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HADAMARD AND DRAGOMIR-AGARWAL INEQUALITIES, THE GENERAL EULER TWO POINT FORMULAE AND CONVEX FUNCTIONS

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Abstract

The general Euler two-point formulae are used with functions possessing various convexity and concavity properties to derive inequalities pertinent to numerical integration.

Key words and phrases: Hadamard inequality, r-convexity, integral inequalities, general Euler two-point formula

1. Introduction

One of the cornerstones of nonlinear analysis is the Hadamard inequality, which states that if $[a,b]\,(a<b)$ is a real interval and $f:[a,b]\to\mathbf{R}$ is a convex function, then

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(t)dt\leq\frac{f(a)+f(b)}{2}. \tag{1.1}$$

Recently, S.S. Dragomir and R.P. Agarwal [4] considered the trapezoid formula for numerical integration of functions f such that $|f'|^q$ is a convex function for some $q\geq 1$. Their approach was based on estimating the difference between the two sides of the right-hand inequality in (1.1). Improvements of their results were obtained in [8]. In particular, the following tool was established.

Suppose $f:I^0\subseteq\mathbf{R}\to\mathbf{R}$ is differentiable on I^0 and such that $|f'|^q$ is convex on $[a,b]$ for some $q\geq 1$, where $a,b\in I^0\,(a<b)$. Then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{b-a}{4}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{1/q}. \tag{1.2}$$

Some generalizations to higher-order convexity and applications of these results are given in [2]. Related results for Euler midpoint, Euler-Simpson, Euler

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twopoint, dual Euler-Simpson, Euler-Simpson 3/8 and Euler-Maclaurin formulae were considered in [9] (see also [3] and [10]).

In this paper we consider some related results using the general Euler two-point formulae. We will use interval [0,1] because of simplicity and since it involves no loss in generality.

The general two-point formula is defined as

$$\int_0^1 f(t)dt = \frac{1}{2}(f(x)+f(1-x))+E(f;x) \tag{1.3}$$

with $E(f;x)$ being the remainder and $x \in [0,1/2]$.

In [5] A. Guessab and G. Schmeisser have proved, among others, the following theorem and corollary (see also [11]).

Theorem 1. *Let f be a L-Lipschitzian function defined on* [0,1]*. Then, for each* $x \in [0,1/2]$*, the remainder in (1.3) satisfies*

$$|E(f;x)| \leq \frac{4x^2+(1-2x)^2}{4} \cdot L.$$

This inequality is sharp for each admissible x. Eauality is attained if and only if $f = \pm L f_* + c$*, with* $c \in \mathbf{R}$ *and*

$$f_*(t) := \begin{cases} x-t & for\, 0 \leq t \leq x \\ t-x & for\, x \leq t \leq 1/2 \\ f_*(1-t) & for\, 1/2 \leq t \leq 1. \end{cases}$$

Corollary 1. *Let f be a differentiable function defined on* [0,1]*, such that f′ is L-Lipschitzian function. If* $0 \leq x \leq 1/4$*, then*

$$|E(f;x)| \leq \frac{4x^3+6(1-2x)x^2-(1-2x)^3+2[(1-2x)^2-4x^2]^{3/2}}{12} \cdot L.$$

This inequality is sharp for each admissible x. Equality is attained for $f(t) = \pm L \int f_*'(t)dt + +c_1 t + c_0$*, with* $c_0, c_1 \in \mathbf{R}$ *and*

$$f_*'(t) := \begin{cases} 1/2-2\gamma-t & for\, 0 \leq t \leq 1/2-\gamma \\ t-1/2 & for\, 1/2-\gamma \leq t \leq 1/2+\gamma \\ 1/2+2\gamma-t & for\, 1/2+\gamma \leq t \leq 1. \end{cases}$$

where $\gamma = \sqrt{1-4x}/2$.

If $1/4 \le x \le 1/2$, then

$$|E(f;x)| \le \frac{4x^3+6(1-2x)x^2-(1-2x)^3}{12}\cdot L.$$

This inequality is sharp for each admissible x. Equality is attained for $f(t)=\pm\frac{1}{2}Lt^2+$ $+c_1 t+c_0$, *with* $c_0, c_1 \in \mathbf{R}$.

