

The Barber Paradox: On its Paradoxicality and its Relationship to Russell's Paradox

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ABSTRACT: The Barber Paradox is often introduced as a popular version of Russell's paradox, though some philosophers and logicians (e.g. Church) have denied their similarity, even calling the Barber paradox a pseudoparadox. In the first part of the paper, we demonstrate mainly that in the standard (Quinean) definition of a paradox the Barber paradox is a clear-cut example of a non-paradox. In the second part of the paper, we examine a probable source of the paradoxicality of the Barber Paradox, which is found in a certain ambivalence in terms of meaning. The two different readings of the crucial phrase yield distinct existential assumptions which produce the paradoxical conclusion.

KEYWORDS: Degrees of paradoxicality, pseudoparadoxicality, Russell's paradox, the Barber paradox.

1. Introduction

One of the paradoxical aspects of the *Barber paradox* (BP) is that it is not a paradox, though many people still think it is.¹ It is also paradoxical that its authorship is often attributed to Russell,² even though he did not invent it,³ and even warned against it as a false analogy to Russell's paradox (RP).⁴

¹ Including some philosophers that address paradoxes, e.g., Rescher (2001: 143–147).

² Cf., e.g., the popular text Joyce (2002).

³ The probable author of the BP is the Austrian philosopher Mally. This guess originates from Church (1963: 106), who mentioned it in his short review of a paper by Johann Mokre. Mokre attributed the *Postman Paradox* ("Postabholer", "Postzusteller", cf. Mokre 1952: 82) to Mally: Consider a villager who volunteers to bring the mail to all those who do not go to get their own mail – does he bring the mail to himself? (Mokre 1952: 89).

⁴ Russell mentioned the BP explicitly as a false analogy to RP in his large study "The Philosophy of Logical Atomism" published in *Monist* 1918–19, cf. Russell (2010: 101).

Finally, it is also somewhat paradoxical that despite what I have previously written, I will suggest a way to construe the BP as a real paradox.

This paper begins with an exposition of both RP and the BP, mainly showing the ways in which they are not similar. We will see that, according to the standard Quinean definition of a paradox, the BP is simply not a paradox at all. Then, I will attempt to reveal a possible source of the paradoxicality of the BP, which is based on a confusion of meaning. The degree of paradoxicality will be studied for a modified BP.

2. (Dis)similarities between Russell's paradox and the Barber paradox

I will operate from the standard *definition of a paradox* adopted from Quine's seminal article "The Ways of Paradox" (1966):⁵ a paradox is an *argument* whose conclusion contradicts a widely shared opinion or, as I will call it, a naïve theory which forms a usually hidden premise of the argument. To illustrate, the naïve theory of truth is, in the form of a certain general claim, a hidden premise of the Liar paradox. A paradox as an argument is usually reconstructed from its formulation, which is often rather incomplete, leaving the formation of the paradox to the imagination of a reader.

On the standard construal of paradoxes, the solution of a paradox consists in a justified refutation of the problematic premise (or naïve theory) in question or in a justified refutation of some inferential step applied within the argument. By an approach to a certain group of paradoxes we may consider a proposal of a critical theory (as we may call it) which should replace the naïve one, together with an explanation of the paradoxicality of the paradoxes in question. (The notion of critical theory is usually not mentioned in the standard construal of paradoxes, yet it is involved in it; in some cases, critical theory takes form of an axiomatic theory.) For example, Tarski's approach to the Liar Paradox explains the source of its paradoxicality and offers a critical theory of truth which does not lead to a paradox.

All these claims could be made more accurate, but let's accept them as they are.⁶

⁵ The Quinean definition has been popularized by Sainsbury in his well-known introductory book *Paradoxes*, in which he defines a paradox as: "an apparently unacceptable conclusion derived by apparently acceptable reasoning from apparently acceptable premises" (Sainsbury 1995: 1).

⁶ Some doubts about the Quinean definition have been recently expressed, e.g., by Lycan (2010), yet they will not be considered here. I can raise some other doubts and some further doubts were presented to me by an anonymous reviewer; neither these doubts can be discussed here.

RP, discovered by Russell at the very beginning of the 20th century, is the crucial modern paradox.⁷ Since a detailed description of RP does not lie within the scope of this paper, I am only going to state a few of its features which are needed for its comparison with the BP.

The hidden premise of RP is naïve set theory. More precisely, it is an opinion incorporated within it, i.e. the claim that a set can be defined by any condition (predicate⁸). This is usually articulated as the (naïve, unrestricted) Axiom of Comprehension (or Abstraction)

$$\forall \varphi \exists S \forall x ((x \in S) \leftrightarrow \varphi(x)).$$

In other words, for any condition φ there is a set S of individuals satisfying the condition φ .

