On some Gronwall type inequalities with iterated integrals

YEOL JE CHO∗, SEVER S. DRAGOMIR† and YOUNG-HO KIM‡

Abstract. The main objective of the present paper is to establish some new Gronwall type inequalities involving iterated integrals.

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1. Introduction

Let \( u : [\alpha, \alpha + h] \to \mathbb{R} \) be a continuous real-valued function satisfying the inequality

\[
0 \leq u(t) \leq \int_{\alpha}^{t} [a + bu(s)] \, ds, \quad \text{for all } t \in [\alpha, \alpha + h],
\]

where \( a, b \) are nonnegative constants. Then \( u(t) \leq ahe^{bh} \) for \( t \in [\alpha, \alpha + h] \). This result was proved by T. H. Gronwall [8] in the year 1919, and it is a prototype for the study of several integral inequalities of Volterra type, and also for obtaining explicit bounds of the unknown function. Among the several publications on this subject, the paper of Bellman [3] is very well known. It is clear that Bellman’s result contains that of Gronwall. This is the reason why inequalities of this type were called “Gronwall-Bellman inequalities” or “Inequalities of Gronwall type”. The Gronwall type integral inequalities provide a necessary tool for the study of the theory of differential equations, integral equations and inequalities of various types (see Gronwall [8] and Giuliano [9]). Some applications of this result to the study of stability of the solution of linear and nonlinear differential equations may be found in Bellman [3]. Some applications to existence and uniqueness theory of differential equations may be found in Nemyckii-Stepanov [13], Bihari [4], and

∗Department of Mathematics Education, The Research Institute of Natural Sciences, College of Education, Gyeongsang National University, Chinju 660-701, Republic of Korea, e-mail: yjcho@gsnu.ac.kr
†School of Computer Science and Mathematics, Victoria University of Technology, PO Box 14428, MCMC, Melbourne, Victoria 8001, Australia, e-mail: sever@matilda.vu.edu.au
‡Department of Applied Mathematics, Changwon National University, Changwon, Kyung-Nam 641-773, Republic of Korea, e-mail: yhkim@sarim.changwon.ac.kr

Langenhop [10]. During the past few years several authors (see references below and some of the references cited therein) have established several Gronwall type integral inequalities in two or more independent real variables. Of course, such results have application in the theory of partial differential equations and Volterra integral equations.

Bainov and Simeonov [1, p. 101] proved the following interesting integral inequality involving iterated integrals:

Let \( u(t), a(t), \) and \( b(t) \) be nonnegative continuous functions in \( J = [\alpha, \beta] \), and suppose that

\[
\begin{align*}
\int_a^t k_1(t, t_1)u(t_1)\,dt_1 + \cdots \\
+ \int_a^t \left( \int_{t_1}^{t_{i-1}} \cdots \left( \int_{t_{i-2}}^{t_{i+1}} k_{n}(t, t_1, \ldots , t_n)u(t_n)\,dt_n \right) \cdots \right)\,dt_i
\end{align*}
\]

for all \( t \in J \), where \( k_i(t_1, \ldots , t_i) \) are nonnegative continuous functions in \( J_{i+1}, i = 1, 2, \ldots , n \), which are nondecreasing in \( t \in J \) for all fixed \( (t_1, \ldots , t_i) \in J_i, i = 1, 2, \ldots , n \). Then, for all \( t \in J \)

\[
\begin{align*}
u(t) \leq a(t) + b(t) & \int_a^t \tilde{R}[a](t, s)\exp \left( \int_s^t \tilde{R}[b](t, \tau)\,d\tau \right)\,ds, \\
\end{align*}
\]

where, for all \( (t, s) \in J_2 \)

\[
\begin{align*}
\tilde{R}[w](t, s) = k_1(t, s)w(s) + & \int_a^s k_2(t, s, t_2)w(t_2)\,dt_2 \\
+ & \sum_{i=3}^n \int_a^s \left( \int_{t_2}^{t_{i-1}} \cdots \left( \int_{t_{i-2}}^{t_{i+1}} k_{i}(t, s, t_2, \ldots , t_i)w(t_i)\,dt_i \right) \cdots \right)\,dt_i
\end{align*}
\]

for each continuous function \( w(t) \) in \( J \).

