Classification of complete left-invariant affine structures on the oscillator group

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Abstract. The goal of this paper is to provide a method, based on the theory of extensions of left-symmetric algebras, for classifying left-invariant affine structures on a given solvable Lie group of low dimension. To illustrate our method better, we shall apply it to classify all complete left-invariant affine structures on the oscillator group.

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1. Introduction

It is a well known result (see [1, 19]) that a simply connected Lie group $G$ which admits a complete left-invariant affine structure, or equivalently $G$ acts simply transitively by affine transformations on $\mathbb{R}^n$, must be solvable. It is also well known that not every solvable (even nilpotent) Lie group can admit an affine structure [3].

The goal of this paper is to provide a method for classifying all complete left-invariant affine structures on a given solvable Lie group of low dimension. Since the classification has been completely achieved up to dimension four in the nilpotent case (see [10, 14, 17]), we will illustrate our method by applying it to the remarkable solvable non-nilpotent 4-dimensional Lie group $O_4$ known as the oscillator group.

Since complete left-invariant affine structures on a Lie group $G$ are in one-to-one correspondence with complete (in the sense of [22]) left-symmetric structures on its Lie algebra $\mathcal{G}$ [14], we will carry out the classification in terms of complete left-symmetric structures on the oscillator algebra $O_4$.

The paper is organized as follows. In Section 2, we will recall the notion of extensions of Lie algebras and its relationship to the notion of $G$-kernels. In Section 3, we will give some necessary definitions and basic results on left-symmetric algebras and their extensions. In Section 4, given a complete left-symmetric algebra $A_4$ whose associated Lie algebra is $O_4$, we will use the complexification of $A_4$ and some results in [5] and [15] to show first that $A_4$ is not simple. Precisely, we will show that $A_4$ has a proper two-sided ideal whose associated Lie algebra is isomorphic to the

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Now we define a linear map $\phi$ if the notion of extension is known (see, for instance, books [8] and [13]). In light of [21], we will briefly describe the notion of extensions of a Lie algebra $G$ by an abelian Lie algebra $A$ is well known (see, for instance, books [8] and [13]). In light of [21], we will briefly describe the notion of extension $\tilde{G}$ of a Lie algebra $G$ by a Lie algebra $A$ which is not necessarily abelian.

Suppose that a vector space extension $\tilde{G}$ of a Lie algebra $G$ by another Lie algebra $A$ is known, and we want to define a Lie structure on $\tilde{G}$ in terms of the Lie structures of $G$ and $A$. Let $\sigma : G \rightarrow \tilde{G}$ be a section, that is, a linear map such that $\pi \circ \sigma = id$. Then the linear map $\Psi : (a, x) \mapsto i(a) + \sigma(x)$ from $A \oplus G$ onto $\tilde{G}$ is an isomorphism of vector spaces. For $(a, x)$ and $(b, y)$ in $A \oplus G$, a commutator on $\tilde{G}$ must satisfy

\begin{equation}
[i(a) + \sigma(x), i(b) + \sigma(y)] = i([a, b]) + [\sigma(x), i(b)]
+ [i(a), \sigma(y)] + [\sigma(x), \sigma(y)]
\end{equation}

Now we define a linear map $\phi : G \rightarrow \text{End}(A)$ by

\begin{equation}
\phi(x)a = [\sigma(x), i(a)]
\end{equation}

On the other hand, since $\pi([\sigma(x), \sigma(y)]) = \pi([x, y])$, it follows that there exists an alternating bilinear map $\omega : G \times G \rightarrow A$ such that $[\sigma(x), \sigma(y)] = \sigma[x, y] + \omega(x, y)$. 

2. Extensions of Lie algebras

The notion of extensions of a Lie algebra $G$ by an abelian Lie algebra $A$ is well known (see, for instance, books [8] and [13]). In light of [21], we will briefly describe here the notion of extension $\tilde{G}$ of a Lie algebra $G$ by a Lie algebra $A$ which is not necessarily abelian.
To sum up, by means of the isomorphism above, \( \tilde{G} \cong A \oplus G \) and its elements may be denoted by \((a, x)\) with \(a \in A\) and \(x\) is simply characterized by its coordinates in \(G\). The commutator defined by (1) is now given by

\[
[(a, x), (b, y)] = ([a, b] + \phi (x) b - \phi (y) a + \omega (x, y), [x, y]),
\]

for all \((a, x) \in \tilde{G} \cong A \oplus G\).

It is easy to see that this is actually a Lie bracket (i.e., it verifies the Jacobi identity) if and only if the following three conditions are satisfied

1. \(\phi (x) [b, c] = [\phi (x) b, c] + [b, \phi (x) c]\),
2. \([\phi (x), \phi (y)] = \phi ([x, y]) + ad_{\omega (x, y)}\),
3. \(\omega ([x, y], z) - \omega (x, [y, z]) + \omega (y, [x, z]) = \phi (x) \omega (y, z) + \phi (y) \omega (z, x) + \phi (z) \omega (x, y)\).

**Remark 1.** We see that condition (1) above is equivalent to say that \(\phi (x)\) is a derivation of \(A\). In other words, \(G\) is actually acting by derivations, that is, \(\phi : G \to \text{Der} (A)\). Condition (2) indicates clearly that if \(A\) is supposed to be abelian, then \(A\) becomes a \(G\)-module in a natural way, because in this case the linear map \(\phi : G \to \text{Der} (A)\) given by \(\phi (x) a = [\sigma (x), i (a)]\) is well defined. Condition (3) is equivalent to the fact that, if \(A\) is abelian, \(\omega\) is a 2-cocycle (i.e., \(\delta \phi \omega = 0\), where \(\delta \phi\) refers to the coboundary operator corresponding to the action \(\phi\)).

If now \(\sigma' : G \to \tilde{G}\) is another section, then \(\sigma' - \sigma = \tau\) for some linear map \(\tau : G \to A\), and it follows that the corresponding morphism and the 2-cocycle are \(\phi' = \phi + ad \circ \tau\) and \(\omega' = \omega + \delta \tau + \frac{1}{2} [\tau, \tau]\), respectively, where \(ad\) stands here and below (if there is no ambiguity) for the adjoint representation in \(A\), and where \([\tau, \tau]\) has the following meaning: Given two linear maps \(\alpha, \beta : G \to A\), we define

\[
[\alpha, \beta] (x, y) = [\alpha (x), \beta (y)] - [\alpha (y), \beta (x)].
\]

In particular, we have \(\frac{1}{2} [\tau, \tau] (x, y) = [\tau (x), \tau (y)]\). Note here that the Lie algebra \(A\) is not necessarily abelian. Therefore, \(\omega' - \omega\) is a 2-coboundary if and only if \([\tau (x), \tau (y)] = 0\) for all \(x, y \in G\). Equivalently, \(\omega' - \omega\) is a 2-coboundary if and only if \(\tau\) has its range in the center \(Z (A)\) of \(A\). In that case, we get \(\omega' - \omega = \delta \tau \in B^2 (A, Z (A))\), the group of 2-coboundaries for \(G\) with values in \(Z (A)\).

To overcome all these difficulties, we proceed as follows. Let \(C^2 (G, A)\) be the abelian group of all 2-cocycles, i.e., alternating bilinear mappings \(G \times G \to A\). For a given \(\phi : G \to \text{Der} (A)\), let \(T_\phi \in C^2 (G, A)\) be defined by

\[
T_\phi (x, y) = [\phi (x), \phi (y)] - \phi ([x, y]), \quad \text{for all } x, y \in G.
\]

If there exists some \(\omega \in C^2 (G, A)\) such that \(T_\phi = ad \circ \omega\) and \(\delta \omega = 0\), then the pair \((\phi, \omega)\) is called a factor system for \((G, A)\). Let \(Z^2 (G, A)\) be the set of all factor systems for \((G, A)\). It is well known that the equivalence classes of extensions of a Lie algebra \(G\) by a Lie algebra \(A\) are in one-to-one correspondence with the elements of the quotient space \(Z^2 (G, A) / C^1 (G, A)\), where \(C^1 (G, A)\) is the space of linear maps from \(G\) into \(A\) (see, for instance, [21], Theorem II.7). Note that if we assume that \(A\) is abelian, then we meet the well known result (see, for instance, [7]) stating...
that for a given action $\phi : G \to \text{End}(A)$, the equivalence classes of extensions of $G$ by $A$ are in one-to-one correspondence with the elements of the second cohomology group

$$H^2_\phi(G, A) = Z^2_\phi(G, A) / B^2_\phi(G, A).$$

In the present paper, we will be concerned with the special case where $A$ is non-abelian and $G$ is $\mathbb{R}$, and henceforth the cocycle $\omega$ is identically zero.

**Remark 2.** It is worth noticing that the construction above is closely related to the notion of $G$-kernels considered for Lie algebras firstly in [20].

