (12k + 8)/7 \) for \( k \equiv 4 \pmod{7} \) and \( \gamma_t(P_6 \Box P_k) = (12k + 12)/7 \) for \( k \equiv 6 \pmod{7} \).

Let now \( k \equiv x \pmod{7} \), \( x \in \{0, 2, 3, 5\} \). Notice that in Figure 1 a), for \( x \in \{0, 3\} \), and in Figure 1 b), for \( x \in \{2, 5\} \), vertices (3, k) and (4, k) are only not totally
dominated vertices in these patterns. (Notice also that in all three patterns we get two or more vertices which are not totally dominated by \( D \).) Hence \( D' = D \cup \{(3, k), (4, k)\} \) is a total dominating set of \( P_b \square P_k \).

\[ \mathbf{Figure 1: The efficient open dominating set is given by the white vertices} \]

For \( x = 0 \), we have \( k = 7n \) and there is no remainder, but two additional vertices in \( D' \). Thus \( \gamma_t(P_b \square P_k) \leq 12n + 2 = (12k + 14)/7 \).

For \( x = 2 \), we have \( k = 7n + 2 \) and in the remainder \( P_b \square P_2 \) there are six additional vertices in \( D' \). Thus \( \gamma_t(P_b \square P_k) \leq 12n + 6 = (12k + 18)/7 \).

For \( x = 3 \), we have \( k = 7n + 3 \) and in the remainder \( P_b \square P_3 \) there are eight additional vertices in \( D' \). Thus \( \gamma_t(P_b \square P_k) \leq 12n + 8 = (12k + 20)/7 \).

For \( x = 5 \), we have \( k = 7n + 5 \) and in the remainder \( P_b \square P_5 \) there are twelve additional vertices in \( D' \). Thus \( \gamma_t(P_b \square P_k) \leq 12n + 12 = (12k + 24)/7 \).

We still need to show the lower bounds for \( x \in \{0, 2, 3, 5\} \). Let \( k = 7n + r \) for some integers \( n \geq 1 \) and \( r \in \{0, 2, 3, 5\} \). Notice that, considering the symmetry, \( P_b \square P_k \) can be partitioned into \( n - 1 \) consecutive blocks \( B_i \), \( i \in \{1, \ldots, n - 1\} \), isomorphic to \( P_b \square P_7 \) and one final block \( Y \) isomorphic to \( P_b \square P_7 + r \), with \( r \in \{0, 2, 3, 5\} \). Let \( D \) be a \( \gamma_t(P_b \square P_k) \)-set and for every block \( B_i \), let \( B'_i \) be the subset of \( B_i \) obtained from \( B_i \) by deleting its last \( P_b \)-layer. We denote by \( L_i \) this last \( P_b \)-layer of \( B_i \). We will show that there are at least twelve vertices in \( D \cap B_i \) to totally dominate each \( B'_i \).

Let \( i = 1 \). It is not hard to see that \( B'_1 \) can be totally dominated in \( B_1 \) by twelve vertices, only if all these twelve vertices lie in \( B'_2 \). Clearly, we need at least four vertices in the first two \( P_b \)-layers to totally dominate \( P_b^2 \). We have three possibilities in Figure 1 and there are two additional possibilities, where we have exactly four vertices of \( D \) in the first two \( P_b \)-layers. These two are:

\[ A_1 = \{(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)\}, \]
\[ A_2 = \{(2, 1), (2, 2), (5, 1), (5, 2), (1, 4), (3, 4), (4, 4), (6, 4), (1, 5), (6, 5), (3, 6), (4, 6)\}. \]

If we have more than four vertices of \( D \) in the first two \( P_b \)-layers, then this is even easier to see. Also, each of these sets does not totally dominate the whole set \( B_1 \) and
at least two vertices of \( B_1 \) are not totally dominated by them. Notice that we can exchange the last two vertices of \( A_2 \) for \{(2, 7), (3, 7), (4, 7), (5, 7)\}. Hence the whole set \( B_1 \) is totally dominated, but then we use fourteen vertices. We will denote such a set by \( A_2' \). If we set \( A_1' = A_1 \cup \{(2, 7), (5, 7)\} \), then we also get a total dominating set of \( B_1 \) with fourteen vertices. Now, if \( |B_1 \cap D| = 12 \), then \( L_1 \cap D = \emptyset \) and, if \( |B_1 \cap D| = 14 \), then \( L_1 \cap D \) contains two vertices in the case \( A_1' \subset D \) or four vertices when \( A_2' \subset D \). The remaining option \( |B_1 \cap D| > 14 \) leads to a contradiction with \( D \) being a \( \gamma_t(G) \)-set as can be seen later from the context.