In this paper we consider simple inequalities involving iterated integrals in inequality (1.1) for functions when the function \( u \) on the right-hand side of inequality (1.1) is replaced by the function \( u^p \) for some \( p \). We also provide some integral inequalities involving iterated integrals.

2. The main results

In this section we state and prove some new nonlinear integral inequalities involving iterated integrals. Throughout the paper all functions which appear in inequalities are assumed to be real-valued.

Before considering our first integral inequality, we need the following lemmas, which appeared in [1, p. 2, p. 38].

**Lemma 2.1.** Let \( b(t) \) and \( f(t) \) be continuous functions for \( t \geq \alpha \), let \( v(t) \) be a differentiable function for \( t \geq \alpha \), and suppose

\[
v'(t) \leq b(t)v(t) + f(t), \quad t \geq \alpha
\]
and \( v(\alpha) \leq v_0 \). Then, for all \( t \geq \alpha \),

\[
v(t) \leq v_0 \exp \left( \int^t_\alpha b(s) \, ds \right) + \int^t_\alpha f(s) \exp \left( \int^t_s b(\tau) \, d\tau \right) \, ds.
\]

**Lemma 2.2.** Let \( v(t) \) be a positive differential function satisfying the inequality

\[
v'(t) \leq b(t)v(t) + k(t)v^p(t), \quad t \in J = [\alpha, \beta],
\]

where functions \( b \) and \( k \) are continuous in \( J \), and \( p \geq 0, p \neq 1 \), is a constant. Then

\[
v(t) \leq \exp \left( \int^t_\alpha b(s) \, ds \right) \left[ v^q(\alpha) + q \int^t_\alpha k(s) \exp \left( -q \int^s_\alpha b(\tau) \, d\tau \right) \, ds \right]^{1/q}
\]

for all \( t \in [\alpha, \beta_1] \), where \( \beta_1 \) is chosen so that the expression between \([...]\) is positive in the subinterval \([\alpha, \beta_1] \).

In the next theorems we consider some simple inequalities involving iterated integrals.

Let \( \alpha < \beta \), and set

\[
J_i = \{(t_1, t_2, \ldots, t_i) \in \mathbb{R}^i : \alpha \leq t_i \leq \cdots \leq t_1 \leq \beta, i = 1, \ldots, n\}.
\]

**Theorem 2.1.** Let \( u(t), a(t) \) and \( b(t) \) be nonnegative continuous functions in \( J = [\alpha, \beta] \) and let \( p > 1 \) be a constant. Suppose that \( \frac{a(t)}{b(t)} \) is nondecreasing in \( J \) and

\[
u(t) \leq a(t) + b(t) \int^t_\alpha k_1(t, t_1)u^p(t_1) \, dt_1 + \cdots \\
+ \int^t_\alpha \left( \int^{t_i}_{t_{i-1}} \cdots \left( \int^{t_2}_{t_1} k_n(t, t_1, \cdots, t_n)u^p(t_n) \, dt_n \right) \cdots \right) \, dt_1 \] (2.1)

for any \( t \in J \), where \( k_i(t, t_1, \ldots, t_i) \) are nonnegative continuous functions in \( J_{i+1} \) for \( i = 1, 2, \ldots, n \), which are nondecreasing in \( t \) for all fixed \( (t_1, \ldots, t_i) \in J_i \), \( i = 1, 2, \ldots, n \). Then, for any \( t \in [\alpha, \beta_1] \)

\[
u(t) \leq a(t) \left[ 1 - (p - 1) \int^t_\alpha \left( \frac{a(s)}{b(s)} \right)^{p-1} R[|b^p|(t, s)] \, ds \right]^{1/(p-1)} \] (2.2)

where, for any \((t, s) \in J_2\),

\[
\beta_p = \sup \{ t \in J : (p - 1) \int^t_\alpha \left( \frac{a(s)}{b(s)} \right)^{p-1} R[|b^p|(t, s)] \, ds < 1 \},
\]

and

\[
R[u](t, s) = k_1(t, s)w(s) + \int^s_t k_2(t, s, t_2)w(t_2) \, dt_2 \\
+ \sum_{i=3}^n \int^s_t \cdots \left( \int^{t_{i-1}}_t k_i(t, s, t_2, \cdots, t_i)w(t_i) \, dt_i \right) \, dt_2,
\]
for each continuous function \( u(t) \) in \( J \).

