THE PROBLEM OF DIOPHANTUS FOR INTEGERS OF
\(\mathbb{Q}(\sqrt{-3})\)

Zrinka Franušić and Ivan Soldo

Abstract. We solve the problem of Diophantus for integers of the quadratic field \(\mathbb{Q}(\sqrt{-3})\) by finding a \(D(z)\)-quadruple in \(\mathbb{Z}[(1 + \sqrt{-3})/2]\) for each \(z\) that can be represented as a difference of two squares of integers in \(\mathbb{Q}(\sqrt{-3})\), up to finitely many possible exceptions.

1. Introduction and preliminaries

Let \(R\) be a commutative ring with unity 1 and \(n \in R\). The set of nonzero and distinct elements \(\{a_1, a_2, a_3, a_4\}\) in \(R\) such that \(a_i a_j + n\) is a perfect square in \(R\) for \(1 \leq i < j \leq 4\) is called a Diophantine quadruple with the property \(D(n)\) in \(R\) or just a \(D(n)\)-quadruple. If \(n = 1\) then a quadruple with a given property is called a Diophantine quadruple. The problem of constructing such sets was first studied by Diophantus of Alexandria who found the rational quadruple \(\{1, 33/16, 17/4, 105/16\}\) with the property \(D(1)\). Fermat found the first Diophantine quadruple in the ring of integers \(\mathbb{Z}\) - the set \(\{1, 3, 8, 120\}\).

The problem on the existence of \(D(n)\)-quadruples has been studied in different rings, but mainly in rings of integers of numbers fields. The following assertion is shown to be true in many cases: There exists a \(D(n)\)-quadruple if and only if \(n\) can be represented as a difference of two squares, up to finitely many exceptions. In the ring \(\mathbb{Z}\) one part of the assertion is proved independently by several authors (Brown, Gupta, Singh, Mohanty, Ramansamy, see [6, 25, 27]), and another by Dujella in [7]. The set of possible exceptions \(S = \{-4, -3, -1, 3, 5, 8, 12, 20\}\) is still an open problem studied by many authors. The conjecture is that for \(n \in S\) there does not exist a Diophantine quadruple with the property \(D(n)\).

In the ring of integers \(\mathbb{Z}\) well studied is the case of \(n = -1\). There is a conjecture that \(D(-1)\)-quadruple does not exist in \(\mathbb{Z}\). That is known as the \(D(-1)\)-quadruple conjecture and it was presented explicitly in [11] for the first time. While it is conjectured that \(D(-1)\)-quadruples do not exist in integers

2010 Mathematics Subject Classification. 11D09, 11R11.

Key words and phrases. Diophantine quadruples, quadratic field.
characterization of integers that can be represented as a difference of two squares of integers in

to be more precise, assuming the solvability of certain Pellian equation (and only if

that a similar statement is true for rings of integers of some real quadratic elements in

a \(\mathbb{Z}\) [9]. Namely, if \(a + bi\) is not representable as a difference of the squares of two elements in \(\mathbb{Z}[i]\), and in contrary if \(a + bi\) is not of such form and \(a + bi \not\in \{\pm 2, \pm 1 \pm 2i, \pm 4i\}\), then \(D(a + bi)\)-quadruple exists. Franušić in \([20–22]\) found that a similar statement is true for rings of integers of some real quadratic fields, i.e. it can be seen that there exist infinitely many \(D(n)\)-quadruples if and only if \(n\) can be represented as a difference of two squares of integers. To be more precise, assuming the solvability of certain Pellian equation \((x^2 - dy^2 = \pm 2 \text{ or } x^2 - dy^2 = 4 \text{ in odd numbers})\) we are able to obtain an effective characterization of integers that can be represented as a difference of two squares of integers in \(\mathbb{Q}(\sqrt{d})\) and then apply some polynomial formulas for Diophantine quadruples in a combination with elements of a small norm. Also, in \([23]\) the existence problem in the ring of integers of the pure cubic field \(\mathbb{Q}(\sqrt{2})\) has been completely solved.