2. The general Euler two-point for mulae

In the recent paper [11] the following identities, named the general Euler two-point formulae, have been proved. Let $f \in C^n([0,1],\mathbf{R})$ for some $n \ge 3$ and let $x \in [0,1/2]$. If $n=2r-1$, $r \ge 2$, then

$$\int_0^1 f(t)dt = \frac{1}{2}[f(x)+f(1-x)] - T_{r-1}(f) + \frac{1}{2(2r-1)!}\int_0^1 f^{(2r-1)}(t)F_{2r-1}^x(t)dt, \quad (2.1)$$

while for $n=2r$, $r\ge 2$ we have

$$\int_0^1 f(t)dt = \frac{1}{2}[f(x)+f(1-x)] - T_{r-1}(f) + \frac{1}{2(2r)!}\int_0^1 f^{(2r)}(t)F_{2r}^x(t)dt \quad (2.2)$$

and

$$\int_0^1 f(t)dt = \frac{1}{2}[f(x)+f(1-x)] - T_{r}(f) + \frac{1}{2(2r)!}\int_0^1 f^{(2r)}(t)G_{2r}^x(t)dt. \quad (2.3)$$

Here we define $T_0(f)=0$ and for $1 \le m \le \lfloor n/2 \rfloor$

$$T_m(f) = \sum_{k=1}^{m} \frac{B_{2k}(x)}{(2k)!}[f^{(2k-1)}(1) - f^{(2k-1)}(0)],$$

$$G_n^x(t) = B_n^*(x-t) + B_n^*(1-x-t)$$

and

$$F_n^x(t) = B_n^*(x-t) + B_n^*(1-x-t) - B_n(x) - B_n(1-x),$$

where $B_k(\cdot)$, $k \ge 0$, is the k-th Bernoulli polynomial and $B_k = B_k(0) = B_k(1)(k\ge 0)$ the k-th Bernoulli number. By $B_k^*(\cdot)(k \ge 0)$ we denote the function of period one such that $B_k^*(x) = B_k(x)$ for $0 \le x \le 1$.

It was proved in [11] that $F_n^x(1-t)=(-1)^n F_n^x(t), (-1)^{r-1}F_{2r-1}^x(t)\geq 0, (-1)^r F_{2r}^x(t)\geq 0$ for $x\in\left[0,\frac{1}{2}-\frac{1}{2\sqrt{3}}\right]$ and $t\in[0,1/2]$, and $(-1)^r F_{2r-1}^x(t)\geq 0$, $(-1)^{r-1}F_{2r}^x(t)\geq 0$ for $x\in\left[\frac{1}{2\sqrt{3}},\frac{1}{2}\right]$ and $t\in[0,1/2]$. Also

$$\int_0^1 \left|F_{2r-1}^x(t)\right|dt=\frac{2}{r}\left|B_{2r}\left(\frac{1}{2}-x\right)-B_{2r}(x)\right|,$$
$$\int_0^1 \left|F_{2r}^x(t)\right|dt=2\left|B_{2r}(x)\right|$$

and

$$\int_0^1 \left|G_{2r}^x(t)\right|dt\leq 4\left|B_{2r}(x)\right|.$$

Theorem 2. *Suppose* $f:[0,1]\to\mathbf{R}$ *is* $(2r+2)$*-convex for* $r\geq 2$. *Then for* $x\in\left[0,\frac{1}{2},-\frac{1}{2\sqrt{3}}\right]$ *the inequality*

$$\begin{aligned}\frac{\left|B_{2r}(x)\right|}{(2r)!}f^{(2r)}\left(\frac{1}{2}\right)&\leq(-1)^r\left\{\int_0^1 f(t)dt-\frac{1}{2}\left[f(x)+f(1-x)\right]+T_{r-1}(f)\right\}\\&\leq\frac{\left|B_{2r}(x)\right|}{(2r)!}\frac{f^{(2r)}(0)+f^{(2r)}(1)}{2}\end{aligned}\tag{2.4}$$

holds while for $x\in\left[\frac{1}{2\sqrt{3}},\frac{1}{2}\right]$ *we have*

$$\begin{aligned}\frac{\left|B_{2r}(x)\right|}{(2r)!}f^{(2r)}\left(\frac{1}{2}\right)&\leq(-1)^{r-1}\left\{\int_0^1 f(t)dt-\frac{1}{2}\left[f(x)+f(1-x)\right]+T_{r-1}(f)\right\}\\&\leq\frac{\left|B_{2r}(x)\right|}{(2r)!}\frac{f^{(2r)}(0)+f^{(2r)}(1)}{2}.\end{aligned}\tag{2.5}$$