**Proof.** For a fixed \( T \in (\alpha, \beta) \) and \( \alpha \leq t \leq T \) we have

\[
    u(t) \leq a(t) + b(t)v(t),
\]

where

\[
v(t) = \int_{\alpha}^{t} k_1(T, t_1)u^p(t_1) \, dt_1 + \cdots
\]

\[
    + \int_{\alpha}^{t} \left( \int_{\alpha}^{t_1} \cdots \left( \int_{\alpha}^{t_n-1} k_n(T, t_1, \ldots, t_n)u^p(t_n) \, dt_n \right) \cdots \right) dt_1.
\]

Since \( \frac{\partial k_i}{\partial t_i}(T, t_1, \ldots, t_i) = 0 \) for \( i = 1, \ldots, n \) and \( t \in [\alpha, T] \), we have

\[
v'(t) = R[u^p](T, t) \leq (R[b^p](T, t)) \left( \frac{a(t)}{b(t)} + v(t) \right)^p,
\]

that is,

\[
v'(t) \leq Q(T, t)[a(T)/b(T) + v(t)],
\]

where \( Q(T, t) = (R[b^p](T, t))[a(t)/b(t) + v(t)]^{p-1} \). Lemma 2.1 and (2.4) imply

\[
v(t) + \frac{a(T)}{b(T)} \leq \frac{a(T)}{b(T)} \exp \left( \int_{\alpha}^{t} Q(T, s) \, ds \right), \quad \alpha \leq t \leq T.
\]

Hence, for \( t = T \),

\[
v(T) + \frac{a(T)}{b(T)} \leq \frac{a(T)}{b(T)} \exp \left( \int_{\alpha}^{T} Q(T, s) \, ds \right).
\]

From (2.5), we successively obtain

\[
v(T) + \frac{a(T)}{b(T)} \leq \left[ \frac{a(t)}{b(t)} \right]^{p-1} \exp \left( \int_{\alpha}^{t} (p-1)Q(T, s) \, ds \right),
\]

\[
Q(T, t) \leq (R[b^p](T, t)) \left[ \frac{a(t)}{b(t)} \right]^{p-1} \exp \left( \int_{\alpha}^{t} (p-1)Q(T, s) \, ds \right),
\]

\[
Z(T, t) \leq (p-1)(R[b^p](T, t)) \left[ \frac{a(t)}{b(t)} \right]^{p-1} \exp \left( \int_{\alpha}^{t} (p-1)Q(T, s) \, ds \right),
\]

where \( Z(T, t) = (p-1)Q(T, t) \). Consequently, we have

\[
Z(T, s) \exp \left( - \int_{\alpha}^{s} Z(T, s) \, ds \right) \leq (p-1)R[b^p](T, s) \left[ \frac{a(s)}{b(s)} \right]^{p-1}
\]

or

\[
\frac{d}{ds} \left[ - \exp \left( - \int_{\alpha}^{s} Z(T, \tau) \, d\tau \right) \right] \leq (p-1)R[b^p](T, s) \left[ \frac{a(s)}{b(s)} \right]^{p-1}.
\]
Integrating this from $\alpha$ to $t$ yields

$$1 - \exp\left(-\int_{\alpha}^{t} Z(T, s) \, ds\right) \leq (p - 1) \int_{\alpha}^{t} \left(\frac{a(s)}{b(s)}\right)^{p-1} R[b^p](T, s) \, ds,$$

from which we conclude that

$$\exp\left(\int_{\alpha}^{t} Q(T, s) \, ds\right) \leq \left[1 - (p - 1) \int_{\alpha}^{t} \left(\frac{a(s)}{b(s)}\right)^{p-1} R[b^p](T, s) \, ds\right]^{\frac{1}{p-1}}.$$