The case of complex quadratic fields is more demanding because the set of elements with a small norm is poor (while in the real case there exist infinitely many units). A group of authors \([1, 16, 28]\) worked on the problem of the existence of \(D(z)\)-quadruples in \(\mathbb{Z}[\sqrt{-2}]\) and contributed that the problem is almost completely solved. As a prominent case, there appear the case \(z = -1\), which could not be solved by the standard method via polynomial formulas. In \([29]\) and \([30]\) Soldo studied \(D(-1)\)-triples of the form \(\{1, b, c\}\) and the existence of \(D(-1)\)-quadruples of the form \(\{1, b, c, d\}\) in the ring \(\mathbb{Z}[\sqrt{-t}]\), \(t > 0\), for \(b = 2, 5, 10, 17, 26, 37\) or \(50\). He proved a more general result i.e. if positive integer \(b\) is a prime, twice prime or twice prime squared such that \(\{1, b, c\}\) is a \(D(-1)\)-triple in the ring \(\mathbb{Z}[\sqrt{-t}]\), \(t > 0\), then \(c\) has to be an integer. As a consequence of this result, he showed that for \(t \not\in \{1, 4, 9, 16, 25, 36, 49\}\) there does not exist a subset of \(\mathbb{Z}[\sqrt{-t}]\) of the form \(\{1, b, c, d\}\) with the property that the product of any two of its distinct elements diminished by 1 is a square of an element in \(\mathbb{Z}[\sqrt{-t}]\). For those exceptional cases of \(t\) he showed that there exist infinitely many \(D(-1)\)-quadruples of the form \(\{1, b, -c, d\}, c, d > 0\) in \(\mathbb{Z}[\sqrt{-t}]\).
In this paper, we verify assertion on the existence of \( D(z) \)-quadruples in complex quadratic field \( \mathbb{Q}(\sqrt{-3}) \), i.e. in the corresponding ring of integers \( \mathbb{Z}[(1 + \sqrt{-3})/2] \). In other words, we show the following theorems.

**Theorem 1.1.** There exists a \( D(z) \)-quadruple in the ring of integers of the quadratic field \( \mathbb{Q}(\sqrt{-3}) \) if and only if \( z \) can be represented as a difference of two squares of integers in \( \mathbb{Q}(\sqrt{-3}) \), up to possible exceptions \( z \in \{ -1, 3, \frac{1}{2} - \frac{1}{2}\sqrt{-3}, \frac{1}{2} + \frac{1}{2}\sqrt{-3} \} \).

**Theorem 1.2.** There exists a \( D(z) \)-quadruple in the ring \( \mathbb{Z}[\sqrt{-3}] \) if and only if \( z \) can be represented as a difference of two squares of elements in \( \mathbb{Z}[\sqrt{-3}] \), up to possible exceptions \( z \in \{ -4, -1, 3, 2 - 2\sqrt{-3}, 2 + 2\sqrt{-3} \} \).

Although we have mentioned that the case of complex quadratic fields is rather complicated, observe that the Pellian equation \( x^2 - dy^2 = 4 \) is solvable for \( d = -3 \) in \( \mathbb{Z} \) (the only solution is \( 1 + \sqrt{-3} \)). To begin with, we will list briefly all statements that we require for the proofs of Theorem 1.1 and Theorem 1.2.

**Lemma 1.3 ([8, Theorem 1]).** Let \( R \) be a commutative ring with the unity \( 1 \) and \( m, k \in R \). The set
\[
\{ m, m^2(3k + 1)^2 + 2k, m^2(3k + 2)^2 + 2k + 2, 9m(2k + 1)^2 + 8k + 4 \}
\]
has the \( D(2m(2k + 1) + 1) \)-property.

The set (1.1) is a \( D(2m(2k + 1) + 1) \)-quadruple if it contains no equal elements or elements equal to zero.

**Lemma 1.4.** If \( u \) is an element of a commutative ring \( R \) with the unity \( 1 \) and \( \{ w_1, w_2, w_3, w_4 \} \) is a \( D(w) \)-quadruple in \( R \), then \( \{ w_1u, w_2u, w_3u, w_4u \} \) is a \( D(wu^2) \)-quadruple in \( R \).

**Lemma 1.5 ([14, Theorem 1]).** An integer \( z \in \mathbb{Q}(\sqrt{-3}) \) can be represented as a difference of two squares of elements in \( \mathbb{Z}[\sqrt{-3}] \) if and only if is one of the following forms
\[
2m + 1 + 2n\sqrt{-3}, \ 4m + 4n\sqrt{-3}, \ 4m + 2 + (4n + 2)\sqrt{-3}, \ m, n \in \mathbb{Z}.
\]

**Lemma 1.6 ([14, Theorem 2]).** An integer \( z \in \mathbb{Q}(\sqrt{-3}) \) can be represented as a difference of two squares of elements in \( \mathbb{Z}[(1 + \sqrt{-3})/2] \) if and only if is one of the following forms
\[
2m + 1 + 2n\sqrt{-3}, \ 2m + (2n + 1)\sqrt{-3}, \ 4m + 4n\sqrt{-3}, \ 4m + 2 + (4n + 2)\sqrt{-3},
\]
\[
\frac{2m + 1}{2} + \frac{2n + 1}{2}\sqrt{-3}, \ m, n \in \mathbb{Z}.
\]
Lemma 1.7 ([22, Lemma 5]). For each \(M, N \in \mathbb{Z}\), there exist \(k \in \mathbb{Z}\) such that