If f is $(2r+2)$*-concave, the inequalities are reversed.*

Proof. For $x\in\left[0,\frac{1}{2},-\frac{1}{2\sqrt{3}}\right]$ from (2.2) we have

$$\begin{aligned}&(-1)^r\left\{\int_0^1 f(t)dt-\frac{1}{2}\left[f(x)+f(1-x)\right]+T_{r-1}(f)\right\}\\&=(-1)^r\frac{1}{2(2r)!}\int_0^1 f^{(2r)}(t)F_{2r}^x(t)dt=\frac{1}{2(2r)!}\int_0^1 f^{(2r)}(t)\left|F_{2r}^x(t)\right|dt\\&=\frac{1}{2(2r)!}\int_0^1 f^{(2r)}((1-t)\cdot 0+t\cdot 1)\left|F_{2r}^x(t)\right|dt\end{aligned}\tag{2.6}$$

Using the discrete Jensen inequality for the convex function $f^{(2r)}$, we have

$$\int_0^1 f^{(2r)}((1-t)\cdot 0+t\cdot 1)\left|F_{2r}^x(t)\right| dt$$

$$\leq f^{(2r)}(0)\int_0^1 (1-t)\left|F_{2r}^x(t)\right| dt+f^{(2r)}(1)\int_0^1 t\left|F_{2r}^x(t)\right| dt$$

$$=\left|B_{2r}(x)\right|\left(f^{(2r)}(0)+f^{(2r)}(1)\right) \tag{2.7}$$

since $\int_0^1(1-t)\left|F_{2r}^x(t)\right| dt=\frac{1}{2}\int_0^1\left|F_{2r}^x(t)\right| dt$. So, the second inequality in (2.4) follows.

By Jensen's integral inequality we have

$$\int_0^1 f^{(2r)}((1-t)\cdot 0+t\cdot 1)\left|F_{2r}^x(t)\right| dt$$

$$\geq\left(\int_0^1\left|F_{2r}^x(t)\right| dt\right)f^{(2r)}\left(\frac{\int_0^1((1-t)\cdot 0+t\cdot 1)\left|F_{2r}^x(t)\right| dt}{\int_0^1\left|F_{2r}^x(t)\right| dt}\right)$$

$$=2\left|B_{2k}(x)\right|f^{(2r)}\left(\frac{1}{2}\right). \tag{2.8}$$

The first inequality in (2.4) now follows from (2.6).

The proof of inequality (2.5) is similar.

Remark 1. *If in Theorem 2 we chose $x=0, 1/2, 1/3$, we get generalizations of Hadamard inequality for Euler trapezoid, Euler midpoint and Euler two-point Newton-Cotes formulae respectively (see* [2], [3] *and* [9]*).*

Theorem 3. *Suppose $f:[0,1]\to\mathbf{R}$ is n-times differentiable and $x\in\left[0,\frac{1}{2}-\frac{1}{2\sqrt{3}}\right]\cup\left[\frac{1}{2\sqrt{3}},\frac{1}{2}\right]$.*

(a) If $\left|f^{(n)}\right|^q$ is convex for some $q\geq 1$, then for $n=2r-1$, $r\geq 2$, we have

$$\left|\int_0^1 f(t)dt-\frac{1}{2}\left[f(x)+f(1-x)\right]+T_{r-1}(f)\right|$$

$$\leq\frac{2}{(2r)!}\left|B_{2r}\left(\frac{1}{2}-x\right)-B_{2r}(x)\right|\left[\frac{\left|f^{(2r-1)}(0)\right|^q+\left|f^{(2r-1)}(1)\right|^q}{2}\right]^{1/q}. \tag{2.9}$$