This, together with (2.3) and (2.5), implies

$$u(t) \leq a(t) \left[1 - (p - 1) \int_{\alpha}^{t} \left(\frac{a(s)}{b(s)}\right)^{p-1} R[b^p](T, s) \, ds\right]^{\frac{1}{p-1}}.$$

In particular, for $T = t$ we find (2.2). This completes the proof. \hfill $\Box$

**Theorem 2.2.** Let $u(t)$ and $b(t)$ be nonnegative continuous functions in $J = [\alpha, \beta]$, and suppose that

$$u(t) \leq b(t) \left[a(t) + \int_{\alpha}^{t} k_1(t, t_1)u^p(t_1) \, dt_1 + \cdots + \int_{\alpha}^{t} \left(\int_{\alpha}^{t_1} \cdots \left(\int_{\alpha}^{t_{n-1}} k_n(t, t_1, \ldots, t_n)u^p(t_n) \, dt_n\right) \cdots \right) \, dt_1\right]$$

for $t \in J$, where $p \geq 0$, $p \neq 1$ is a constant, $a(t) > 0$ is a nondecreasing continuous function in $t \in J$, and $k_i(t, t_1, \ldots, t_i)$ are nonnegative continuous functions in $J_{i+1}$, $i = 1, 2, \ldots, n$, which are nondecreasing in $t \in J$ for all fixed $(t_1, \ldots, t_i) \in J_i$, $i = 1, 2, \ldots, n$. Then

$$u(t) \leq b(t) \left[a^q(t) + q \int_{\alpha}^{t} R[b^p](T, s) \, ds\right]^{1/q} \quad (2.6)$$

for any $t \in [\alpha, \beta_1)$, where $q = 1 - p$ and $\beta_1$ is chosen so that the expression between $[\ldots]$ is positive in the subinterval $[\alpha, \beta_1)$.

**Proof.** For a fixed $T \in (\alpha, \beta]$ and $\alpha \leq t \leq T$ we have

$$u(t) \leq b(t)v(t)$$

$$\equiv b(t) \left[a(T) + \int_{\alpha}^{t} k_1(T, t_1)u^p(t_1) \, dt_1 + \cdots + \int_{\alpha}^{t} \left(\int_{\alpha}^{t_1} \cdots \left(\int_{\alpha}^{t_{n-1}} k_n(T, t_1, \ldots, t_n)u^p(t_n) \, dt_n\right) \cdots \right) \, dt_1\right].$$

Since $v(\alpha) = a(T)$, $v(t)$ is nondecreasing and continuous in $J$ and $\frac{\partial k_i}{\partial t}(T, t_1, \ldots, t_i) \equiv 0$ for $i = 1, \ldots, n$ and $t \in [\alpha, T]$, we have

$$v'(t) = R[a^p](T, t) \leq R[b^p v^p](T, t) \leq (R[b^p](T, t)) v^p(T, t). \quad (2.7)$$
\textbf{Lemma 2.1} \textbf{Lemma 2.2} \\
\alpha for $t$ for all $\tau$ for all $\tau$ for all $\tau$ for all $\tau$

\begin{align*}
\text{Lemma 2.1 and (2.7)} & \text{ imply } \\
v(t) & \leq \left[ a^q(T) + q \int_{\alpha}^{t} R[b^p](T, s) \, ds \right]^{1/q} \\
\text{from which we obtain } & \\
u(t) & \leq b(t) \left[ a^q(T) + q \int_{\alpha}^{t} R[b^p](T, s) \, ds \right]^{1/q}
\end{align*}

for $\alpha \leq t \leq T$. In particular, for $T = t$ we find (2.6). This completes the proof. \hfill \Box