1. \(2M + 1 + 2N\sqrt{-3} = 2m(k + 1) + 1\), where \(m = \frac{1}{2} + \frac{1}{2}\sqrt{-3}\),
2. \(4M + 3 + (4N + 2)\sqrt{-3} = 2m(2k + 1) + 1\), where \(m = 1 + \sqrt{-3}\),
3. \(2M + (2N + 1)\sqrt{-3} = m(2k + 1) + 1\), where \(m = 1 + \sqrt{-3}\),
4. \(2M + 1 + (2N + 1)\sqrt{-3} = m(2k + 1) + 1\), where \(m = \frac{1}{2} + \frac{1}{2}\sqrt{-3}\),
5. \(\frac{2M + 1}{2} + \frac{2N + 1}{2}\sqrt{-3} = \frac{m}{2}(2k + 1) + 1\), where \(m = 1 + \sqrt{-3}\).

By using Lemmas 1.3, 1.4 and 1.7, we effectively construct Diophantine quadruples for integers of the forms given in Lemmas 1.5 and 1.6. The following assertion gives the nonexistence of a \(D(z)\)-quadruple in \(\mathbb{Z}[(1 + \sqrt{-3})/2]\) if \(z\) cannot be represented as a difference of two squares in \(\mathbb{Z}[(1 + \sqrt{-3})/2]\), i.e. if and only if \(z\) is of the form \(4m + 2 + 4n\sqrt{-3}\), \(4m + (4n + 2)\sqrt{-3}\), \(2m + 1 + (2n + 1)\sqrt{-3}\).

Lemma 1.8 ([22, Theorem 2]). If \(z\) has one of the forms

\(4m + 2 + 4n\sqrt{-3}\), \(4m + (4n + 2)\sqrt{-3}\), \(2m + 1 + (2n + 1)\sqrt{-3}\),

where \(m, n \in \mathbb{Z}\), then a \(D(z)\)-quadruple in \(\mathbb{Z}[(1 + \sqrt{-3})/2]\) does not exist.

The nonexistence of a \(D(z)\)-quadruple in \(\mathbb{Z}[(\sqrt{-3})/2]\) if \(z\) cannot be represented as a difference of two squares in \(\mathbb{Z}[\sqrt{-3}]\) follows partially from Lemma 1.8 (if \(z = 4m + 2 + 4n\sqrt{-3}\) or \(z = 4m + (4n + 2)\sqrt{-3}\)) and from the following assertion.

Lemma 1.9. Let \(d \in \mathbb{Z}\) is not a perfect square. Then there is no \(D(m + (2n + 1)\sqrt{d})\)-quadruple in the ring \(\mathbb{Z}[\sqrt{d}]\).

Proof. The proof of Proposition 1 in [1] given for \(d = 2\) can be immediately rewritten for an arbitrary \(d\).

Let us denote the set

\[D_4 = \{mu, (m(3k+1)^2+2k)u, (m(3k+2)^2+2k+2)u, (9m(2k+1)^2+8k+4)u\}\].

According to Lemmas 1.3 and 1.4, \(D_4\) is \(D((2m(2k + 1) + 1)u^2)\)-quadruple if it contains no equal elements or elements equal to zero. This polynomial formula combining with specific values of \(m\) and \(u\) solves our problem, up to finitely many cases. Our results are listed in the tables of the following subsections.

2.1. \(D(2m + 1 + 2n\sqrt{-3})\)-quadruples.

In this subsection, for integers \(A\) and \(B\), we will separate the cases of \(z = 4A + 3 + (4B + 2)\sqrt{-3}\) and \(z = 4A + 1 + 4B\sqrt{-3}\) to corresponding subcases.
It is easy to check that for those exceptions of \( z \) in Table 1, the polynomial formula \( D(z) \) gives the set with two equal elements, or some element is equal to zero. Therefore, in those exceptions of \( z \) (and all further exceptions), we used the method for the first time described in [8] (but only for quadruples in \( \mathbb{Z} \), to construct \( D(z) \)-quadruples with all distinct elements, of the form \( \{u, v, r, u + v + 2r, u + 4v + 4r\} \), for some \( u, v, r \in \mathbb{Z}[\sqrt{-3}] \), or \( u, v, r \in \mathbb{Z}^{[1+\sqrt{-3}]} \), respectively. Except in cases of \( z = -1, 3 \), we found the following \( D(z) \)-quadruples in \( \mathbb{Z}[\sqrt{-3}] \):