If $n=2r$, $r\geq 2$, then

$$\left|\int_0^1 f(t)dt-\frac{1}{2}\left[f(x)+f(1-x)\right]+T_{r-1}(f)\right|\leq\frac{\left|B_{2r}(x)\right|}{(2r)!}\left[\frac{\left|f^{(2r)}(0)\right|^q+\left|f^{(2r)}(1)\right|^q}{2}\right]^{1/q} \tag{2.10}$$

and we also have

$$\left|\int_0^1 f(t)dt-\frac{1}{2}\left[f(x)+f(1-x)\right]+T_{r}(f)\right|\leq\frac{2\left|B_{2r}(x)\right|}{(2r)!}\left[\frac{\left|f^{(2r)}(0)\right|^q+\left|f^{(2r)}(1)\right|^q}{2}\right]^{1/q} \tag{2.11}$$

(b) If $\left|f^{(n)}\right|^q$ is concave, then for $n=2r-1$, $r\geq 2$, we have

$$\left|\int_0^1 f(t)dt-\frac{1}{2}\left[f(x)+f(1-x)\right]+T_{r-1}(f)\right|$$

$$\leq\frac{2}{(2r)!}\left|\left[B_{2r}\left(\frac{1}{2}-x\right)-B_{2r}(x)\right]f^{(2r-1)}\left(\frac{1}{2}\right)\right|. \tag{2.12}$$

If $n=2r$, $r\geq 2$, then

$$\left|\int_0^1 f(t)dt-\frac{1}{2}\left[f(x)+f(1-x)\right]+T_{r-1}(f)\right|\leq\frac{1}{(2r)!}\left|B_{2r}(x)\cdot f^{(2r)}\left(\frac{1}{2}\right)\right| \tag{2.13}$$

and we also have

$$\left|\int_0^1 f(t)dt-\frac{1}{2}\left[f(x)+f(1-x)\right]+T_{r}(f)\right|\leq\frac{2}{(2r)!}\left|B_{2r}(x)\cdot f^{(2r)}\left(\frac{1}{2}\right)\right|. \tag{2.14}$$

Proof. First, let $n=2r-1$ for some $r\geq 2$. Then for convex function $\left|f^{(2r)}\right|^q$ by Hölder inequality and using Jesen inequality we have

$$\left|\int_0^1 f(t)dt-\frac{1}{2}\left[f(x)+f(1-x)\right]+T_{r-1}(f)\right|$$

$$\leq\frac{1}{2(2r-1)!}\int_0^1\left|F_{2r-1}^x(t)\right|\cdot\left|f^{(2r-1)}(t\cdot 1+(1-t)\cdot 0)\right|dt$$

$$\leq\frac{1}{2(2r-1)!}\left(\int_0^1\left|F_{2r-1}^x(t)\right|\,dt\right)^{1-1/q}\times\left(\int_0^1\left|F_{2r-1}^x(t)\right|\cdot\left|f^{(2r-1)}(t\cdot 1+(1-t)\cdot 0)\right|^q dt\right)^{1/q}$$

$$\leq \frac{1}{2(2r-1)!}\left(\int_0^1 \left|F_{2r-1}^x(t)\right| dt\right)^{1-1/q} \times \left(\int_0^1 \left|F_{2r-1}^x(t)\right| \cdot \left[t\left|f^{(2r-1)}(1)\right|^q + (1-t)\left|f^{(2r-1)}(0)\right|^q\right] dt\right)^{1/q}$$

$$= \left(\frac{1}{2(2r-1)!}\int_0^1 \left|F_{2r-1}^x(t)\right| dt\right)^{1-1/q} \times$$

$$\times\left[\frac{1}{2(2r-1)!}\left|f^{(2r-1)}(1)\right|^q \int_0^1 t\left|F_{2r-1}^x(t)\right| dt + \frac{1}{2(2r-1)!}\left|f^{(2r-1)}(0)\right|^q \int_0^1 (1-t)\left|F_{2r-1}^x(t)\right| dt\right]^{1/q}$$

$$= \left(\frac{2}{(2r)!}\left|B_{2r}\left(\frac{1}{2}-x\right) - B_{2r}(x)\right|\right)^{1-1/q} \times$$