\textbf{Theorem 2.3.} Let $u(t)$, $a(t)$ and $b(t)$ be nonnegative continuous functions in $J = [\alpha, \beta]$. Suppose that

\begin{align*}
u(t) & \leq a(t) + b(t) \left[ \int_{\alpha}^{t} k_1(t, t_1) u^p(t_1) \, dt_1 + \ldots \\
& \quad + \int_{\alpha}^{t} \left( \int_{\alpha}^{t_1} \cdots \left( \int_{\alpha}^{t_{n-1}} k_n(t, t_1, \ldots, t_n) u^p(t_n) \, dt_n \right) \ldots \right) dt_1 \right]
\end{align*}

for all $t \in J$, where $0 < p \leq 1$ be a constant, $\frac{a(t)}{b(t)} \geq 1$ is nondecreasing in $J$ and $k_i(t, t_1, \ldots, t_i)$ are nonnegative continuous functions in $J_i + 1, i = 1, 2, \ldots, n$, which are nondecreasing in $t \in J$ for all fixed $(t_1, \ldots, t_i) \in J_i, i = 1, 2, \ldots, n$. Then

\begin{equation}
u(t) \leq a(t) \exp \left( \int_{\alpha}^{t} R[b^p](t, \tau) \, d\tau \right) \tag{2.8} \end{equation}

for all $t \in [\alpha, \beta]$. \hfill \textbf{Proof.} For a fixed $T \in (\alpha, \beta)$ and $\alpha \leq t \leq T$, we have

\begin{align*}u(t) & \leq a(t) + b(t) w(t) \\
& \equiv a(t) + b(t) \left[ \int_{\alpha}^{t} k_1(T, t_1) u^p(t_1) \, dt_1 + \ldots \\
& \quad + \int_{\alpha}^{t} \left( \int_{\alpha}^{t_1} \cdots \left( \int_{\alpha}^{t_{n-1}} k_n(T, t_1, \ldots, t_n) u^p(t_n) \, dt_n \right) \ldots \right) dt_1 \right].
\end{align*}

Since $w(\alpha) = 0$, $w(t)$ is nondecreasing and continuous in $J$ and $\frac{\partial w}{\partial t}(T, t_1, \ldots, t_i) \equiv 0$ for $i = 1, \ldots, n$ and $t \in [\alpha, T]$, we have

\begin{align*}w'(t) & = R[u^p](T, t) \leq R[b^p](T, t) \left( \frac{a(t)}{b(t)} + w(t) \right) \tag{2.9} \\
& \leq R[b^p](T, t) \left( \frac{a(t)}{b(t)} + w(t) \right)
\end{align*}

\textbf{Lemma 2.1} and (2.9) imply

\begin{equation}w(t) \leq \int_{\alpha}^{t} R[b^p](T, s) \left( \frac{a(s)}{b(s)} \right) \exp \left( \int_{s}^{t} R[b^p](T, \tau) \, d\tau \right) \, ds, \tag{2.9} \end{equation}
from which we obtain
\[ u(t) \leq a(t) + b(t) \int_{\alpha}^{t} R[b^p](T, s) \left( \frac{a(s)}{b(s)} \right) \exp \left( \int_{s}^{t} R[b^p](T, \tau) d\tau \right) ds. \] (2.10)

Indeed, (2.10) implies that
\[ u(t) \leq a(t) \left[ 1 + \int_{\alpha}^{t} R[b^p](T, s) \exp \left( \int_{s}^{t} R[b^p](T, \tau) d\tau \right) ds \right] = a(t) \exp \left( \int_{\alpha}^{t} R[b^p](T, \tau) d\tau \right) \]
for \( \alpha \leq t \leq T \). In particular, for \( T = t \) we find (2.8). This completes the proof. \( \square \)

**Theorem 2.4.** Let \( u, f_1, \ldots, f_n \) be nonnegative continuous functions in \( J = [\alpha, \beta] \), and suppose that
\[ u(t) \leq a \left[ 1 + \int_{\alpha}^{t} f_1(t_1) \left( \int_{\alpha}^{t_1} f_2(t_2) \left( \int_{\alpha}^{t_2} f_3(t_3) \left( \cdots \int_{\alpha}^{t_{n-1}} f_n(t_n) dt_n \right) \cdots \right) dt_n \right) dt_1 \right] \]
for all \( t \in J \), where \( a \geq 1 \) and \( 0 < p \leq 1 \) are constants. Then
\[ u(t) \leq a R_1(t), \quad t \in J, \]
(2.12)
where
\[ R_n(t) = \exp \left( \int_{\alpha}^{t} f_n(s) ds \right), \quad t \in J, \]
and
\[ R_n(t) = 1 + \int_{\alpha}^{t} f_i(t) R_{i+1}(s) \exp \left( \int_{\alpha}^{s} f_i(\tau) d\tau \right) ds \]
for all \( t \in J \) and \( i = n - 1, \ldots, 1 \).