### 3. Left-symmetric algebras

The notion of a left-symmetric algebra arises naturally in various areas of mathematics and physics. It originally appeared in the works of Vinberg [23] and Koszul [16] concerning convex homogeneous cones and bounded homogeneous domains, respectively. It also appears, for instance, in connection with Yang-Baxter equation and integrable hydrodynamic systems (cf. [4, 12, 18]). A left-symmetric algebra $(A, \cdot)$ is a finite-dimensional algebra $A$ in which the products, for all $x, y, z \in A$, satisfy the identity

$$(x \cdot y) \cdot z - x \cdot (y \cdot z) = (y \cdot x) \cdot z - y \cdot (x \cdot z) \quad (4)$$

It is clear that an associative algebra is a left-symmetric algebra. Actually, if $A$ is a left-symmetric algebra and $(x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z)$ is the associator of $x, y, z$, then we can see that (4) is equivalent to $(x, y, z) = (y, x, z)$. This means that the notion of a left-symmetric algebra is a natural generalization of the notion of an associative algebra. If $A$ is a left-symmetric algebra, then the commutator

$$[x, y] = x \cdot y - y \cdot x \quad (5)$$

defines the structure of a Lie algebra on $A$, called the associated Lie algebra. Conversely, if $G$ is a Lie algebra with a left-symmetric product $\cdot$ satisfying (5), then we say that the left-symmetric structure is compatible with the Lie structure on $G$.

On the other hand, let $G$ be a Lie group with a left-invariant flat affine connection $\nabla$, and define a product $\cdot$ on the Lie algebra $G$ of $G$ by

$$x \cdot y = \nabla_x y, \text{ for all } x, y \in G. \quad (6)$$

Then, conditions on the connection $\nabla$ for being flat and torsion-free are now equivalent to conditions (4) and (5), respectively. Conversely, suppose that $G$ is endowed with a left-symmetric product $\cdot$ which is compatible with the Lie bracket of $G$. In this case, in order to obtain a left-invariant flat affine structure on $G$, we can define an operator $\nabla$ on $G$ according to identity (6) and then extend it by left-translations to the whole Lie group $G$. To sum up, the left-invariant flat affine structures on $G$ are in one-to-one correspondence with the left-symmetric structures on $G$ compatible with the Lie structure.

Let now $A$ be a left-symmetric algebra, and let $L_x$ and $R_x$ be the left and right multiplications by the element $x$, that is, $L_x y = x \cdot y$ and $R_x y = y \cdot x$. We say that
A is complete if \( R_x \) is a nilpotent operator, for all \( x \in A \). It turns out that, for a given simply connected Lie group \( G \) with Lie algebra \( \mathcal{G} \), the complete left-invariant flat affine structures on \( G \) are in one-to-one correspondence with the complete left-symmetric structures on \( \mathcal{G} \) compatible with the Lie structure (see, for example, [14]). It is also known that an \( n \)-dimensional simply connected Lie group admits a complete left-invariant flat affine structure if and only if it acts simply transitively on \( \mathbb{R}^n \) by affine transformations (see [14]). A simply connected Lie group acting simply transitively on \( \mathbb{R}^n \) by affine transformations must be solvable according to [1], but it is worth noticing that there exist solvable (even nilpotent) Lie groups which do not admit affine structures (see [3]).

We close this section by fixing some notations which we will use in what follows. For a left-symmetric algebra \( A \), we can easily check that the subset

\[
T(A) = \{ x \in A : L_x = 0 \}
\]

(7)
is a two-sided ideal in \( A \). Geometrically, if \( G \) is a Lie group which acts simply transitively on \( \mathbb{R}^n \) by affine transformations, then \( T(G) \) corresponds to the set of translational elements in \( G \), where \( G \) is endowed with the complete left-symmetric product corresponding to the action of \( G \) on \( \mathbb{R}^n \). It has been conjectured in [1] that every nilpotent Lie group \( G \) which acts simply transitively on \( \mathbb{R}^n \) by affine transformations contains a translation which lies in the center of \( G \), but this conjecture turned out to be false (see [9]).

3.1. Extensions of left-symmetric algebras

In this section, we will briefly discuss the problem of an extension of a left-symmetric algebras. To our knowledge, this notion has been considered for the first time in [14]. Suppose we are given a vector space \( A \) as an extension of a left-symmetric algebra \( K \) by another left-symmetric algebra \( E \). We want to define a left-symmetric structure on \( A \) in terms of the left-symmetric structures given on \( K \) and \( E \). In other words, we want to define a left-symmetric product on \( A \) for which \( E \) becomes a two-sided ideal in \( A \) such that \( A/E \cong K \); or equivalently, \( 0 \to E \to A \to K \to 0 \) becomes a short exact sequence of left-symmetric algebras.

Theorem 1 (See [14]). There exists a left-symmetric structure on \( A \) extending a left-symmetric algebra \( K \) by a left-symmetric algebra \( E \) if and only if there exist two linear maps \( \lambda, \rho : K \to \text{End}(E) \) and a bilinear map \( g : K \times K \to E \) such that, for all \( x, y, z \in K \) and \( a, b \in E \), the following conditions are satisfied.

\[
\begin{align*}
(i) & \quad \lambda_x (a \cdot b) = \lambda_x (a) \cdot b + a \cdot \lambda_x (b) - \rho_x (a) \cdot b, \\
(ii) & \quad \rho_x ([a, b]) = a \cdot \rho_x (b) - b \cdot \rho_x (a), \\
(iii) & \quad [\lambda_x, \lambda_y] = \lambda_{[x,y]} + L_{g(x,y)-g(y,x)}, \text{ where } L_{g(x,y)-g(y,x)} \text{ denotes the left multiplication in } E \text{ by } g(x,y) - g(y,x), \\
(iv) & \quad [\lambda_x, \rho_y] = \rho_{x \cdot y} - \rho_y \circ \rho_x + R_{g(x,y)}, \text{ where } R_{g(x,y)} \text{ denotes the right multiplication in } E \text{ by } g(x,y),
\end{align*}
\]
(v) \( g(x, y \cdot z) - g(y, x \cdot z) + \lambda_x(g(y, z)) - \lambda_y(g(x, z)) - g([x, y], z) - \rho_z(g(x, y) - g(y, x)) = 0. \)

If the conditions of Theorem 1 are fulfilled, then the extended left-symmetric product on \( A \cong K \times E \) is given by

\[
(x, a) \cdot (y, b) = (x \cdot y, a \cdot b + \lambda_x(b) + \rho_y(a) + g(x, y)).
\]

(8)

It is remarkable that if the left-symmetric product of \( E \) is trivial, then the conditions of Theorem 1 simplify to the following three conditions:

(i) \( [\lambda_x, \lambda_y] = \lambda_{[x,y]}, \) i.e., \( \lambda \) is a representation of Lie algebras,

(ii) \( [\lambda_x, \rho_y] = \rho_{x \cdot y} - \rho_y \circ \rho_x, \)

(iii) \( g(x, y \cdot z) - g(y, x \cdot z) + \lambda_x(g(y, z)) - \lambda_y(g(x, z)) - g([x, y], z) - \rho_z(g(x, y) - g(y, x)) = 0. \)

In this case, \( E \) becomes a \( K \)-bimodule and the extended product given in (8) simplifies, too. Recall that if \( K \) is a left-symmetric algebra and \( V \) is a vector space, then we say that \( V \) is a \( K \)-bimodule if there exist two linear maps \( \lambda, \rho : K \to \text{End}(V) \) which satisfy conditions (i) and (ii) stated above.

Let \( K \) be a left-symmetric algebra, and let \( V \) be a \( K \)-bimodule. Let \( L^p(K, V) \) be the space of all \( p \)-linear maps from \( K \) to \( V \), and define two coboundary operators \( \delta_1 : L^1(K, V) \to L^2(K, V) \) and \( \delta_2 : L^2(K, V) \to L^3(K, V) \) as follows: For a linear map \( h \in L^1(K, V) \) we set

\[
\delta_1 h(x, y) = \rho_y(h(x)) + \lambda_x(h(y)) - h(x \cdot y),
\]

(9)

and for a bilinear map \( g \in L^2(K, V) \) we set

\[
\delta_2 g(x, y, z) = g(x, y \cdot z) - g(y, x \cdot z) + \lambda_x(g(y, z)) - \lambda_y(g(x, z)) - g([x, y], z) - \rho_z(g(x, y) - g(y, x)).
\]

(10)

It is straightforward to check that \( \delta_2 \circ \delta_1 = 0 \). Therefore, if we set \( Z^2_{\lambda, \rho}(K, V) = \ker \delta_2 \) and \( B^2_{\lambda, \rho}(K, V) = \text{Im} \delta_1 \), we can define a notion of second cohomology for the actions \( \lambda \) and \( \rho \) by simply setting \( H^2_{\lambda, \rho}(K, V) = Z^2_{\lambda, \rho}(K, V) / B^2_{\lambda, \rho}(K, V) \). As in the case of extensions of Lie algebras, we can prove that for given linear maps \( \lambda, \rho : K \to \text{End}(V) \), the equivalence classes of extensions \( 0 \to V \to A \to K \to 0 \) of \( K \) by \( V \) are in one-to-one correspondence with the elements of the second cohomology group \( H^2_{\lambda, \rho}(K, V) \). We close this subsection with the following lemma on completeness of left-symmetric algebras (see [6, Proposition 3.4]).