- \( \{3 + \sqrt{-3}, 1 - \sqrt{-3}, -2, -5 - 3\sqrt{-3}\} \) is the \( D(3 + 2\sqrt{-3}) \)-quadruple,
- \( \{3 - \sqrt{-3}, 1 + \sqrt{-3}, -2, -5 + 3\sqrt{-3}\} \) is the \( D(3 - 2\sqrt{-3}) \)-quadruple,
- \( \{1 + 3\sqrt{-3}, -1 + \sqrt{-3}, 2, 1 - \sqrt{-3}\} \) is the \( D(-1 - 2\sqrt{-3}) \)-quadruple,
- \( \{1 - 3\sqrt{-3}, -1 - \sqrt{-3}, 2, 1 + \sqrt{-3}\} \) is the \( D(-1 + 2\sqrt{-3}) \)-quadruple,
- \( \{8, 1 + \sqrt{-3}, 1 - \sqrt{-3}, -4\} \) is the \( D(5) \)-quadruple.
- \( \{\sqrt{-3}, 3\sqrt{-3}, 8\sqrt{-3}, 120\sqrt{-3}\} \) is the \( D(-3) \)-quadruple,
- \( \{1, 3, 8, 120\} \) is the \( D(1) \)-quadruple,
- \( \{6, -2 + 2\sqrt{-3}, -2 + 2\sqrt{-3}, -14\} \) is the \( D(9) \)-quadruple,
- \( \{2 + 2\sqrt{-3}, 1 + \sqrt{-3}, 1 - \sqrt{-3}, 2 - 2\sqrt{-3}\} \) is the \( D(-7) \)-quadruple.

### 2.2. \( D(2m + (2n + 1)\sqrt{-3}) \)-quadruples.

<table>
<thead>
<tr>
<th>( z )</th>
<th>( k )</th>
<th>( m )</th>
<th>( n )</th>
<th>( D(z) ) in ( \mathbb{Z}[\sqrt{-3}] )</th>
<th>exceptions of ( z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 4A + 3 + 4B\sqrt{-3} )</td>
<td>( A + B\sqrt{-3} )</td>
<td>1</td>
<td>1</td>
<td>( 2 + \sqrt{-3} )</td>
<td>( -1, 3 )</td>
</tr>
<tr>
<td>( 4A + 1 + (4B + 2)\sqrt{-3} )</td>
<td>( \frac{12A + 32B + 5 + 12\sqrt{-3}}{12} )</td>
<td>1</td>
<td>1</td>
<td>( 2 + \sqrt{-3} )</td>
<td>( -1, 3 )</td>
</tr>
<tr>
<td>( 8A + 3 + (8B + 2)\sqrt{-3} )</td>
<td>( 1 + \sqrt{-3} )</td>
<td>1</td>
<td>1</td>
<td>( 2 + 2\sqrt{-3} )</td>
<td>( -1, 3 )</td>
</tr>
<tr>
<td>( 8A + 7 + (8B + 6)\sqrt{-3} )</td>
<td>( 1 + \sqrt{-3} )</td>
<td>1</td>
<td>1</td>
<td>( 2 + 2\sqrt{-3} )</td>
<td>( -1, 3 )</td>
</tr>
<tr>
<td>( 8A + 9 + (8B + 2)\sqrt{-3} )</td>
<td>( 1 + \sqrt{-3} )</td>
<td>1</td>
<td>1</td>
<td>( 2 + 2\sqrt{-3} )</td>
<td>( -1, 3 )</td>
</tr>
<tr>
<td>( 8A + 5 + (8B + 4)\sqrt{-3} )</td>
<td>( A + B\sqrt{-3} )</td>
<td>2</td>
<td>1</td>
<td>( 5, -3 )</td>
<td>( -1, 3 )</td>
</tr>
<tr>
<td>( 8A + 1 + (8B + 4)\sqrt{-3} )</td>
<td>( A + B\sqrt{-3} )</td>
<td>4</td>
<td>1</td>
<td>( 1, 9, -7 )</td>
<td>( -1, 3 )</td>
</tr>
<tr>
<td>( 8A + 1 + (8B + 4)\sqrt{-3} )</td>
<td>( A + B\sqrt{-3} )</td>
<td>4</td>
<td>1</td>
<td>( 1, 9, -7 )</td>
<td>( -1, 3 )</td>
</tr>
</tbody>
</table>