As is well known, Russell challenged this opinion with an attempt to define set \mathbf{R} , Russell's set, with the help of the expression:

the set of all (and only those) sets which are not members of themselves

which can be formalized as $\{S \mid S \notin S\}$. Its core is the condition $S \notin S$, which is quite legitimate in naïve set theory. However, Russell showed that there are complications with this. If we assume that \mathbf{R} contains itself, we are then forced to conclude that it does not contain itself because \mathbf{R} contains only those sets which are not members of themselves; on the other hand, if we assume that \mathbf{R} does not contain itself, we must conclude that it does contain itself; in both cases there is a contradiction. To avoid the paradox, approaches such as Russell's or Church's type theory and Zermelo-Fraenkel's set theory (which is one of the critical theories concerning sets) do not allow an unrestricted version of the comprehension principle according to which any condition (predicate) defines a set.

A formulation similar to the following one is used to induce the BP:⁹ In a certain village, there is a barber who shaves all (and only those) villagers

⁷ There is a great amount of literature discussing RP. An important recent contribution is the compilation Link (2004).

⁸ Though there is a difference between predicates and conditions, we can abstract from it in this text.

⁹ One of the BP's popular variants is the *Catalogue Paradox* formulated by the French mathematician Gonseth (1936: 254): In a certain library, there is a catalogue which lists all catalogues which do not list themselves – does this catalogue list itself? Gonseth (p. 255) described this paradox as having both set and Barber variants."§106. La structure du paradoxe des catalogues." pp. 255–257. A very similar version is the *bibliography of all bibliographies* (cf., e.g., Rescher 2001: 144, where nearly a dozen such phrases are offered). Another popular version is the unnamed *paradox concerning secretaries* which was published by Johnston (1940: 474) as a deliberately popular variant of the RP: There is a club C having a regulation according to which only a secretary of a club to which she is not eligible to join can become a

who do not shave themselves – does the barber shave himself? If we assume that the barber does not shave himself, we conclude that he must also shave himself; if we assume that the barber does shave himself, we conclude that he cannot; in both cases, a contradiction is concluded.

Many sum up the case of the barber by simply stating that such a barber, *just like* set \mathbf{R} , cannot exist. (We will focus on the existential issues in the next section.) Another similarity between the BP and RP is based on the structural similarity between the key expressions from the formulation setting of the paradox, i.e. the description

the individual who shaves all (and only those) individuals who do not shave themselves.

However, there are also some important *dissimilarities*. Firstly, there is a huge difference between the sets which should be determined by the two crucial expressions. In the case of the BP the *empty set* of individuals is specified – no individual is its member. In other words, the formula

$$\forall y(\mathbf{Sh}(x,y) \leftrightarrow \sim \mathbf{Sh}(y,y)), \text{ where } x \text{ is free}$$

is unsatisfiable because the condition in question requires somebody who only shaves himself when he does not shave himself. However, no possible common value of variables x and y can have both \mathbf{R} and non- \mathbf{R} to itself. This led Thomson (1962: 104) to the right conclusion that $\sim \exists x \forall y (\mathbf{Sh}(x,y) \leftrightarrow \sim \mathbf{Sh}(y,y))$ is a theorem of classical logic.¹⁰ In the case of RP, on the other hand, the empty set is not specified. As documented in the above considerations: *no set* is specified at all.

This important ‘empty set/no set’ difference has a connection with the following dissimilarity. If one ascertains that no such village barber can possibly exist, one gives up the unsupported empirical assumption that there is such a barber. The conclusion that there really is no such barber is highly acceptable. On the other hand, the non-existence of Russell’s set \mathbf{R} leads to the questioning of the reasonable mathematical assumption that such a set as set \mathbf{R} should exist. A conclusion that there really is no such set \mathbf{R} can be difficult to accept. (This explanation was already offered by Quine (1966: 12) and we will partly return to such considerations in the next section.)

Employing the standard definition of a paradox, the *crucial dissimilarity* between the two paradoxes consists in the fact that the premise of RP as

member of this club C – is the secretary of C eligible to join C? In German-speaking regions (cf., e.g., Mokre 1952: 82, Brendel 1992: 8) one can also often meet the *Self-murderer’s Paradox* (“Selbstmörder”).