**Proof.** We set
\[ u_1(t) = a + L_1[u^p](t), \quad u_{j+1}(t) = u_j + L_{j+1}[u^p](t) \]
for all \( t \in J \) and \( j = 1, \ldots, n - 1 \), where
\[ L_k[u^p](t) = \int_{\alpha}^{t} f_k(t_k) u^p(t_k) dt_k + \cdots + \int_{\alpha}^{t} f_k(t_k) \left( \int_{\alpha}^{t} f_{k+1}(t_{k+1}) \left( \cdots \int_{\alpha}^{t} f_n(t_n) dt_n \right) \cdots \right) dt_k \]
for all \( t \in J \) and \( k = 1, \ldots, n \). Now, (2.11) implies
\[ u(t) \leq u_1(t). \]
(2.13)
Taking into account that
\[ u_k(t) \leq u_{k+1}(t), \]
\[ (L_k[u^p])' = f_k(u^p(t) + L_{k+1}[u^p]), \quad k = 1, \ldots, n - 1, \]
and
\[ (L_n[u^p])' = f_n(t)u^p(t). \]

We successively find
\[ u'_1(t) = (L_1[u^p](t))' = f_1[u^p(t) + L_2[u^p]] \leq f_1u_1(t) + L_2[u^p], \]
\[ u'_k(t) \leq (f_1 + \cdots + f_{k-1})u_1(t) + f_ku_k(t), \quad k = 2, \ldots, n - 1, \quad (2.14) \]
\[ u'_n(t) \leq (f_1 + \cdots + f_n)u_n(t). \]

Since \( u_k(\alpha) = a, k = 1, \ldots, n \), (2.14) gives, by successive application of Lemma 2.1,
\[ u_k(t) \leq aR_k(t)\exp\left(\int_\alpha^t \sum_{j=1}^{k-1} f_j(s) \, ds\right), \quad k = n, n-1, \ldots, 1. \]

For \( k = 1 \) this and (2.13) imply (2.12). \( \square \)

**Remark 2.1.** In the case when \( a \geq 0, p = 1 \), the inequality given in (2.11) reduces to the inequality established earlier by Ráb in [16] (see also [1, Theorem 11.6, p. 102]).

**Corollary 2.1.** Let \( u, f, g \) be nonnegative continuous functions in \( J = [\alpha, \beta] \), \( u_0 \geq 1 \) and suppose that
\[ u(t) \leq u_0 + \int_\alpha^t f(s) \left[ u^p(s) + \int_\alpha^s g(\tau)u^p(\tau) \, d\tau\right] \, ds \]
for all \( t \in J \), where \( 0 < p \leq 1 \) is a constant. Then
\[ u(t) \leq u_0 \left[ 1 + \int_\alpha^t f(s) \exp\left(\int_\alpha^s (f(\tau) + g(\tau)) \, d\tau\right) \, ds\right] \]
for all \( t \in J \).

**Theorem 2.5.** Let \( u, f_i, i = 1, \cdots, n \) be nonnegative continuous functions in \( J = [\alpha, \beta] \), and suppose that
\[ u(t) \leq a(t) + \int_\alpha^t f_1(t_1)u^p(t_1) \, dt_1 + \cdots \]
\[ + \int_\alpha^t f_1(t_1)\left(\int_\alpha^{t_1} f_2(t_2) \cdots \left(\int_\alpha^{t_{n-1}} f_n(t_n)u^p(t_n) \, dt_n\right) \cdots \right) \, dt_1 \quad (2.15) \]
for all \( t \in J \), where \( a(t) \geq 1 \) is a continuous function in \( J \) and \( 0 < p \leq 1 \) is a constant. Then
\[ u(t) \leq a(t) + \int_\alpha^t f_1(s)[a(s) + v_2(s)] \, ds, \quad (2.16) \]
Lemma 2.1