Table 2
For the exceptions noted in Table 2, we found the following $D(z)$-quadruples in $\mathbb{Z}[\frac{1 + \sqrt{-3}}{2}]$:
- $\{\frac{1}{2} - \frac{1}{2}\sqrt{-3}, -1, \frac{1}{2} - \frac{1}{2}\sqrt{-3}, -\frac{1}{2} + \frac{1}{2}\sqrt{-3}\}$ is the $D(-\sqrt{-3})$-quadruple,
- $\{\frac{1}{2} + \frac{1}{2}\sqrt{-3}, -\frac{1}{2} - \frac{1}{2}\sqrt{-3}, -2, -\frac{23}{2} + \frac{1}{2}\sqrt{-3}\}$ is the $D(2 + \sqrt{-3})$-quadruple,
- $\{\frac{1}{2} + \frac{3}{2}\sqrt{-3}, -1, \frac{1}{2} + \frac{1}{2}\sqrt{-3}, -\frac{3}{2} - \frac{1}{2}\sqrt{-3}\}$ is the $D(\sqrt{-3})$-quadruple,
- $\{\frac{1}{2} - \frac{1}{2}\sqrt{-3}, -\frac{1}{2} + \frac{1}{2}\sqrt{-3}, -2, -\frac{23}{2} - \frac{1}{2}\sqrt{-3}\}$ is the $D(2 - \sqrt{-3})$-quadruple.

2.3. $D(4m + 4n\sqrt{-3})$-quadruples.

<table>
<thead>
<tr>
<th>$z$</th>
<th>$k$</th>
<th>$m$</th>
<th>$u$</th>
<th>$D_4$ in $\mathbb{Z}$</th>
<th>exceptions of $z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$8A + 8B\sqrt{-3}$</td>
<td>$-A+3B-2$</td>
<td>$-A+B\sqrt{-3}$</td>
<td>$\pm$</td>
<td>$1+\sqrt{-3}$</td>
<td>$\mathbb{Z}[\frac{1 + \sqrt{-3}}{2}]$</td>
</tr>
<tr>
<td>$8A+4+(8B+4)\sqrt{-3}$</td>
<td>$-A+3B-1$</td>
<td>$-A+B+1\sqrt{-3}$</td>
<td>$\pm$</td>
<td>$1+\sqrt{-3}$</td>
<td>$\mathbb{Z}[\frac{1 + \sqrt{-3}}{2}]$</td>
</tr>
</tbody>
</table>

Table 3

The set $\{1, 2 - 2\sqrt{-3}, 5, 13 - 4\sqrt{-3}\}$ is the $D(-4 + 4\sqrt{-3})$-quadruple in $\mathbb{Z}[\sqrt{-3}]$ and it is easy to see that there exits infinitely many $D(0)$-quadruples.

We obtain a $D(8A + 4 + 8B\sqrt{-3})$-quadruple by multiplying elements of a $D(2m + 1 + 2n\sqrt{-3})$-quadruple by $u = 2$ except for $z = -4, 12$, but
\[
\{1, \frac{7}{2} + \frac{1}{2}\sqrt{-3}, \frac{7}{2}, \frac{1}{2}\sqrt{-3}, 13\}
\]
is the $D(-4)$-quadruple, and
\[
\{-2, 7 + \sqrt{-3}, 7 - \sqrt{-3}, 30\}
\]
is the $D(12)$-quadruple. Also, a $D(8A + (8B + 4)\sqrt{-3})$-quadruple is obtained by multiplying elements of a $D(2m + (2n + 1)\sqrt{-3})$-quadruple by $u = 2$. Obviously, the resulting sets are subsets of $\mathbb{Z}[\sqrt{-3}]$ (except for $z = -4$).

2.4. $D(4m + 2 + (4n + 2)\sqrt{-3})$-quadruples.

<table>
<thead>
<tr>
<th>$z$</th>
<th>$k$</th>
<th>$m$</th>
<th>$u$</th>
<th>$D_4$ in $\mathbb{Z}$</th>
<th>exceptions of $z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$8A + 2 + (8B + 2)\sqrt{-3}$</td>
<td>$-A - 1 - B\sqrt{-3}$</td>
<td>$\pm$</td>
<td>$1+\sqrt{-3}$</td>
<td>$\mathbb{Z}[\frac{1 + \sqrt{-3}}{2}]$</td>
<td>$-6+2\sqrt{-3}$, $2+2\sqrt{-3}$</td>
</tr>
<tr>
<td>$8A + 6 + (8B + 6)\sqrt{-3}$</td>
<td>$\frac{A+1}{2} + \frac{B+1}{2}\sqrt{-3}$</td>
<td>$\pm$</td>
<td>$1+\sqrt{-3}$</td>
<td>$\mathbb{Z}[\frac{1 + \sqrt{-3}}{2}]$</td>
<td>$-6-2\sqrt{-3}$, $2-2\sqrt{-3}$</td>
</tr>
<tr>
<td>$8A + 2 + (8B + 6)\sqrt{-3}$</td>
<td>$-(A+1) - (B+1)\sqrt{-3}$</td>
<td>$\pm$</td>
<td>$1 - \sqrt{-3}$</td>
<td>$\mathbb{Z}[\frac{1 + \sqrt{-3}}{2}]$</td>
<td>$-6-2\sqrt{-3}$, $2-2\sqrt{-3}$</td>
</tr>
<tr>
<td>$8A + 6 + (8B + 2)\sqrt{-3}$</td>
<td>$\frac{A+1}{2} + \frac{B+1}{2}\sqrt{-3}$</td>
<td>$\pm$</td>
<td>$1 - \sqrt{-3}$</td>
<td>$\mathbb{Z}[\frac{1 + \sqrt{-3}}{2}]$</td>
<td>$-6-2\sqrt{-3}$, $2-2\sqrt{-3}$</td>
</tr>
</tbody>
</table>