¹⁰ In Kalish et al. (2000), it is presented as the theorem (T269): $\sim \exists y \forall x (\mathbf{R}(x,y) \leftrightarrow \sim \mathbf{R}(x,x))$.

an argument is the naïve set theory (the naïve Axiom of Comprehension), while the BP as an argument does not contain the set theory as its premise. Moreover, the BP contains *no naïve theory*, thus there is nothing which could be contradicted by the conclusion (remember “para-doxa”). Hence, the BP is *not a paradox* at all. This difference between the two arguments can be easily recognized in the fact that RP has the effect of refuting of the naïve set theory, while the BP does not lead us to the refutation of any naïve theory.

Vigorous refutations of the BP as an analogy of RP can be found at various places in the specialized literature, though the explanation just given is not, as far as I know, presented there. One of the first dismissals, important from the viewpoint of its future acceptance by other writers, is by Grelling (1936: 481), who wrote in negative response to Perelman (1936: 205), who considered the BP to be a proper logical paradox. Grelling even wrote that the solution to the paradox (“joke”) can perhaps be a useful exercise for young students of logic. It is then not surprising that the BP, and its version by Johnston (cf. above), was already labelled a *pseudoparadox* by Church (1940: 157).

3. The source of the paradoxicality of the Barber Paradox studied with reference to its modifications

When Quine (1966: 12) emphatically wrote “we had never positively believed in such a barber,” he missed, I think, something important about the problem. It is also a bit inconvenient to just throw the BP away onto the pseudoparadox dump. Since the BP intuitively seems to be a paradox, any serious approach to paradoxes should investigate this fact. Such an attempt is made in this section, in which we encounter the phenomenon of the degree of paradoxicality.

Firstly, I am going to show how to build a proper paradox from the usual formulation of the BP. One has to explicitly add a certain problematic premise, i.e. that every property, including

‘being an x such that for every y , x shaves y if and only if y does not shave y ’,

has at least one bearer, that an individual exists that instantiates the property.¹¹ This is contradicted by the conclusion that there can be no such individual. Of course, the theory of the instantiation of properties is hardly

¹¹ Instantiation is a relation which links individuals to properties; whenever an individual bears (possess, displays) a property, the property is instantiated by the individual. For example, when Fido happens to be a dog, he instantiates the property ‘being a dog’. The theory of instantiation we discuss above adds a special point: to be a property, the property is to be instantiated (a property without a bearer is not a property).

generally held when considering issues pertaining to paradoxes. Nevertheless, this has no effect on what I intend to show, namely that the BP *would be a proper paradox* with such an additional premise.¹²

An explanation as to why some think the BP is a “simpler version” of RP might be that they insert the additional premise into their reconstruction of the BP. The theory that every property has an instance is correlative with the view that every condition (predicate) defines a set, thus the internal similarity of RP and the BP, which we noted as lacking in the preceding section, would be preserved. Since the theory of the instantiation of properties is unlikely to have adherents, this explanation of the BP’s paradoxicality can be sidelined.

The supplemented BP is apparently akin to the Horned Man paradox.¹³ The Horned Man is not a paradox at all, unless one holds the rather uncommon premise that one really has everything that one has not lost. Thus, whether something is *considered a paradox* depends partly on the degree of recognition of the problematic naïve theory. The degree of paradoxicality of the Horned Man paradox is very low, while the degree of paradoxicality of the recent Hypergame paradox¹⁴ is rather high. The degree of paradoxicality of RP is smaller than that of the Hypergame paradox, since it has nearly become a logician’s platitude (to use Quine’s own wording¹⁵) that some conditions do not define a set.¹⁶

Now let us focus on the source of the paradoxicality of the BP as it is normally conceived. For my findings it is convenient to recall that during ordinary human communication speakers often assume that we are speak-

¹² Note also the fact that one formulation of a paradox may give rise to more than one paradox as an argument, or a paradox and a non-paradox. If some reader of an incomplete formulation is unable to reconstruct a paradox in his mind, then she sees no paradox. For example, the readers such as Grelling or Church read the formulation of the BP in such a way that there is clearly no paradox (as we will see later, they disambiguated the BP’s crucial phrase such a way that a paradoxical conclusion is excluded).

¹³ What you have not lost, you have. But you have not lost horns. Therefore, you have horns.

¹⁴ See, e.g., Zwicker (1987). To sketch out the formulation at least partly: the moves in the Hypergame consist in the playing of games and the problem arises if the Hypergame itself is chosen in some move. The paradox affects game theory, and not set theory as it might seem from my simplifying description.

¹⁵ “One man’s antinomy can be another man’s veridical paradox, and one man’s veridical paradox can be another man’s platitude” (Quine 1966: 12).