**Lemma 1.** Let \( 0 \to E \to A \to K \to 0 \) be a short exact sequence of left-symmetric algebras. Then, \( A \) is complete if and only if \( E \) and \( K \) are so.
3.2. Central extensions of left-symmetric algebras

The notion of central extensions known for Lie algebras may analogously be defined for left-symmetric algebras. Let $A$ be a left-symmetric extension of a left-symmetric algebra $K$ by another left-symmetric algebra $E$, and let $G$ be the Lie algebra associated to $A$. Define the center of $A$ to be $C(A) = T(A) \cap Z(G)$, that is,

$$C(A) = \{ x \in A : x \cdot y = y \cdot x = 0, \text{ for all } y \in A \}, \tag{11}$$

where $Z(G)$ is the center of the Lie algebra $G$ and $T(A)$ is the two-sided ideal of $A$ defined by (7).

**Definition 1.** The extension $0 \rightarrow E \xrightarrow{j} A \xrightarrow{\pi} K \rightarrow 0$ of left-symmetric algebras is said to be central (resp. exact) if $i(E) \subseteq C(A)$ (resp. $i(E) = C(A)$).

**Remark 3.** It is not difficult to show that if the extension $0 \rightarrow E \xrightarrow{j} A \xrightarrow{\pi} K \rightarrow 0$ is central, then both the left-symmetric product and the $K$-bimodule on $E$ are trivial (i.e., $a \cdot b = 0$ for all $a, b \in E$, and $\lambda = \rho = 0$). It is also easy to show that if $[g]$ is the cohomology class associated to this extension, and if

$$I_{[g]} = \{ x \in K : x \cdot y = y \cdot x = 0 \text{ and } g(x, y) = g(y, x) = 0, \text{ for all } y \in K \},$$

then the extension is exact if and only if $I_{[g]} = 0$ (see [14]). We note here that $I_{[g]}$ is well defined because any other element in $[g]$ takes the form $g + \delta_1 h$, with $\delta_1 h(x, y) = -h(x \cdot y)$.

Let now $K$ be a left-symmetric algebra, and $E$ a trivial $K$-bimodule. Denote by $(A, [g])$ the central extension $0 \rightarrow E \rightarrow A \rightarrow K \rightarrow 0$ corresponding to the cohomology class $[g] \in H^2(K, E)$. Let $(A, [g])$ and $(A', [g'])$ be two central extensions of $K$ by $E$, and let $\mu \in Aut(E) = GL(E)$ and $\eta \in Aut(K)$, where $Aut(E)$ and $Aut(K)$ are the groups of left-symmetric automorphisms of $E$ and $K$, respectively. It is clear that if $h \in L^1(K, E)$, then the linear mapping $\psi : A \rightarrow A'$ defined by $\psi(x, a) = (\eta(x), \mu(a) + h(x))$ is an isomorphism provided $g'(\eta(x), \eta(y)) = \mu(g(x, y)) - \delta_1 h(x, y)$ for all $x, y \in K \times K$, i.e. $\eta^* [g'] = \mu_* [g]$. This allows us to define an action of the group $G = Aut(E) \times Aut(K)$ on $H^2(K, E)$ by setting

$$(\mu, \eta) \cdot [g] = \mu_* \eta^* [g], \tag{12}$$

or equivalently, $(\mu, \eta) \cdot g(x, y) = \mu(g(\eta(x), \eta(y)))$ for all $x, y \in K$.

Denoting the set of all exact central extensions of $K$ by $E$ by

$$H^2_{ex}(K, E) = \{ [g] \in H^2(K, E) : I_{[g]} = 0 \},$$

and the orbit of $[g]$ by $G_{[g]}$, it turns out that the following result is valid (see [14]).

**Proposition 1.** Let $[g]$ and $[g']$ be two classes in $H^2_{ex}(K, E)$. Then, the central extensions $(A, [g])$ and $(A', [g'])$ are isomorphic if and only if $G_{[g]} = G_{[g']}$. In other words, the classification of the exact central extensions of $K$ by $E$ is, up to left-symmetric isomorphism, the orbit space of $H^2_{ex}(K, E)$ under the natural action of $G = Aut(E) \times Aut(K)$. 


3.3. Complexification of a real left-symmetric algebra

Let $A$ be a real left-symmetric algebra of dimension $n$, and let $A^C$ denote the real vector space $A \oplus A$. Let $J : A \oplus A \to A \oplus A$ be the linear map on $A \oplus A$ defined by $J(x,y) = (-y,x)$. For $\alpha + i\beta \in \mathbb{C}$ and $x,x',y,y' \in A$, we define

\[
(\alpha + i\beta)(x,y) = (\alpha x - \beta y, \alpha y + \beta x),
\]
\[
(x,y) : (x',y') = (xx' - yy', xy' + yx').
\]

We endow the set $A^C$ with the componentwise addition, multiplication by complex numbers defined by (13), and the product defined by (14). It is then straightforward to verify that $A^C$, when endowed with the product defined by (14), becomes a complex left-symmetric algebra called the complexification of $A$. The left-symmetric algebra $A$ can be identified with the set of elements in $A^C$ of the form $(x,0)$, where $x \in A$. If $e_1,\ldots,e_n$ is a basis of $A$, then the elements $(e_1,0),\ldots,(e_n,0)$ form a basis of the complex vector space $A^C$. It follows that $\dim_{\mathbb{C}}(A^C) = \dim_{\mathbb{R}}(A)$.

Since $A^C$ is a left-symmetric algebra, we know that the commutator $[(x,y),(x',y')] = (x,y) : (x',y') - (x',y') : (x,y)$ defines a Lie algebra $G^C$ on $A^C$. Computing this commutator, we get the following lemma.

**Lemma 2.** The complex Lie algebra $G^C$ associated to the complex left-symmetric algebra $A^C$ is isomorphic to the complexification of the Lie algebra $G$ associated to the left-symmetric algebra $A$.

Therefore, if $e_1,\ldots,e_n$ is a basis of $A$, then the elements $(e_1,0),\ldots,(e_n,0)$ form a basis of $G^C$, and the structural constants of $G^C$ are real since they coincide with the structural constants of $G$ in the basis $e_1,\ldots,e_n$.

4. Left-symmetric structures on the oscillator algebra

Recall that the Heisenberg group $H_3$ is the 3-dimensional Lie group diffeomorphic to $\mathbb{R} \times \mathbb{C}$ with the group law

\[
(v_1, z_1) : (v_2, z_2) = (v_1 + v_2 + \frac{1}{2} \text{Im}(\overline{v_1} z_2), z_1 + z_2),
\]

for all $v_1, v_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{C}$. Let $\lambda > 0$, and let $G = \mathbb{R} \rtimes H_3$ be equipped with the group law

\[
(t_1, v_1, z_1) : (t_2, v_2, z_2) = (t_1 + t_2, v_1 + v_2 + \frac{1}{2} \text{Im}(\overline{v_1} z_2 e^{i\lambda t_1}), z_1 + z_2 e^{i\lambda t_1}),
\]

for all $t_1, t_2 \in \mathbb{R}$ and $(v_1, z_1), (v_2, z_2) \in H_3$. This is a 4-dimensional Lie group with Lie algebra $\mathcal{G}$ having a basis $\{e_1, e_2, e_3, e_4\}$ such that

\[
[e_1, e_2] = e_3, \ [e_4, e_1] = \lambda e_2, \ [e_4, e_2] = -\lambda e_1,
\]

and all the other brackets are zero. It follows that the derived series is given by

\[
\mathcal{D}^1\mathcal{G} = [\mathcal{G}, \mathcal{G}] = \text{span}\{e_1, e_2, e_3\}, \ \mathcal{D}^2\mathcal{G} = \text{span}\{e_3\}, \ \mathcal{D}^3\mathcal{G} = \{0\}.
\]
and therefore \( \mathcal{G} \) is a (non-nilpotent) 3-step solvable Lie algebra. When \( \lambda = 1 \), \( G \) is known as the oscillator group. We will denote it by \( O_4 \), and we shall denote its Lie algebra by \( \mathcal{O}_4 \) and call it the oscillator algebra.

From now on, \( A_4 \) will be a complete real left-symmetric algebra whose associated Lie algebra is \( O_4 \). We begin by proving the following proposition which will be crucial to the classification of complete left-symmetric structures on \( O_4 \).