Table 4
While the polynomial formula $D_4$ gives sets with two equal elements, for those exceptions of $z$ of Table 4, we found the following $D(z)$-quadruples in $\mathbb{Z}[\frac{1}{2} + \sqrt{-3}]$:

- $\{-\frac{1}{2} + \frac{1}{2}\sqrt{-3}, -9 - 2\sqrt{-3}, -\frac{45}{2} - \frac{1}{2}\sqrt{-3}, -\frac{85}{2} - \frac{1}{2}\sqrt{-3}\}$ is the $D(-6 + 2\sqrt{-3})$-quadruple,
- $\{-\frac{1}{2} + \frac{1}{2}\sqrt{-3}, -\frac{5}{2} + \frac{3}{2}\sqrt{-3}, -1 + 2\sqrt{-3}, -\frac{13}{2} + \frac{13}{2}\sqrt{-3}\}$ is the $D(2 + 2\sqrt{-3})$-quadruple,
- $\{-\frac{1}{2} - \frac{1}{2}\sqrt{-3}, -9 + 2\sqrt{-3}, -\frac{45}{2} + \frac{1}{2}\sqrt{-3}, -\frac{85}{2} + \frac{1}{2}\sqrt{-3}\}$ is the $D(-6 - 2\sqrt{-3})$-quadruple,
- $\{-\frac{1}{2} - \frac{1}{2}\sqrt{-3}, -\frac{5}{2} - \frac{3}{2}\sqrt{-3}, -1 - 2\sqrt{-3}, -\frac{13}{2} - \frac{13}{2}\sqrt{-3}\}$ is the $D(2 - 2\sqrt{-3})$-quadruple.

2.5. $D(2m+1 + 2n\sqrt{-3})$-quadruples.

We derive $D(2m+1 + 2n\sqrt{-3})$-quadruples from $D(2m + 1 + 2n\sqrt{-3})$ and $D(2m + (2n+1)\sqrt{-3})$-quadruples by multiplying them by $\frac{1 + \sqrt{-3}}{2}$ and $\frac{1 - \sqrt{-3}}{2}$.

- Multiplying the elements of a $D(2m + 1 + 2n\sqrt{-3})$-quadruple by $u = \frac{1 + \sqrt{-3}}{2}$ we obtain a $D((2m + 1 + 2n\sqrt{-3})u^2)$-quadruple except for $z = \frac{1}{2} + \frac{1}{2}\sqrt{-3}, -\frac{5}{2} + \frac{3}{2}\sqrt{-3}$. The number $(2m + 1 + 2n\sqrt{-3})u^2$ is of the form $2A + \frac{1}{2} + 2B + \frac{1}{2}\sqrt{-3}$ and for given $A, B \in \mathbb{Z}$ the equation

$$2m + 1 + 2n\sqrt{-3})u^2 = \frac{2A + 1}{2} + \frac{2B + 1}{2}\sqrt{-3}$$

has an integer solution $(m, n \in \mathbb{Z})$ if and only if $-A + 3B \equiv 1 \pmod{4}$ and $A + B \equiv 3 \pmod{4}$, i.e. $(A, B) \pmod{4} \in \{(0, 3), (1, 2), (2, 1), (3, 0)\}$. Concerning exceptions, the set

$$\{\frac{1}{2} + \frac{1}{2}\sqrt{-3}, -\frac{5}{2} + \frac{3}{2}\sqrt{-3}, -1 + 2\sqrt{-3}, -\frac{15}{2} + \frac{15}{2}\sqrt{-3}\}$$

represents the $D(-\frac{1}{2} + \frac{1}{2}\sqrt{-3})$-quadruple, while we could not find the $D(\frac{1}{2} - \frac{1}{2}\sqrt{-3})$-quadruple.