$$\left[\frac{1}{(2r)!}\left|B_{2r}\left(\frac{1}{2}-x\right) - B_{2r}(x)\right|\left|f^{(2r-1)}(1)\right|^q + \frac{1}{(2r)!}\left|B_{2r}\left(\frac{1}{2}-x\right) - B_{2r}(x)\right|\left|f^{(2r-1)}(0)\right|^q\right]^{1/q}$$

$$= \frac{2}{(2r)!}\left|B_{2r}\left(\frac{1}{2}-x\right) - B_{2r}(x)\right|\left[\frac{\left|f^{(2r-1)}(0)\right|^q + \left|f^{(2r-1)}(1)\right|^q}{2}\right]^{1/q}.$$

On the other hand, if $\left|f^{(2r-1)}\right|^q$ is concave, then

$$\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f(x)+f(1-x)\right] + T_{r-1}(f)\right|$$

$$\leq \frac{1}{2(2r-1)!}\int_0^1 \left|F_{2r-1}^x(t)\right| \cdot \left|f^{(2r-1)}(t)\right| dt$$

$$\leq \frac{1}{2(2r-1)!}\left(\int_0^1 \left|F_{2r-1}^x(t)\right| dt\right) \times \left[\left|f^{(2r-1)}\left(\frac{\int_0^1 \left|F_{2r-1}^x(t)\right|((1-t)\cdot 0 + t\cdot 1)dt}{\int_0^1 \left|F_{2r-1}^x(t)\right| dt}\right)\right|^q\right]^{1/q}$$

$$= \frac{2}{(2r)!}\left|\left[B_{2r}\left(\frac{1}{2}-x\right) - B_{2r}(x)\right] f^{(2r-1)}\left(\frac{1}{2}\right)\right|$$

so the inequality (2.9) and (2.12) are completely proved.

The proofs of the inequalities (2.10), (2.13), (2.11) and (2.14) are similar.

Remark 2. *For (2.12) to be satisfied it is enough to suppose that* $\left|f^{(2r-1)}\right|$ *is a concave function. If* $|g|^q$ *is concave na* $[0,1]$ *for some* $q \geq 1$*, then for* $x, y \in [0,1]$ *and* $\lambda \in [0,1]$

$$|g(\lambda x+(1-\lambda)y|^q \geq \lambda|g(x)|^q+(1-\lambda)|g(y)|^q \geq (\lambda|g(x)|^q+(1-\lambda)|g(y)|)^q$$

by the power-mean inequality. Therefore $|g|$ *is also concave on* $[0,1]$.

Remark 3. *If in Theorem 3 we chose* $x=0, 1/2, 1/3$, *we get generalizations of Dragomir-Agarwal inequality for Euler trapezoid, Euler midpoint and Euler two-point Newton-Cotes formulae respectively (see* [2], [3], [9] *and* [10]*).*

The resultant formulae in Theorems 2 and 3 when r=2 are of special interest, so we isolate them as corollaries.

Corollary 2. *If* $f:[0,1]\to \mathbf{R}$ *is 6-convex, then for* $x\in\left[0,\frac{1}{2},-\frac{1}{2\sqrt{3}}\right]$ *we have*

$$\frac{1}{24}\left|x^4-2x^3+x^2-\frac{1}{30}\right|f^{(4)}\left(\frac{1}{2}\right)$$

$$\leq \int_0^1 f(t)dt-\frac{1}{2}[f(x)+f(1-x)]+\frac{1}{12}[f'(1)-f'(0)]$$

$$\leq \frac{1}{24}\left|x^4-2x^3+x^2-\frac{1}{30}\right|\frac{f^{(4)}(0)+f^{(4)}(1)}{2}.$$

while for $x\in\left[\frac{1}{2\sqrt{3}},\frac{1}{2}\right]$ *the inequalities*

$$\frac{1}{24}\left|x^4-2x^3+x^2-\frac{1}{30}\right|f^{(4)}\left(\frac{1}{2}\right)$$

$$\leq \frac{1}{2}[f(x)+f(1-x)]-\int_0^1 f(t)dt-\frac{1}{12}[f'(1)-f'(0)]$$

$$\leq \frac{1}{24}\left|x^4-2x^3+x^2-\frac{1}{30}\right|\frac{f^{(4)}(0)+f^{(4)}(1)}{2}.$$

hold.