**Proposition 2.** \( A_4 \) is not simple (i.e., \( A_4 \) contains a proper two-sided ideal).

**Proof.** Assume to the contrary that \( A_4 \) is simple, and let \( \mathcal{A}_4^\mathbb{C} \) be its complexification. By [15], Lemma 2.10, it follows that \( \mathcal{A}_4^\mathbb{C} \) is either simple or a direct sum of two simple ideals having the same dimension. If \( \mathcal{A}_4^\mathbb{C} \) is simple, then we can apply Proposition 5.1 in [5] to deduce that, being simple and complete, \( \mathcal{A}_4^\mathbb{C} \) is necessarily isomorphic to the complex left-symmetric algebra \( L \) having a basis \( \{e_1, e_2, e_3, e_4\} \) such that

\[
\begin{align*}
\{e_1, e_2\} &= e_1, \\
\{e_2, e_3\} &= e_3, \\
\{e_3, e_4\} &= -2e_3,
\end{align*}
\]

and all other products are zero. It follows that the Lie algebra \( \mathcal{G}_4 \) associated to \( B_4 \) admits a basis \( \{e_1, e_2, e_3, e_4\} \) such that

\[
\begin{align*}
\{e_1, e_2\} &= e_1, \\
\{e_2, e_3\} &= e_2, \\
\{e_3, e_4\} &= -2e_3.
\end{align*}
\]

This leads to a contradiction since, according to Lemma 2, \( \mathcal{G}_4 \) should be isomorphic to the complexification of the Lie algebra \( \mathcal{O}_4 \), but this is obviously not the case. This contradiction shows that \( \mathcal{A}_4^\mathbb{C} \) cannot be simple.

If \( \mathcal{A}_4^\mathbb{C} \) is a direct sum of two simple ideals having the same dimension, say \( \mathcal{A}_4^\mathbb{C} = A_1 \oplus A_2 \), it follows that \( \dim A_1 = \dim A_2 = \frac{1}{2} \dim \mathcal{A}_4^\mathbb{C} = 2 \). In this case, by Corollary 4.1 in [5], \( A_1 \) and \( A_2 \) are both isomorphic to the unique two-dimensional complex simple left-symmetric algebra having a basis

\[
B_2 = \langle e_1, e_2 : e_1 \cdot e_1 = 2e_1, e_1 \cdot e_2 = e_2, e_2 \cdot e_2 = e_1 \rangle.
\]

This is a contradiction, since \( A_1 \) and \( A_2 \) are complete but \( B_2 \) is not. This contradiction shows that \( \mathcal{A}_4^\mathbb{C} \) cannot be direct sum of two simple ideals. We deduce that \( A_4 \) is not simple, and this completes the proof of the proposition. \( \Box \)

Before we return to the algebra \( A_4 \), we need to give the following lemmas.

**Lemma 3.** Let \( A \) be a left-symmetric algebra with Lie algebra \( \mathcal{G} \), and \( R \) a two-sided ideal in \( A \). Then, the Lie algebra \( \mathcal{R} \) associated to \( R \) is an ideal in \( \mathcal{G} \).

**Proof.** Let \( x \in \mathcal{R} \) and \( y \in \mathcal{G} \). Since \( R \) is a two-sided ideal, then \( x \cdot y \) and \( y \cdot x \) belong to \( R \). It follows that \( [x, y] = x \cdot y - y \cdot x \in R \), and therefore \( \mathcal{R} \) is an ideal in \( \mathcal{G} \). \( \Box \)

**Lemma 4.** The oscillator algebra \( \mathcal{O}_4 \) contains only two proper ideals which are \( Z(\mathcal{O}_4) \cong \mathbb{R} \) and \( [\mathcal{O}_4, \mathcal{O}_4] \cong \mathcal{H}_3 \).
Proof. It is clear that \( \mathcal{Z} (\mathcal{O}_4) \cong \mathbb{R} \) and \([\mathcal{O}_4, \mathcal{O}_4] \cong \mathcal{H}_3\) are proper ideals in \( \mathcal{O}_4\). If \( \mathcal{I} \) is a proper ideal in \( \mathcal{O}_4 \), then \( \mathcal{I} \) should be unimodular. If \( \dim (\mathcal{I}) = 1 \), then \( \mathcal{I} \) is isomorphic to \( \mathcal{Z} (\mathcal{O}_4) \cong \mathbb{R} \). If \( \dim (\mathcal{I}) = 2 \), then being unimodular, \( \mathcal{I} \) is isomorphic to \( \mathbb{R}^2 \). In particular, \( \mathcal{I} \) contains \( \mathcal{Z} (\mathcal{O}_4) \) and thus \( \mathcal{O}_4 / \mathcal{I} \) is abelian, a contradiction since \( \mathcal{O}_4 \) is not nilpotent. Hence, \( \mathcal{O}_4 \) contains no two-dimensional ideals. If \( \dim (\mathcal{I}) = 3 \), then being unimodular and solvable, \( \mathcal{I} \) is isomorphic to either \( \mathcal{H}_3 \), the Lie algebra \( \mathcal{E} (2) \) of the group of the rigid motions of the plane, or the Lie algebra \( \mathcal{E} (1, 1) \) of the group of the rigid motions of the Minkowski plane. However, it is straightforward to show that \( \mathcal{O}_4 \) cannot be obtained as an extension of \( \mathcal{E} (2) \) or \( \mathcal{E} (1, 1) \). We have therefore proved the lemma.

By the above proposition, \( \mathcal{A}_4 \) is not simple and hence it has a proper two-sided ideal \( I \), so we get a short exact sequence of complete left-symmetric algebras

\[
0 \rightarrow I \xrightarrow{i} \mathcal{A}_4 \xrightarrow{\pi} J \rightarrow 0.
\]

In fact, according to Lemma 1, the completeness of \( I \) and \( J \) comes from that of \( \mathcal{A}_4 \). If \( \mathcal{I} \) is the Lie subalgebra associated to \( I \) then, by Lemma 3, \( \mathcal{I} \) is an ideal in \( \mathcal{O}_4 \). From Lemma 4, it follows that there are two cases to consider according to whether \( \mathcal{I} \) is isomorphic to \( \mathcal{H}_3 \) or \( \mathbb{R} \). Next, we will focus on the case where \( \mathcal{I} \) is isomorphic to \( \mathcal{H}_3 \cong [\mathcal{O}_4, \mathcal{O}_4] \). In this case, the short exact sequence (15) becomes

\[
0 \rightarrow I_3 \xrightarrow{i} \mathcal{A}_4 \xrightarrow{\pi} I_0 \rightarrow 0,
\]

where \( I_3 \) is a complete 3-dimensional left-symmetric algebra whose Lie algebra is \( \mathcal{H}_3 \), and \( I_0 = \{e_0 : e_0 \cdot e_0 = 0\} \) the trivial one-dimensional real left-symmetric algebra. It is easy to prove the following proposition (cf. [10, Theorem 3.5]).

**Proposition 3.** Up to left-symmetric isomorphism, the complete left-symmetric structures on the Heisenberg algebra \( \mathcal{H}_3 \) are classified as follows: There is a basis \( \{e_1, e_2, e_3\} \) of \( \mathcal{H}_3 \) relative to which the left-symmetric product is given by one of the following classes:

(i) \( e_1 \cdot e_1 = pe_3 \), \( e_2 \cdot e_2 = qe_3 \), \( e_1 \cdot e_2 = \frac{1}{2}e_3 \), \( e_2 \cdot e_1 = -\frac{1}{2}e_3 \), where \( p, q \in \mathbb{R} \).

(ii) \( e_1 \cdot e_2 = me_3 \), \( e_2 \cdot e_1 = (m-1)e_3 \), \( e_2 \cdot e_2 = e_1 \), where \( m \in \mathbb{R} \).

**Remark 4.** It is noticeable that the left-symmetric products on \( \mathcal{H}_3 \) belonging to class (i) in Proposition 3 are obtained by central extensions (in the sense of fixed in Subsection 3.1) of \( \mathbb{R}^2 \) endowed with some complete left-symmetric structure by \( I_0 \). However, the left-symmetric products on \( \mathcal{A}_3 \) belonging to class (ii) are obtained by central extensions of the non-abelian two-dimensional Lie algebra \( \mathcal{G}_2 \) endowed with its unique complete left-symmetric structure by \( I_0 \).

Now we return to the short exact sequence (16). First, let \( \sigma : I_0 \rightarrow \mathcal{A}_4 \) be a section, and set \( \sigma (e_0) = x_0 \in \mathcal{A}_4 \). Define two linear maps \( \lambda, \rho \in \text{End} (\mathcal{A}_4) \) by putting \( \lambda (y) = x_0 \cdot y \) and \( \rho (y) = y \cdot x_0 \), and put \( e = x_0 \cdot x_0 \) (clearly \( e \in I_3 \)). Let \( g : I_0 \times I_0 \rightarrow I_4 \) be the bilinear map defined by \( g (e_0, e_0) = e \). It is obvious, using the notation of Subsection 3.1, to verify that \( \delta_2 g = 0 \), i.e. \( g \in Z^2_{\lambda, \rho} (I_0, I_3) \). The extended
left-symmetric product on $I_3 \oplus I_0$ given by (8) turns out to take the simplified form 
$(x, a\epsilon_0) \cdot (y, b\epsilon_0) = (x \cdot y + a\lambda(y) + b\rho(x) + ab\epsilon, 0)$, for all $x, y \in I_3$ and $a, b \in \mathbb{R}$.