- Multiplying the elements of a $D(2m + 1 + 2n\sqrt{-3})$-quadruple by $u = \frac{1 - \sqrt{-3}}{2}$ we obtain a $D((2m + 1 + 2n\sqrt{-3})u^2)$-quadruple except for $z = \frac{1}{2} + \frac{1}{2}\sqrt{-3}, -\frac{5}{2} + \frac{3}{2}\sqrt{-3}$. For given $A, B \in \mathbb{Z}$ the equation (2.1) has an integer solution if and only if $A + 3B \equiv 0 \pmod{4}$ and $A - B \equiv 0 \pmod{4}$, i.e. $(A, B) \pmod{4} \in \{(0, 0), (1, 1), (2, 2), (3, 3)\}$. The set

$$\{\frac{1}{2} - \frac{1}{2}\sqrt{-3}, -\frac{5}{2} + \frac{3}{2}\sqrt{-3}, -1 - 2\sqrt{-3}, -\frac{15}{2} + \frac{15}{2}\sqrt{-3}\}$$

is the $D(-\frac{1}{2} - \frac{1}{2}\sqrt{-3})$-quadruple and we have not detected a $D(\frac{1}{2} + \frac{1}{2}\sqrt{-3})$-quadruple.
• Multiplying the elements of a $D(2m + (2n + 1)\sqrt{-3})$-quadruple by $u = \frac{1+\sqrt{-3}}{2}$ we obtain a $D(\frac{2A+1}{2} + \frac{2B+1}{2}\sqrt{-3})$-quadruple. For given $A, B \in \mathbb{Z}$ the equation

$$ (2m + (2n + 1)\sqrt{-3})u^2 = \frac{2A+1}{2} + \frac{2B+1}{2}\sqrt{-3} $$

has an integer solution if and only if $A+3B \equiv 3 \pmod{4}$ and $A+B \equiv 1 \pmod{4}$, i.e. $(A, B) \pmod{4} \in \{(0, 1), (1, 0), (3, 2), (2, 3)\}$.

• Multiplying the elements of a $D(2m + (2n + 1)\sqrt{-3})$-quadruple by $u = \frac{1-\sqrt{-3}}{2}$ we obtain a $D(\frac{2A+1}{2} + \frac{2B+1}{2}\sqrt{-3})$-quadruple. For given $A, B \in \mathbb{Z}$ the equation (2.2) has an integer solution if and only if $A+3B \equiv 2 \pmod{4}$ and $A+B \equiv 2 \pmod{4}$, i.e. $(A, B) \pmod{4} \in \{(0, 2), (2, 0), (1, 3), (3, 1)\}$.

### 3. $D(z)$ quadruples in $\mathbb{Z}[\sqrt{-3}]$

In the previous section we see that some $D(z)$-quadruples that have been constructed already lie in $\mathbb{Z}[\sqrt{-3}]$ but some of them do not although $z$ can be represented as a difference of squares in $\mathbb{Z}[\sqrt{-3}]$. Here we show that this can be improved.

#### 3.1. $D(2m + 1 + 2n\sqrt{-3})$-quadruples.

<table>
<thead>
<tr>
<th>$z$</th>
<th>$k$</th>
<th>$n$</th>
<th>$u$</th>
<th>$D_k$ in $\mathbb{Z}[\sqrt{-3}]$</th>
<th>exceptions of $z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4A + 1 + (4B + 2)\sqrt{-3}$</td>
<td>$\frac{2A + 4m + 1}{2} + \frac{2B + 4n + 1}{2}\sqrt{-3}$</td>
<td>$\sqrt{-3}$</td>
<td>$\sqrt{-3}$</td>
<td>$\mathbb{Z}[\sqrt{-3}]$</td>
<td>$-3 - 2\sqrt{-3}$</td>
</tr>
</tbody>
</table>

Table 5

The set \{-\sqrt{-3}, -2 + \sqrt{-3}, -2, -8 + 3\sqrt{-3}\} is a $D(-3 - 2\sqrt{-3})$, while the set \{\sqrt{-3}, -2 - \sqrt{-3}, -2, -8 - 3\sqrt{-3}\} is a $D(-3 + 2\sqrt{-3})$-quadruple in $\mathbb{Z}[\sqrt{-3}]$.

#### 3.2. $D(4m + 2 + (4n + 2)\sqrt{-3})$-quadruples.

Since there exist a $D(\frac{2m+1}{2} + \frac{2n+1}{2}\sqrt{-3})$-quadruple in $\mathbb{Z}[(1 + \sqrt{-3})/2]$, by multiplying by 2 the elements of this quadruple we obtain a $D(4m + 2 + (4n + 2)\sqrt{-3})$-quadruple in $\mathbb{Z}[\sqrt{-3}]$, up to $z = 2 - 2\sqrt{-3}, 2 + 2\sqrt{-3}$.
3.3. $D(4m + 4n\sqrt{-3})$-quadruples.