If f is 6-concave, the reversed inequalities apply.

Corollary 3. *Suppose* $f:[0,1]\to\mathbf{R}$ *is 4-times differentiable and* $x\in\left[0,\frac{1}{2},-\frac{1}{2\sqrt{3}}\right]\cup\left[\frac{1}{2\sqrt{3}},\frac{1}{2}\right]$.

(a) If $\left|f^{(3)}\right|^q$ *is convex for some* $q \geq 1$*, then*

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(x)+f(1-x)] + \frac{1}{12}[f'(1)-f'(0)]\right|$$

$$\leq \frac{1}{12}\left|2x^3 - \frac{3}{2}x^2 + \frac{1}{16}\right|\left[\frac{\left|f^{(3)}(0)\right|^q + \left|f^{(3)}(1)\right|^q}{2}\right]^{1/q}$$

and if $\left|f^{(4)}\right|^q$ *is convex for some* $q \geq 1$*, then*

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(x)+f(1-x)] + \frac{1}{12}[f'(1)-f'(0)]\right|$$

$$\leq \frac{1}{24}\left|x^4 - 2x^3 + x^2 - \frac{1}{30}\right|\left[\frac{\left|f^{(4)}(0)\right|^q + \left|f^{(4)}(1)\right|^q}{2}\right]^{1/q}.$$

(b) If $\left|f^{(3)}\right|^q$ *is concave, then*

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(x)+f(1-x)] + \frac{1}{12}[f'(1)-f'(0)]\right|$$

$$\leq \frac{1}{12}\left|\left[2x^3 - \frac{3}{2}x^2 + \frac{1}{16}\right]f^{(3)}\left(\frac{1}{2}\right)\right|$$

and if $\left|f^{(4)}\right|$ *is concave, then*

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(x)+f(1-x)] + \frac{1}{12}[f'(1)-f'(0)]\right|$$

$$\leq \frac{1}{24}\left|\left[x^4 - 2x^3 + x^2 - \frac{1}{30}\right]f^{(4)}\left(\frac{1}{2}\right)\right|.$$

Note that inequalities in Theorems 2 hold for 3 are for $r \geq 2$. Now, we will give some results of the same type in the case when $r=1$.

Theorem 4. *Suppose* $f:[0,1] \to \mathbf{R}$ *is 4-convex. then for* $x \in [0,1/4]$ *the following inequalities hold*

$$\left[\frac{-6x^2+6x-1}{24} + \frac{1}{6}(1-4x)^{3/2}\right]f''\left(\frac{1}{2}\right) \leq \frac{1}{2}[f(x)+f(1-x)] - \int_0^1 f(t)dt$$

$$\leq \left[\frac{-6x^2+6x-1}{24} + \frac{1}{6}(1-4x)^{3/2}\right]\frac{f''(1)+f''(0)}{2},$$

while for $x \in [1/4, 1/2]$ *we have*

$$\frac{-6x^2+6x-1}{24} f''\left(\frac{1}{2}\right) \le \left[+ \frac{1}{6}(1-4x)^{3/2} \right] \le \int_0^1 f(t)dt - \frac{1}{2}\left[f(x)+f(1-x)\right]$$

$$\le \frac{-6x^2+6x-1}{24} \cdot \frac{f''(1)+f''(0)}{2}.$$

Proof. It was proved in [11] that for $x \in [0, 1/4]$ we have

$$\int_0^1 \left|F_2^x(t)\right| dt = \frac{-6x^2+6x-1}{6} + \frac{2}{3}(1-4x)^{3/2},$$

while for $x \in [1/4, 1/2]$

$$\int_0^1 \left|F_2^x(t)\right| dt = \frac{-6x^2+6x-1}{6}.$$

So, using identity (2.2) by to the proof of Theorem 2 we get above inequalities. See also results from Corollary 1.