Let now \( i = 2 \). If \( L_1 \cap D = \emptyset \), then we have the same arguments as for \( B_1 \) and we obtain at least twelve vertices in \( B_2 \cap D \). If \( |L_1 \cap D| = 2 \), then \( (2, 7), (5, 7) \in D \) and \( (2, 8) \) and \( (5, 8) \) are already totally dominated. To totally dominate other vertices of \( P_6^s \) we need additional four vertices from \( P_6^b \) or \( P_6^r \) in \( D \). These vertices have no influence on layers \( P_6^{11}, P_6^{12} \) and \( P_6^{13} \). Hence, we need six additional vertices to totally dominate these layers to finish \( B_2' \). With this we already have at least ten vertices in \( B_2 \cap D \), which gives twelve together with two vertices of \( |L_1 \cap D| \).

Notice that there is a possibility to totally dominate the whole set \( B_2 \), if we have (at least) two additional vertices in \( L_2 \cap D \). If \( |L_1 \cap D| = 4 \), then \( A_2' \subset L_2 \subset D \) and \( (2, 8), (3, 8), (4, 8), (5, 8) \) are dominated by them. In this case there is only one possibility to totally dominate \( B_2' \) with ten additional vertices and this happens when

\[
\{(1, 9), (6, 9), (1, 10), (3, 10), (4, 10), (6, 10), (2, 12), (5, 12), (2, 13), (5, 13)\} \subset D.
\]

For every other option we need more vertices in \( B_2' \cap D \). So, in this case we have fourteen vertices, together with four vertices of \( |L_1 \cap D| \), but from these four vertices of \( |L_1 \cap D| \), two of them must be counted for \( B_1 \). Thus, there are twelve vertices for \( B_1 \) and twelve vertices for \( B_2 \). Notice that there is a possibility to totally dominate the whole set \( B_2 \), if we have (at least) two additional vertices in \( L_2 \cap D \) (these are \( (2, 14) \) and \( (5, 14) \)).

We continue for \( i \in \{3, \ldots, n - 1\} \) and, by using the same procedure, for every \( B_i \) we can find twelve vertices, which totally dominate \( B_i \) and no vertex is counted twice. Hence, \( |D| \geq 12(n - 1) \) and we still need to check \( Y \). If \( L_{n-1} \cap D = \emptyset \), then we are immediately done for \( Y \), since we have

\[
\gamma_t(P_6 \sqcup P_8) = 14, \gamma_t(P_6 \sqcup P_9) = 18, \gamma_t(P_6 \sqcup P_{10}) = 20 \text{ and } \gamma_t(P_6 \sqcup P_{12}) = 24,
\]

for \( r = 0, r = 2, r = 3 \) and \( r = 5 \), respectively. Altogether, we have

\[
|D| \geq 12(n - 1) + 14 = 12n + 2 = \frac{12k + 14}{7},
\]

for \( r = 0 \) and \( k = 7n \). Similarly, we get other values. Now, if

\[
L_{n-1} \cap D = \{(2, 7(n-1)), (5, 7(n-1))\},
\]

then these two vertices have not been counted yet. Also, if \( |L_{n-1} \cap D| = 4 \), then as above, two of these vertices are counted for \( B_{n-1} \) and the other two vertices are still not counted. Hence, in these two cases, by applying the same argument as before, for this situation we easily get the desired values:

\[
|Y \cap D| \geq 12, |Y \cap D| \geq 16, |Y \cap D| \geq 18 \text{ and } |Y \cap D| \geq 22,
\]
for $r = 0, r = 2, r = 3$ and $r = 5$, respectively. By adding the two vertices from $L_{n-1} \cap D$ we get the final solution.

It is straightforward to observe that for $k \equiv x \pmod{7}$, $x \in \{0, 5, 6\}$, the result of Theorem 1 gives smaller values than Equation 1.

References