We have shown in $\S$ 2.3. that $D(8m + (8n + 4)\sqrt{-3})$ and $D(8m + 4 + 8n\sqrt{-3})$-quadruples in $\mathbb{Z}[\sqrt{-3}]$ are obtained by multiplying by 2 the elements of $D(2m + (2n + 1)\sqrt{-3})$ and $D(2m + 2n\sqrt{-3})$-quadruples in $\mathbb{Z}[(1 + \sqrt{-3})/2]$ up to the $D(-4)$-quadruple whose elements are not in $\mathbb{Z}[\sqrt{-3}]$.

The set
\[ \{1, 9k^2 - 8k, 9k^2 - 2k + 1, 36k^2 - 20k + 1\} \]
is $D(8k)$-quadruple ([7]) if $k \neq 0, 1$, so there exists a $D(8m + 8n\sqrt{-3})$-quadruple in $\mathbb{Z}[\sqrt{-3}]$.

It is easy to check that for those exceptions of $z$ in Table 6, the polynomial formula $D_4$ gives the set with two equal elements. Therefore for certain $z$, we found the following $D(z)$-quadruples in $\mathbb{Z}[\sqrt{-3}]$:

- $\{2 + \sqrt{-3}, 2 - 2\sqrt{-3}, 2 - 3\sqrt{-3}, 6 - 11\sqrt{-3}\}$ is the $D(-12 + 4\sqrt{-3})$-quadruple,
- $\{2 - \sqrt{-3}, 2 + 2\sqrt{-3}, 2 + 3\sqrt{-3}, 6 + 11\sqrt{-3}\}$ is the $D(-12 + 4\sqrt{-3})$-quadruple,
- $\{2 + \sqrt{-3}, -2 + 2\sqrt{-3}, -2 + \sqrt{-3}, -10 + 5\sqrt{-3}\}$ is the $D(8)$-quadruple.

**Remark 3.1.** Concerning the list of possible exceptions given in Theorem 1.1 and Theorem 1.2, we can easily observe that $3 = -1 \cdot (\sqrt{-3})^2$, $-4 = -1 \cdot 2^2$, $\frac{1}{2} \pm \frac{1}{2}\sqrt{-3} = -1 \cdot (\frac{1}{2} \pm \frac{1}{2}\sqrt{-3})^2$ and $2 \pm 2\sqrt{-3} = -1 \cdot (1 \pm \sqrt{-3})^2$. So, we are not surprised that the key point lies in an investigation on the existence of $D(-1)$-quadruples in rings $\mathbb{Z}[(1 + \sqrt{-3})/2]$ and $\mathbb{Z}[\sqrt{-3}]$. In an analogy to $D(-1)$-quadruple conjecture in the ring of integers and the problem of existence of $D(-1)$-quadruples in $\mathbb{Z}[\sqrt{-t}]$, $t > 0$ studied in [29] and [30], we might expect that for such $z$ there does not exist a $D(z)$-quadruple.

**Acknowledgements.**
This work has been supported by Croatian Science Foundation under the project no. 6422.

**References**
[26] B. He and A. Togbé, On the \( D(-1) \)-triple \( \{1, k^2 + 1, k^2 + 2k + 2\} \) and its unique \( D(-1) \)-extension, J. Number Theory 131 (2011), 120–137.
[27] S. P. Mohanty and A. M. S. Ramasamy, On \( P_{r,k} \) sequences, Fibonacci Quart. 23 (1985), 36–44.
Diophantov problem za cijele brojeve kvadratnog polja $\mathbb{Q}(\sqrt{-3})$

Zrinka Franušić i Ivan Soldo

Sažetak. Rješavamo Diofantov problem za cijele brojeve kvadratnog polja $\mathbb{Q}(\sqrt{-3})$ konstruiranjem $D(z)$-četvorki u prstenu $\mathbb{Z}[\sqrt{-3}]$ za svaki $z$ koji se može prikazati kao razlika dva kvadrata u $\mathbb{Q}(\sqrt{-3})$, do na konačno mnogo mogućih izuzetaka.

Zrinka Franušić
Department of Mathematics
University of Zagreb
Bijenička cesta 30, HR-10 000 Zagreb
Croatia
E-mail: fran@math.hr

Ivan Soldo
Department of Mathematics
University of Osijek
Trg Ljudevita Gaja 6, HR-31 000 Osijek
Croatia
E-mail: isoldo@mathos.hr

Received: 31.3.2014.