The conditions in Theorem 1 can be simplified to the following conditions:

\begin{align}
\lambda(x \cdot y) &= \lambda(x) \cdot y + x \cdot \lambda(y) - \rho(x) \cdot y \\
\rho([x, y]) &= x \cdot \rho(y) - y \cdot \rho(x) \\
[\lambda, \rho] + \rho^2 &= R_e
\end{align}

Let $\phi : \mathbb{R} \to \text{End} (H_3)$ be the linear map defined by formula (2). As we mentioned in Remark 1, $\mathbb{R}$ acts on $H_3$ by derivations, that is, $\phi : \mathbb{R} \to \text{Der} (H_3)$. In particular, we deduce in view of (3) that $\lambda = D + \rho$ for some derivation $D$ of $H_3$. The derivations of $H_3$ are given by the following lemma, whose proof is straightforward and is therefore omitted.

**Lemma 5.** In a basis $\{e_1, e_2, e_3\}$ of $H_3$ satisfying $[e_1, e_2] = e_3$, a derivation $D$ of $H_3$ takes the form

$$D = \begin{pmatrix} a_1 b_1 & 0 \\ a_2 b_2 & 0 \\ a_3 b_3 a_1 + b_2 \end{pmatrix}.$$  

On the other hand, observe that $(x, a\epsilon_0) \in T(A_4)$ if and only if $(x, a\epsilon_0) \cdot (y, b\epsilon_0) = (0, 0)$ for all $(y, b\epsilon_0) \in I_3 \oplus I_0$, or equivalently, $x \cdot y + a\lambda(y) + b\rho(x) + ab\epsilon = 0$ for all $(y, b\epsilon_0) \in I_3 \oplus I_0$. Since $y$ and $b$ are arbitrary, we conclude that this is also equivalent to say that $(L_x)_{|a_3} = -a\lambda$ and $\rho(x) = -a\epsilon$. In particular, an element $x \in I_3$ belongs to $T(A_4)$ if and only if $(L_x)_{|a_3} = 0$ and $\rho(x) = 0$, or equivalently,

$$I_3 \cap T(A_4) = T(I_3) \cap \ker \rho.$$  

The following lemma will be crucial for the next section.

**Lemma 6.** The center $C(A_4) = T(A_4) \cap Z(O_4)$ is non-trivial.

**Proof.** In view of Proposition 3, we have to consider two cases.

**Case 1.** Assume that there is a basis $\{e_1, e_2, e_3\}$ of $H_3$ relative to which the left-symmetric product of $I_3$ is given by : $e_1 \cdot e_1 = p\epsilon_3, e_2 \cdot e_2 = q\epsilon_3, e_1 \cdot e_2 = \frac{1}{2} \epsilon_3$, $e_2 \cdot e_1 = -\frac{1}{2} \epsilon_3$, where $p, q \in \mathbb{R}$. Substituting $x = e_1$ and $y = e_2$ into (18), we find that the operator $\rho$ takes the form

$$\rho = \begin{pmatrix} \alpha_1 \beta_1 & 0 \\ \alpha_2 \beta_2 & 0 \\ \alpha_3 \beta_3 \gamma_3 \end{pmatrix},$$

with $\gamma_3 = p\beta_1 - q\alpha_2 + \frac{1}{2}(\alpha_1 + \beta_2)$. Since $\lambda = D + \rho$ for some $D \in H_3$, we use Lemma 5 to deduce that

$$\lambda = \begin{pmatrix} \alpha_1 + a_1 \beta_1 + b_1 & 0 \\ \alpha_2 + a_2 \beta_2 + b_2 & 0 \\ \alpha_3 + a_3 \beta_3 + b_1 \gamma_3 + a_1 + b_2 \end{pmatrix}.$$
Since \((L_{c_3})|_{I_3} = 0\) and \(e \in I_3\), then (19), when applied to \(c_3\), gives

\[\gamma_3^2 c_3 = c_3 \cdot e = 0,\]

from which we get \(\gamma_3 = 0\), i.e., \(\rho(c_3) = 0\). It follows from (20) that \(c_3 \in T(A_4)\).

Since \((\mathcal{O}_4) = \mathbb{R} c_3\), we deduce that \(C(A_4) = T(A_4) \cap Z(\mathcal{O}_4) \neq 0\), as required.

**Case 2.** Assume now that there is a basis \(\{e_1, e_2, c_3\}\) of \(H_3\) relative to which the left-symmetric product of \(I_3\) is given by : \(e_1 \cdot e_2 = me_3\), \(e_2 \cdot e_1 = (m - 1)e_3\), \(e_2 \cdot e_2 = e_1\), where \(m\) is a real number.

Substituting successively \(x = e_1\), \(y = e_2\) and \(x = e_2\), \(y = e_3\) into equation (18), we find that the operator \(\rho\) takes the form

\[\rho = \begin{pmatrix}
\alpha_1 & \beta_1 & -\alpha_2 \\
\alpha_2 & \beta_2 & 0 \\
\alpha_3 & \beta_3 & m\beta_2 - (m - 1)\alpha_1
\end{pmatrix},
\]

with \((m - 1)\alpha_2 = 0\).

We claim that \(\alpha_2 = 0\). To prove this, let us assume to the contrary that \(\alpha_2 \neq 0\).

It follows that \(m = 1\), and therefore

\[\rho(e_3) = -\alpha_2 e_1 + \beta_2 e_3\]
\[\rho^2(e_3) = -\alpha_2(\alpha_1 + \beta_2) e_1 - \alpha_2^2 e_2 + (\beta_2^2 - \alpha_2\beta_3) e_3\]

Since \(\alpha_2 \neq 0\), we deduce that \(e_3, \rho(e_3), \rho^2(e_3)\) form a basis of \(I_3\). Since \(\rho\) is nilpotent (by completeness of the left-symmetric structure), it follows that \(\rho^3(e_3) = 0\). In other words, \(\rho\) has the form

\[\rho = \begin{pmatrix}
0 & 0 & 1 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

with respect to the basis \(e'_1 = -\rho(e_3), e'_2 = \rho^2(e_3), e'_3 = -e_3\).

Using the fact that \(\alpha_1 + 2\beta_2 = 0\) which follows from the identity \(\rho^3(e_3) = 0\), we see that \(e'_1 \cdot e'_2 = \alpha_2^3 e'_3\), \(e'_2 \cdot e'_2 = \alpha_2^3 e'_1\), and all other products are zero.

For simplicity, assume without loss of generality that \(\alpha_2 = 1\). Since \(\lambda = D + \rho\) for some \(D \in H_3\), Lemma 5 tells us that, with respect to the basis \(e'_1, e'_2, e'_3\), the operator \(\lambda\) takes the form

\[\lambda = \begin{pmatrix}
a_1 & b_1 & 1 \\
a_2 - 1 & b_2 & 0 \\
a_3 & b_3 & a_1 + b_2
\end{pmatrix}.
\]

Applying formula (19) to \(e'_3\) and recalling that \(e'_3 \cdot e = 0\) since \(e \in I_3\), we deduce that \(a_2 = 1\) and \(b_2 = a_3 = 0\). Then, substituting \(x = y = e'_2\) into equation (17), we get \(a_1 = b_1 = 0\). Thus, the form of \(\lambda\) reduces to

\[\lambda = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & b_3 & 0
\end{pmatrix}.
\]
Now, by setting $e = ae_1 + be_2 + ce_3$ and applying (19) to $e_1$, we get that $b_3 = -b$. By using (8), we deduce that the nonzero left-symmetric products are
\[ e'_1 \cdot e'_2 = e'_3, \quad e'_2 \cdot e'_3 = e'_1, \]
\[ e'_1 \cdot e'_3 = -e'_2, \quad e'_1 \cdot e'_2 = -be'_3, \]
\[ e'_3 \cdot e'_4 = e'_4 \cdot e'_3 = e'_1, \quad e'_4 \cdot e'_2 = e. \]

This implies, in particular, that $\dim \{ O_4, O_4 \} = \dim \{ A_4, A_4 \} = 2$, a contradiction. It follows that $\alpha_2 = 0$, as desired.

We now return to (21). Since $\alpha_2 = 0$, we have
\[ \rho = \begin{pmatrix} \alpha_1 & \beta_1 & 0 \\ 0 & \beta_2 & 0 \\ \alpha_3 & \beta_3 & m_3 - (m - 1) \alpha_1 \end{pmatrix}, \]
and since $\lambda = D + \rho$ for some $D \in H_3$ then, in view of Lemma 5, the operator $\lambda$ takes the form
\[ \lambda = \begin{pmatrix} \alpha_1 + a_1 \beta_1 + b_1 & 0 \\ a_2 & \beta_2 + b_2 & 0 \\ \alpha_3 + a_3 \beta_3 + b_3 a_1 + b_2 + m_2 - (m - 1) \alpha_1 \end{pmatrix}. \]

Once again, by applying (19) to $e_3$ and recalling that $e_3 \cdot e = 0$ since $e \in I_3$, we deduce that $(m_2 - (m - 1) \alpha_1)^2 = 0$, thereby showing that $\rho(e_3) = 0$. Now, in view of (20) we get $e_3 \in T(A_3)$, and since $Z(O_4) = \mathbb{R}e_3$ we deduce that $C(A_4) = T(A_3) \cap Z(O_4) \neq 0$, as desired. This completes the proof of the lemma.