¹⁶ How to measure degree of paradoxicality exactly is not an easy matter – it is like measuring a degree of (say) surprise for a gift or novelty of some news. The degree apparently depends on individuals, but it would be more fruitful to find some other basis (e.g. one’s conceptual background, ‘background context’, assumptions) in order to apply logical methods of measuring e.g. distances between formulas.

ing about existing things: to repeat Strawson's well-known example, when somebody says that his children went to sleep, we usually presuppose that he has some children. The formulation of the BP even begins with the assumption that such a village barber exists, or can exist. After examining the paradox, however, we concluded that there could be no such barber. We are thus putting our finger on the most probable cause of the BP's paradoxicality, which consists in a hidden *ambiguity of meaning*. The two meanings of the crucial expression imply distinct existential assumptions, and this is the root of the contradiction.¹⁷

Due to the *irreflexive reading* of the BP's crucial expression – i.e. “shaves all those who do not shave themselves” – the barber shaves *all others*, everybody except himself. (The suggestion to replace the quantifier “all” with the quantifier “all others”, using thus an addition of $(x \neq y)$ in the antecedent of the biconditional which was formally stated above, was already made by van Benthem (1978: 52).) This is perhaps a natural reading because an artisan or specialist is often understood to be somebody who does something for others.¹⁸ However, the question “Does he shave himself?” led hearers to speculate about the other meaning of the expression: its *reflexive reading*, due to which the artisan can also serve himself likewise.

I have said that the paradoxicality of the BP originates in the fact that the following sentences are contradictory:

- (P) *There exists a barber who shaves (all and only) those who do not shave themselves,*
- (C) *There does not exist a barber who shaves (all and only) those who do not shave themselves.*

However, the premise P and the conclusion C are only contradictory provided they are both read in the reflexive sense. With the reflexive reading, sentence C is *logically (i.e. necessarily) true*. If P is really contradictory to C, P is *logically false*. It thus implies a specific existential assumption as regards the barber, namely that *no such barber can possibly exist*.

Nevertheless, it is rather unnatural to understand P just in the reflexive sense. Especially if one assumes, as the reader of the paradox does, that such a barber can exist. P is thus understood rather as P²:

¹⁷ A little remark on interplay of degree of paradoxicality and ambiguity in meaning is in order. Ambiguity in meaning is a one of the factors which cause paradoxicality thus it also affects its degree. The BP is a good example: having a sharp meaning of “shave” in their mind, Grelling and Church could see no paradox in the BP, while common people can be puzzled; the degree of their puzzlement is to some extent correlative with their ability to disambiguate the crucial phrase of the BP.

¹⁸ On the other hand, some professions involve reflexivity – e.g. a babysitter usually takes care also about her children.

(P') *There exists a barber who shaves all (and only) others who do not shave themselves.*

With this irreflexive reading, P – i.e. P' – is *contingently true*. Its relevant specific existential assumption thus differs from that of the reflexive reading: *such a barber can exist by a contingent chance*.

To sum up, the *paradoxicality of the BP* consists in the fact that the premise P, which is assumed to be possibly true (being rather P'), turned out to be necessarily false, because the meaning of the crucial expression has switched.

One might perhaps utilize the above observation concerning P and P' to say that the BP is analogous to RP after all. In this case, the following argument would be advanced as the explicit form of the BP:

(P'') *Possibly, there is a barber who shaves all (and only) those who do not shave themselves.*

(C'') *Necessarily, there is no barber who shaves all (and only) those who do not shave themselves.*

The explicit form of RP would be quite analogous to this, but mentioning set membership instead of a barber and shaving.

Despite this one similarity, a principal difference still remains between the two paradoxes because one must consider the relevant conditions of being true. The possible existence of such a barber should occur within the empirical world of material beings and the internal consistency or inconsistency involved in the description of the barber is a plain matter. However, Russell's set **R** should exist within the realm of immaterial mathematical entities, in which a consistency question is usually considered to be a *condicio sine qua non*.

4. Conclusion

In the first main section of this paper, I have argued that, according to the standard definition of a paradox, the BP is not a paradox at all. In its usual form, the BP harbours no naïve theory (in the form of a problematic premise) which could be contradicted by its conclusion. The BP thus has only an incidental resemblance to RP. Various similarities and dissimilarities between both paradoxes have been discussed throughout the whole paper.

In the second main section, I have focused on the problem of the degree of paradoxicality. A probable source of the admitted paradoxicality of the BP consists in the ambiguity in the meaning of the verb "shave", and thus the crucial phrase of the BP, which results in different conclusions concerning the possible existence of a certain barber. We have seen then that the nature of

these existential assumptions of the BP differs significantly from that of the assumptions involved in the background of RP.¹⁹

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