where

\[ u_n(t) = \int_\alpha^t (f_1(s) + \cdots + f_n(s))a(s) \exp \left( \int_s^t (f_1(\tau) + \cdots + f_n(\tau)) \, d\tau \right) \, ds \]

and

\[ u_k(t) = \int_\alpha^t \left[ \sum_{j=1}^k f_j(s) \right] a(s) + f_k(s)v_{k+1}(s) \, ds \exp \left( \int_s^t \sum_{j=1}^{k-1} f_j(\tau) \, d\tau \right) \, ds \]

for all \( t \in J \) and \( k = n - 1, \ldots, 2 \).

**Proof.** Let \( L_k[u^p](t) \) be defined as in Theorem 2.4 and put

\[ v_1(t) = L_1[u^p](t), \quad v_{k+1}(t) = v_k + L_{k+1}[u^p](t) \]

for all \( t \in J \) and \( k = 1, \ldots, n-1 \). Then (2.15) implies

\[ u(t) \leq a(t) + v_1(t) \tag{2.17} \]

and we successively find

\[ v_1'(t) = (L_1[u^p](t))' = f_1[u^p(t) + L_2[u^p]] \]

\[ \leq f_1[a(t) + v_1(t) + L_2[u^p]] = f_1[a(t) + v_2(t)] \]

\[ v_k'(t) \leq (f_1 + \cdots + f_{k-1})v_k(t) + (f_1 + \cdots + f_k)a + f_kv_{k+1}(t), \quad (2.18) \]

\[ k = 2, \ldots, n-1, \]

\[ v_n'(t) \leq (f_1 + \cdots + f_n)v_n(t) + (f_1 + \cdots + f_n)a(t). \]

Since \( v_k(\alpha) = 0, k = 1, \ldots, n \), solving the system (2.18) ‘backward’ and applying Lemma 2.1, we arrive at

\[ v_1(t) \leq \int_\alpha^t f_1(s)[a(s) + v_2(s)] \, ds \tag{2.19} \]

where

\[ v_k(t) = \int_\alpha^t \left[ \sum_{j=1}^k f_j(s) \right] a(s) + f_k(s)v_{k+1}(s) \, ds \exp \left( \int_s^t \sum_{j=1}^{k-1} f_j(\tau) \, d\tau \right) \, ds, \]

for all \( t \in J \), \( k = n - 1, \ldots, 2 \), and

\[ u_n(t) = \int_\alpha^t (f_1(s) + \cdots + f_n(s))a(s) \exp \left( \int_s^t (f_1(\tau) + \cdots + f_n(\tau)) \, d\tau \right) \, ds. \]

The inequalities (2.17) and (2.19) imply (2.16). This completes the proof. \( \square \)

**Remark 2.2.** In the case when \( a(t) \geq 0, p = 1 \), the inequality given in (2.15) reduces to the inequality established earlier by Young in [18] (see also [1, Theorem 11.7, p. 103]).
Corollary 2.2. Let \( u, f, g, h \) be nonnegative continuous functions in \( J = [\alpha, \beta] \). Suppose that

\[
u(t) \leq u_0 + \int_{\alpha}^{t} (f(s)u^p(s) + h(s)) \, ds + \int_{\alpha}^{t} f(s) \left( \int_{\alpha}^{s} g(\tau)u^p(\tau) \, d\tau \right) \, ds
\]

for all \( t \in J \), where \( u_0 + \int_{\alpha}^{t} h(s) \, ds \geq 1 \) is a continuous function in \( J \) and \( 0 < p \leq 1 \) is a constant. Then

\[
u(t) \leq u_0 + \int_{\alpha}^{t} h(s) \, ds + \int_{\alpha}^{t} f(s) \left[ u_0 + \int_{\alpha}^{s} h(\tau) \, d\tau \right] \exp \left( \int_{\alpha}^{s} (f(\tau) + g(\tau) \tau) \, d\tau \right) \, ds.
\]

Proof. Indeed, the proof follows from Theorem 2.5 with \( f_1 = f, f_2 = g \) and \( a(t) = u_0 + \int_{\alpha}^{t} h(s) \, ds \).

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References

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