5. Classification

We know from Section 4 that $A_4$ has a proper two-sided ideal $I$ which is isomorphic to either the trivial one-dimensional real left-symmetric algebra $I_0 = \{ e_0 : e_0 \cdot e_0 = 0 \}$ or a 3-dimensional left-symmetric algebra $I_3$ (as described in Proposition 3) whose associated Lie algebra is the Heisenberg algebra $H_3$. In the case where $I \cong I_3$, we know by Lemma 6 that $C(A_4) \neq \{ 0 \}$. Since in our situation $\dim Z(O_4) = 1$, it follows that $C(A_4) \cong I_0$, so that we have a central short exact sequence of left-symmetric algebras of the form
\[ 0 \to I_0 \to A_4 \to I_3 \to 0. \] (22)

In general, one has that the center of a left-symmetric algebra is a part of the center of the associated Lie algebra, and therefore the following lemma is proved.

**Lemma 7.** The Lie algebra associated to $I_3$ is isomorphic to the Lie algebra $E(2)$ of the group of Euclidean motions of the plane.

Recall that $E(2)$ is solvable non-nilpotent and has a basis $\{ e_1, e_2, e_3 \}$ which satisfies $[e_1, e_2] = e_3$ and $[e_1, e_3] = -e_2$.

In the case where $I \cong I_0$, we know by Lemma 3 that the associated Lie algebra is $I \cong \mathbb{R}$. Since, by Lemma 4, $O_4$ has only two proper ideals which are $Z(O_4) \cong \mathbb{R}$ and
\[ [\mathcal{O}_4, \mathcal{O}_4] \cong \mathcal{H}_3, \] it follows that \( \mathcal{I} \cong \mathbb{R} \) coincides with the center \( Z(\mathcal{O}_4) \). We deduce from this that, if \( \mathcal{J} \) denotes the Lie algebra of the left-symmetric algebra \( J \) in the short exact sequence (15), then \( \mathcal{J} \) is isomorphic to \( \mathcal{E}(2) \). Therefore, we have a short sequence of left-symmetric algebras which looks like (22), except that it would not necessarily be central. But, as we will see a little later, this is necessarily a central extension (i.e., \( I \cong C(\mathcal{A}_4) \cong I_0 \)).

To summarize, each complete left-symmetric structure on \( \mathcal{O}_4 \) may be obtained by an extension of a complete 3-dimensional left-symmetric algebra \( \mathcal{A}_3 \) whose associated Lie algebra is \( \mathcal{E}(2) \) by \( I_0 \). Next, we shall determine all the complete left-symmetric structures on \( \mathcal{E}(2) \). These are described by the following lemma that we state without proof (see [10], Theorem 4.1).

**Lemma 8.** Up to left-symmetric isomorphism, any complete left-symmetric structure on \( \mathcal{E}(2) \) is isomorphic to the following one which is given in a basis \( \{e_1, e_2, e_3\} \) of \( \mathcal{E}(2) \) by the relations
\[
e_1 \cdot e_2 = e_3, \quad e_1 \cdot e_3 = -e_2, \quad e_2 \cdot e_2 = e_3 \cdot e_3 = \varepsilon e_1.
\]

There are exactly two non-isomorphic conjugacy classes according to whether \( \varepsilon = 0 \) or \( \varepsilon \neq 0 \).

From now on, \( \mathcal{A}_3 \) will denote the vector space \( \mathcal{E}(2) \) endowed with one of the complete left-symmetric structures described in Lemma 8. The extended Lie bracket on \( \mathcal{E}(2) \oplus \mathbb{R} \) is given by
\[
[[x, a], (y, b)] = ([x, y], \omega(x, y)),
\]
with \( \omega \in Z^2(\mathcal{E}(2), \mathbb{R}) \). The extended left-symmetric product on \( \mathcal{A}_3 \oplus I_0 \) is given by
\[
(x, ae_0) \cdot (y, be_0) = (x \cdot y, b \lambda_x(e_0) + a \rho_y(e_0) + g(x, y)),
\]
with \( \lambda, \rho : \mathcal{A}_3 \to \text{End}(I_0) \) and \( g \in Z^2_{\lambda, \rho}(\mathcal{A}_3, I_0) \).

As we have noticed in Section 3, \( I_0 \) is an \( \mathcal{A}_3 \)-bimodule, or equivalently, the conditions in Theorem 1 simplify to the following conditions:

(i) \( \lambda_{[x, y]} = 0 \),

(ii) \( \rho_{x \cdot y} = \rho_y \circ \rho_x \),

(iii) \( g(x, y \cdot z) - g(y, x \cdot z) + \lambda_x(g(y, z)) - \lambda_y(g(x, z)) - g([x, y], z) - \rho_z(g(x, y) - g(y, x)) = 0 \).

By using (23) and (24), we deduce from \( [[x, a], (y, b)] = (x, ae_0) \cdot (y, be_0) - (y, be_0) \cdot (x, ae_0) \) that
\[
\omega(x, y) = g(x, y) - g(y, x) \text{ and } \lambda = \rho.
\]
By applying identity (ii) above to \( e_i \cdot e_i, 1 \leq i \leq 3 \), we deduce that \( \rho = 0 \), and a fortiori \( \lambda = 0 \). In other words, the extension \( \mathcal{A}_4 \) is always central (i.e., \( I \cong C(\mathcal{A}_4) \)) even in the case where \( \mathcal{I} \cong \mathbb{R} \). In fact, we have

**Claim 1.** The extension \( 0 \to I_0 \to \mathcal{A}_4 \to \mathcal{A}_3 \to 0 \) is exact.
We obtain \( \lambda = e^\delta \). To deduce that \( x \in A \)
where \( A \)
we can now determine the extended left-symmetric structure on \( h \)
where \( I \)
we recall from Subsection 3.1 that the extension given by the short sequence
(22) is exact, i.e., \( \iota (I_0) = C (A_4) \), if and only if \( I_{[g]} = 0 \), where
\[
I_{[g]} = \{ x \in A_3 : x \cdot y = y \cdot x = 0 \text{ and } g (x , y ) = g (y , x ) = 0, \text{ for all } y \in A_3 \}.
\]
To show that \( I_{[g]} = 0 \), let \( x \) be an arbitrary element in \( I_{[g]} \), and put \( x = ae_1 + be_2 + ce_3 \in I_{[g]} \). Now, by computing all the products \( x \cdot e_i = e_i \cdot x = 0, 1 \leq i \leq 3 \), we easily deduce that \( x = 0 \).

Our aim is to classify complete left-symmetric structures on \( O_4 \), up to left-symmetric isomorphisms. By Proposition 1, the classification of exact central extensions of \( A_3 \) by \( I_0 \) is nothing but the orbit space of \( H^2_{\text{ex}} (A_3, I_0) \) under the natural action of \( G = Aut (I_0) \times Aut (A_3) \). Accordingly, we must compute \( H^2_{\text{ex}} (A_3, I_0) \).

Since \( I_0 \) is a trivial \( A_3 \)-bimodule, we see first from (9) and (10) that the coboundary operator \( \delta \) simplifies as follows:
\[
\delta_1 h (x, y ) = - h (x \cdot y ) , \quad \delta_2 g (x , y , z ) = g (x , y \cdot z ) - g (y , x \cdot z ) - g ([x , y ] , z ) ,
\]
where \( h \in L^1 (A_3, I_0) \) and \( g \in L^2 (A_3, I_0) \).

In view of Lemma 8, there are two cases to be considered.