Theorem 5. *Suppose* $f : [0,1] \to \mathbf{R}$ *is 2-times differentiable.*

(a) If $|f'|^q$ *is convex for some* $q \ge 1$*, then for* $x \in [0, 1/2]$ *we have*

$$\left| \int_0^1 f(t)dt - \frac{1}{2}\left[f(x)+f(1-x)\right] \right| \le \frac{8x^2-4x+1}{4} \left[\frac{|f'(0)|^q + |f'(1)|^q}{2} \right]^{1/q}$$

If $|f''|^q$ *is convex for some* $q \ge 1$ *and* $x \in [0, 1/4]$*, then*

$$\left| \int_0^1 f(t)dt - \frac{1}{2}\left[f(x)+f(1-x)\right] \right|$$

$$\le \left[\frac{-6x^2+6x-1}{24} + \frac{1}{6}(1-4x)^{3/2} \right] \left[\frac{|f''(0)|^q + |f''(1)|^q}{2} \right]^{1/q}.$$

while for $x \in [1/4, 1/2]$ *we have*

$$\left| \int_0^1 f(t)dt - \frac{1}{2}\left[f(x)+f(1-x)\right] \right| \le \left[\frac{-6x^2+6x-1}{24} \right] \left[\frac{|f''(0)|^q + |f''(1)|^q}{2} \right]^{1/q}$$

(b) If $|f'|$ is concave for some $q \geq 1$, then for $x \in [0, 1/2]$ we have

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(x)+f(1-x)]\right| \leq \frac{8x^2-4x+1}{4}\left|f'\left(\frac{1}{2}\right)\right|.$$

If $|f''|$ is concave for some $q \geq 1$ and $x \in [0, 1/4]$, then

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(x)+f(1-x)]\right| \leq \left[\frac{-6x^2+6x-1}{24}+\frac{1}{6}(1-4x)^{3/2}\right]\left|f''\left(\frac{1}{2}\right)\right|,$$

while for $x \in [1/4, 1/2]$ we have

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(x)+f(1-x)]\right| \leq \left[\frac{-6x^2+6x-1}{24}\right]\left|f''\left(\frac{1}{2}\right)\right|.$$

Proof. It was proved in [11] that for $x \in [0, 1/2]$

$$\int_0^1 |F_1^x(t)|dt = \frac{8x^2-4x+1}{2},$$

so, using identity (2.1) and similar by the proof of Theorem 3 we get first inequalities in (a) and (b). Second inequality in (a) and (b) we prove similarly. See also results from Theorem 1 and Corollary 1.

Remark 4. *For $x=0$, $x=1/3$ and $x=1/2$ in above theorems we get the results from* [9] *and* [10].

For $x=1/4$ we get two-point Maclaurin formula and then we have

$$\frac{1}{192}f^{(2)}\left(\frac{1}{2}\right) \leq \int_0^1 f(t)dt - \frac{1}{2}\left[f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)\right] \leq \frac{1}{192}\,\frac{f^{(2)}(0)+f^{(2)}(1)}{2}.$$

If $|f'|^q$ is convex for some $q \geq 1$, then

$$\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)\right]\right| \leq \frac{1}{8}\left[\frac{|f'(0)|^q+|f'(1)|^q}{2}\right]^{1/q}$$

and if $|f^{(2)}|^q$ is convex for some $q \geq 1$, then

$$\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)\right]\right| \leq \frac{1}{192}\left[\frac{|f^{(2)}(0)|^q+|f^{(2)}(1)|^q}{2}\right]^{1/q}.$$

If $|f'|$ is concave, then

$$\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)\right]\right| \leq \frac{1}{8}\left|f'\left(\frac{1}{2}\right)\right|$$

and if $|f^{(2)}|$ is concave for some $q \geq 1$, then

$$\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)\right]\right| \leq \frac{1}{192}\left|f^{(2)}\left(\frac{1}{2}\right)\right|.$$

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Hadamaradova i Dragomir-Agarwalova nejednakost, općenite Eulerove formule dviju točaka i konveksne funkcije

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Abstract. Korištene su općenite Eulerove formule dviju točaka za funkcije s raznim svojstvima konveksnosti ili konkavnosti za izvođenje nekih nejednakosti u numeričkoj integraciji.

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