**Case 1.** \( A_3 = \langle e_1, e_2, e_3 : e_1 \cdot e_2 = e_3, e_1 \cdot e_3 = - e_2 \rangle \).

In this case, using the first formula above for \( \delta_1 \), we get
\[
\delta_1 h = \begin{pmatrix}
0 & h_{12} & h_{13} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} ,
\]
where \( h_{12} = - h (e_3) \) and \( h_{13} = h (e_2) \). Similarly, using the second formula above for \( \delta_2 \), we verify easily that if \( g \) is a cocycle (i.e. \( \delta_2 g = 0 \)) and \( g_{ij} = g (e_i, e_j) \), then
\[
g = \begin{pmatrix}
g_{11} & g_{12} & g_{13} \\
g_{21} & g_{22} & g_{23} \\
g_{31} & g_{32} & g_{33}
\end{pmatrix} ,
\]
that is, \( g_{21} = g_{31} = 0, g_{32} = - g_{23} \), and \( g_{33} = g_{22} \). We deduce that, in the basis above, the class \([g] \in H^2 (A_3, \mathbb{R}) \) of a cocycle \( g \) takes the simplified form
\[
g = \begin{pmatrix}
\alpha & 0 & 0 \\
0 & \beta & \gamma \\
0 & - \gamma & \beta
\end{pmatrix} .
\]
We can now determine the extended left-symmetric structure on \( A_4 \). By setting \( \tilde{e}_i = (e_i, 0) \), \( 1 \leq i \leq 3 \), and \( \tilde{e}_4 = (0, 1) \), and using formula (24) which (since \( \lambda = \rho = 0 \)) reduces to
\[
(x, ae_0) \cdot (y, be_0) = (x \cdot y, g (x, y)) ,
\]
we obtain
\[
\begin{align*}
\tilde{e}_1 \cdot \tilde{e}_1 &= \alpha \tilde{e}_4 , & \tilde{e}_2 \cdot \tilde{e}_2 &= \tilde{e}_3 \cdot \tilde{e}_3 &= \beta \tilde{e}_4 \\
\tilde{e}_1 \cdot \tilde{e}_2 &= \tilde{e}_3 , & \tilde{e}_1 \cdot \tilde{e}_3 &= - \tilde{e}_2 , \\
\tilde{e}_2 \cdot \tilde{e}_3 &= \gamma \tilde{e}_4 , & \tilde{e}_3 \cdot \tilde{e}_2 &= - \gamma \tilde{e}_4 .
\end{align*}
\]
and all the other products are zero. We observe here that we should have \( \gamma \neq 0 \), given that the underlying Lie algebra is \( \mathcal{O}_4 \). We denote by \( A_4(\alpha, \beta, \gamma) \) the Lie algebra \( \mathcal{O}_4 \) endowed with the above complete left-symmetric product.

Let now \( A_4(\alpha, \beta, \gamma) \) and \( A_4(\alpha', \beta', \gamma') \) be two arbitrary left-symmetric structures on \( \mathcal{O}_4 \) given as above, and let \([g]\) and \([g']\) be the corresponding classes in \( H^2_{\text{ext}}(A_3, I_0) \). By Proposition 1, we know that \( A_4(\alpha, \beta, \gamma) \) is isomorphic to \( A_4(\alpha', \beta', \gamma') \) if and only if the exists \((\mu, \eta) \in Aut(I_0) \times Aut(A_3)\) such that for all \( x, y \in A_3 \), we have

\[
g'(x, y) = \mu(g(\eta(x), \eta(y))). \tag{28}
\]

We shall first determine \( Aut(I_0) \times Aut(A_3) \). We have \( Aut(I_0) \cong \mathbb{R}^* \), and it is easy too to determine \( Aut(A_3) \). Indeed, recall that the unique left-symmetric structure of \( A_3 \) is given by \( e_1 \cdot e_2 = e_3, \ e_1 \cdot e_3 = -e_2, \) and let \( \eta \in Aut(A_3) \) be given in the basis \( \{e_1, e_2, e_3\} \) by

\[
\eta = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & \gamma \\ 0 & -\gamma & \beta \end{pmatrix}.
\]

From the identity \( \eta(e_3) = \eta(e_1 \cdot e_2) = \eta(e_1) \cdot \eta(e_2) \), we get \( c_1 = 0, \ c_2 = -a_1b_3, \) and \( c_3 = a_1b_2 \). From the identity \( -\eta(e_2) = \eta(e_1 \cdot e_3) = \eta(e_1) \cdot \eta(e_3) \) we get \( b_1 = 0, \ b_2 = a_1c_3, \) and \( b_3 = -a_1e_2 \). Since \( \det \eta \neq 0 \), we deduce that \( a_1 = \pm 1 \). It follows, by setting \( \varepsilon = \pm 1 \), that \( b_3 = -\varepsilon c_2 \) and \( c_3 = \varepsilon b_2 \). From the identity \( \eta(e_1) \cdot \eta(e_1) = \eta(e_1 \cdot e_1) = 0 \), we obtain \( a_2 = a_3 = 0 \). Therefore, \( \eta \) takes the form

\[
\eta = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & b_2 & c_2 \\ 0 & -\varepsilon c_2 & \varepsilon b_2 \end{pmatrix}, \quad b_2^2 + c_2^2 \neq 0.
\]

We now apply formula (28). For this we recall first that in the basis above the classes \([g]\) and \([g']\) corresponding to \( A_4(\alpha, \beta, \gamma) \) and \( A_4(\alpha', \beta', \gamma') \), respectively, have the forms

\[
g = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & \gamma \\ 0 & -\gamma & \beta \end{pmatrix} \quad \text{and} \quad g' = \begin{pmatrix} \alpha' & 0 & 0 \\ 0 & \beta' & \gamma' \\ 0 & -\gamma' & \beta' \end{pmatrix},
\]

respectively. From \( g'(e_1, e_1) = \mu g(\eta(e_1), \eta(e_1)) \), we get

\[
\alpha' = \mu \alpha, \tag{29}
\]

and from \( g'(e_2, e_2) = \mu g(\eta(e_2), \eta(e_2)) \), we get

\[
\beta' = \mu \left( b_2^2 + c_2^2 \right) \beta. \tag{30}
\]

Similarly, from \( g'(e_2, e_3) = \mu g(\eta(e_2), \eta(e_3)) \) we get

\[
\gamma' = \mu \varepsilon \left( b_2^2 + c_2^2 \right) \gamma. \tag{31}
\]

Recall here that \( \mu \neq 0, \ \gamma \neq 0, \) and \( b_2^2 + c_2^2 \neq 0 \).

**Claim 2.** Each \( A_4(\alpha, \beta, \gamma) \) is isomorphic to some \( A_4(\alpha', \beta', 1) \). Precisely, \( A_4(\alpha, \beta, \gamma) \) is isomorphic to \( A_4\left(\varepsilon \frac{\alpha}{\gamma}, \varepsilon \frac{\beta}{\gamma}, 1\right) \).
Proof. By (29), (30), and (31), \( A_4 (\alpha, \beta, \gamma) \) is isomorphic to \( A_4 (\alpha', \beta', 1) \) if and only if there exists \( \mu \in \mathbb{R}^* \) and \( b, c \in \mathbb{R} \), with \( b^2 + c^2 \neq 0 \), such that
\[
\alpha' = \mu \alpha, \beta' = \mu (b^2 + c^2) \beta, 1 = \mu c (b^2 + c^2) \gamma.
\]

Now, by taking \( b^2 + c^2 = 1 \) (for instance, \( b = \cos \theta_0 \) and \( c = \sin \theta_0 \) for some \( \theta_0 \)), the third equation yields \( \mu = \frac{\delta}{\gamma} \). Substituting the value of \( \mu \) in the two first equations, we deduce that \( \alpha' = \frac{\alpha}{\gamma} \) and \( \beta' = \frac{\beta}{\gamma} \). Consequently, each \( A_4 (\alpha, \beta, \gamma) \) is isomorphic to \( A_4 \left( \frac{\alpha}{\gamma}, \frac{\beta}{\gamma}, 1 \right) \).

Case 2. \( A_3 = \langle e_1, e_2, e_3 : e_1 \cdot e_2 = e_3, e_1 \cdot e_3 = -e_2, e_2 \cdot e_2 = e_3 \cdot e_3 = e_1 \rangle \). Similarly to the first case, we get
\[
\delta_1 h = \begin{pmatrix}
0 & h_{12} & h_{13} \\
0 & h_{22} & 0 \\
0 & 0 & h_{22}
\end{pmatrix}
\text{ and } g = \begin{pmatrix}
0 & g_{12} & g_{13} \\
g_{22} & 0 & g_{23} \\
g_{22} & 0 & g_{22}
\end{pmatrix},
\]
where \( h_{12} = -h(e_3), h_{13} = h(e_2), h_{22} = -h(e_1), \) and \( g_{ij} = g(e_i, e_j) \). It follows that in this case the class \( [g] \in H^2 (A_3, \mathbb{R}) \) of a cocycle \( g \) takes the reduced form
\[
g = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \gamma \\
0 & -\gamma & 0
\end{pmatrix}, \quad \gamma \neq 0.
\]

By setting \( \tilde{e}_i = (e_i, 0), 1 \leq i \leq 3, \) and \( \tilde{e}_4 = (0, 1) \), and using formula (26) we find that the nonzero relations are
\[
\tilde{e}_1 \cdot \tilde{e}_2 = \tilde{e}_3, \quad \tilde{e}_1 \cdot \tilde{e}_3 = -\tilde{e}_2, \quad \tilde{e}_2 \cdot \tilde{e}_2 = \tilde{e}_3 \cdot \tilde{e}_3 = \tilde{e}_1
\]
\[
\tilde{e}_2 \cdot \tilde{e}_3 = \gamma \tilde{e}_4, \quad \tilde{e}_3 \cdot \tilde{e}_2 = -\gamma \tilde{e}_4, \quad \gamma \neq 0.
\]

We can now state the main result of this paper.

Theorem 2. Let \( A_4 \) be a complete non-simple real left-symmetric algebra whose associated Lie algebra is \( \mathcal{O} (4) \). Then \( A_4 \) is isomorphic to one of the following left-symmetric algebras:

(i) \( A_4 (s, t) \): There exist real numbers \( s, t, \) and a basis \( \{e_1, e_2, e_3, e_4\} \) of \( \mathcal{O} (4) \) relative to which the nonzero left-symmetric relations are
\[
e_1 \cdot e_1 = se_4, \quad e_2 \cdot e_2 = e_3 \cdot e_3 = te_4
\]
\[
e_1 \cdot e_2 = e_3, \quad e_1 \cdot e_3 = -e_2
\]
\[
e_2 \cdot e_3 = \frac{1}{2} e_4, \quad e_3 \cdot e_2 = -\frac{1}{2} e_4
\]

The conjugacy class of \( A_4 (s, t) \) is given as follows: \( A_4 (s', t') \) is isomorphic to \( A_4 (s, t) \) if and only if \( (s', t') = (\alpha s, \pm t) \) for some \( \alpha \in \mathbb{R}^* \).
(ii) $B_4$: There is a basis $\{e_1, e_2, e_3, e_4\}$ of $O(4)$ relative to which the nonzero left-symmetric relations are

\[
e_1 \cdot e_2 = e_3, \quad e_1 \cdot e_3 = -e_2, \quad e_2 \cdot e_2 = e_3 \cdot e_3 = e_1 \\
e_2 \cdot e_3 = \frac{1}{2}e_4, \quad e_3 \cdot e_2 = -\frac{1}{2}e_4.
\]

**Proof.** According to the discussion above, there are two cases to be considered.

**Case 1.** $A_3 = (e_1, e_2, e_3 : e_1 \cdot e_2 = e_3, \ e_1 \cdot e_3 = -e_2)$.

In this case, Claim 2 asserts that $A_4$ is isomorphic to some $A_4(\alpha, \beta, 1)$; and according to equations (27), we know that there is a basis $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$ of $O_4$ relative to which the nonzero relations for $A_4(\alpha, \beta, 1)$ are:

\[
\tilde{e}_1 \cdot \tilde{e}_1 = \alpha \tilde{e}_4, \quad \tilde{e}_2 \cdot \tilde{e}_2 = \tilde{e}_3 \cdot \tilde{e}_3 = \beta \tilde{e}_4 \\
\tilde{e}_1 \cdot \tilde{e}_2 = \tilde{e}_3, \quad \tilde{e}_1 \cdot \tilde{e}_3 = -\tilde{e}_2, \\
\tilde{e}_2 \cdot \tilde{e}_3 = \tilde{e}_4, \quad \tilde{e}_3 \cdot \tilde{e}_2 = -\tilde{e}_4.
\]

Now, it is clear that by setting $s = \frac{\alpha}{\beta}, \ t = \frac{\gamma}{\beta}, \ e_i = \tilde{e}_i$ for $1 \leq i \leq 3$, and $e_4 = 2\tilde{e}_4$, we get the desired two-parameter family $A_4(s, t)$. On the other hand, we see from (29), (30), and (31) that $A_4(s', t')$ is isomorphic to $A_4(s, t)$ if and only if there exists $\alpha \in \mathbb{R}^*$ and $b, c \in \mathbb{R}$, with $b^2 + c^2 \neq 0$, such that

\[
s' = \alpha s, \\
t' = \alpha (b^2 + c^2) t, \\
1 = \alpha e (b^2 + c^2).
\]

From the third equation, we get $b^2 + c^2 = \frac{1}{\alpha^2}$, and by substituting the latter into the second equation, we get $t' = \alpha t$. In other words, we have shown that $A_4(s', t')$ and $A_4(s, t)$ are isomorphic if and only if there exists $\alpha \in \mathbb{R}^*$ such that $s' = \alpha s$ and $t' = \alpha t$.

**Case 2.** $A_3 = (e_1, e_2, e_3 : e_1 \cdot e_2 = e_3, \ e_1 \cdot e_3 = -e_2, \ e_2 \cdot e_2 = e_3 \cdot e_3 = e_1)$.

In this case, by (32), there is a basis $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$ of $O_4$ relative to which the nonzero relations in $A_4$ are:

\[
\tilde{e}_1 \cdot \tilde{e}_2 = \tilde{e}_3, \quad \tilde{e}_1 \cdot \tilde{e}_3 = -\tilde{e}_2, \quad \tilde{e}_2 \cdot \tilde{e}_2 = \tilde{e}_3, \quad \tilde{e}_3 = \tilde{e}_1 \\
\tilde{e}_2 \cdot \tilde{e}_3 = \gamma \tilde{e}_4, \quad \tilde{e}_3 \cdot \tilde{e}_2 = -\gamma \tilde{e}_4, \gamma \neq 0.
\]

By setting $e_i = \tilde{e}_i$ for $1 \leq i \leq 3$, and $e_4 = 2\gamma \tilde{e}_4$, we see that $A_4$ is isomorphic to $B_4$. This finishes the proof of the main theorem.

**Remark 5.** Recall that a left-symmetric algebra $A$ is called Novikov if it satisfies the condition $(x \cdot y) \cdot z = (x \cdot z) \cdot y$, for all $x, y, z \in A$.

Novikov left-symmetric algebras were introduced in [2] (see also [24] for some important results concerning this). We note here that $A_4(s, 0)$ is Novikov and that $B_4$ is not.
We can explicitly compute the exponential map \( \exp : O_4 \to O_4 \) of the oscillator group in the parametrization given in Section 4. Details of the argument are left to the reader (see [11]). It is given by
\[
\exp(v, z, t) = \begin{cases} 
  (v + \frac{|z|^2}{2t}, (1 - \frac{\sin 2t}{2t})z, \frac{\sin t}{t}), & t \neq 0 \\
  (v, z, 0), & t = 0
\end{cases}
\]

On the other hand, we note that the mapping \( X \mapsto (L_X, X) \) is a Lie algebra representation of \( O_4 \) in \( \text{aff}(\mathbb{R}^4) = \text{End}(\mathbb{R}^4) \oplus \mathbb{R}^4 \). By using the exponential map of the affine group \( \text{Aff}(\mathbb{R}^4) = \text{GL}(\mathbb{R}^4) \ltimes \mathbb{R}^4 \), Theorem 2 can now be stated, in terms of simply transitive actions of subgroups of \( \text{Aff}(\mathbb{R}^4) \), as follows. To state it, define the continuous functions
\[
\begin{align*}
  f(x) &= \begin{cases} 
    \frac{\sin x}{x}, & x \neq 0 \\
    1, & x = 0
  \end{cases}, \\
  g(x) &= \begin{cases} 
    \frac{1 - \cos x}{x}, & x \neq 0 \\
    0, & x = 0
  \end{cases}, \\
  h(x) &= \begin{cases} 
    \frac{x - \sin x}{x^2}, & x \neq 0 \\
    0, & x = 0
  \end{cases}, \\
  k(x) &= \begin{cases} 
    \frac{1 - \cos x}{x}, & x \neq 0 \\
    0, & x = 0
  \end{cases},
\end{align*}
\]

and set
\[
\begin{align*}
  \Phi_t(x) &= \left(\frac{y}{2} + tz\right)g(x) - \left(\frac{z}{2} - ty\right)f(x), \\
  \Psi_t(x) &= \left(\frac{y}{2} + tz\right)f(x) + \left(\frac{z}{2} - ty\right)g(x).
\end{align*}
\]

**Theorem 3.** Suppose that the oscillator group \( O_4 \) acts simply transitively by affine transformations on \( \mathbb{R}^4 \). Then, as a subgroup of \( \text{Aff}(\mathbb{R}^4) = \text{GL}(\mathbb{R}^4) \ltimes \mathbb{R}^4 \), \( O_4 \) is conjugate to one of the following subgroups:

\( G_4 \)

(i) \[
G_4 = \begin{cases} 
  \begin{bmatrix} 
    1 & yf(x) + zg(x) & zf(x) - yg(x) & 0 \\
    0 & \cos x & -\sin x & 0 \\
    0 & \sin x & \cos x & 0 \\
    0 & \Phi_0(x) & \Psi_0(x) & 1
  \end{bmatrix} : x, y, z, w \in \mathbb{R} \\
\end{cases}
\]

(ii) \[
G_4(s, t) = \begin{cases} 
  \begin{bmatrix} 
    1 & 0 & 0 & 0 \\
    0 & \cos x - \sin x & 0 & 0 \\
    0 & \sin x & \cos x & 0 \\
    sx & \Phi_t(x) & \Psi_t(x) & 1
  \end{bmatrix} \times 
  \begin{bmatrix} 
    x \\
    yf(x) - zg(x) \\
    zf(x) + yg(x) \\
    w + \frac{1}{2}x^2 + (y^2 + z^2)\left(\frac{k(t)}{t} + tk(x)\right)
  \end{bmatrix}, \\
\end{cases} : x, y, z, w \in \mathbb{R}
\]

where \( s, t \in \mathbb{R} \). The only pairs of conjugate subgroups in \( \text{Aff}(\mathbb{R}^4) \) are \( G_4(s, t) \) and \( G_4(\alpha s, \pm t) \) where \( \alpha \in \mathbb{R}^* \).
References