ČEBYŠEV SETS IN HYPERSPACES OVER A MINKOWSKI SPACE

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Dedicated to Professor Sibe Mardešić on the occasion of his 80th birthday

Abstract. In this paper we extend our previous results on Čebyšev sets in hyperspaces over a Euclidean n-space to hyperspaces over a Minkowski space.

The notion of Čebyšev set has been studied mainly for normed linear spaces (see [4, 13]), but it can be considered for arbitrary metric spaces (see [13, Appendix II]). A subset $A$ of a metric space $(X, \rho)$ is a Čebyšev set in this space provided that for every point of $X$ there is a unique nearest point in $A$. The function $\xi_A : X \to A$ which assigns to $x \in X$ the unique nearest point of $A$ is called metric projection.

Čebyšev sets in $K^0_n$ (the space of convex bodies in $\mathbb{R}^n$), $K^n_n$ (the space of nonempty compact convex sets) and $O^n_n$ (the space of compact, strictly convex sets), all endowed with the Hausdorff metric $d_H$ associated with the Euclidean metric, were studied in [2, 5].

The present paper is closely related to [2] and [5]. Its purpose is to extend previous results on hyperspaces over a Minkowski space.

1. Preliminaries

Let $\| \cdot \|_o$ be the Euclidean norm in $\mathbb{R}^n$:

$$\|x\|_o := \sqrt{x \cdot x},$$

where $\cdot$ is the usual scalar product.

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Let $B^n$ and $S^{n-1}$ be the Euclidean unit ball and unit sphere:

$$B^n := \{ x \in \mathbb{R}^n \mid \|x\|_0 \leq 1 \}, \quad \text{and} \quad S^{n-1} := \{ x \in \mathbb{R}^n \mid \|x\|_0 = 1 \}.$$ 

As usually, bd, cl, and int are boundary, closure, and interior, and conv$A$ is the convex hull of $A$. For distinct $a, b$, let $\Delta(a, b)$ be the segment with endpoints $a, b$.

For any subset $A$ of $\mathbb{R}^n$, we shall use the symbol $[A]$ to denote the set of singletons in $A$:

$$[A] := \{ \{x\} \mid x \in A \}.$$

Thus, in particular, $[\mathbb{R}^n]$ is the set of singletons in $\mathbb{R}^n$.

We shall use the symbol $\subset$ for strict inclusion:

$$X \subset Y \iff X \subseteq Y \text{ and } X \neq Y.$$ 

A Minkowski space is a finite dimensional Banach space $(M, \| \cdot \|)$ (see [14]). Thus, up to an isomorphism, every $n$-dimensional Minkowski space is a normed linear space $(\mathbb{R}^n, \| \cdot \|)$.

Let $B$ be the unit ball determined by $\| \cdot \|$:

$$B := \{ x \in \mathbb{R}^n \mid \|x\| \leq 1 \}.$$ 

Then $B$ is a convex body symmetric at 0. Conversely, every convex body $A$ symmetric at 0 determines a norm, $\| \cdot \|_A$, usually referred to as the Minkowski functional:

$$\|x\|_A := \inf \{ t \in \mathbb{R}_+ \mid x \in tA \}$$

(see [14, p. 17]). In particular, $\| \cdot \|_B = \| \cdot \|$ and the unit ball determined by $\| \cdot \|_B$ coincides with $B$.

Note that if the unit ball is strictly convex, then so are all the balls in $(M, \| \cdot \|)$.

We shall need the following lemma.

**Lemma 1.1.** Let $x_1, x_2 \in \mathbb{R}^n$ and $\alpha \geq \frac{1}{2} \|x_1 - x_2\|$; let $(x_1 + \alpha B) \cap (x_2 + \alpha B)$ not be a singleton. If $\alpha_0$ is the radius of the smallest ball, $B_{\alpha_0}$, with centre $x_0 = \frac{1}{2}(x_1 + x_2)$, containing $(x_1 + \alpha B) \cap (x_2 + \alpha B)$, then

(i) $\alpha_0 \leq \alpha$;

(ii) $\alpha_0 < \alpha$ if bd$B$ does not contain any segment parallel to $x_1 - x_2$.

**Proof.** Let $p \in (x_1 + \alpha B) \cap (x_2 + \alpha B) \cap \text{bd}(x_0 + \alpha_0 B)$. Then there exist $b_0 \in \text{bd}B$ and distinct $b_1, b_2 \in B$ such that

$$p = x_0 + \alpha_0 b_0 = x_1 + \alpha b_1 = x_2 + \alpha b_2.$$

Thus $\alpha(b_1 + b_2) = \alpha_0 b_0$, whence $\frac{\alpha_0}{\alpha} = \frac{\|b_1 + b_2\|}{2} \leq 1$. This proves (i).

If bd$B$ does not contain $\Delta(b_1, b_2)$, then the inequality is strict. This proves (ii).
Let us first consider the family $C^n$ of nonempty compact subsets of $\mathbb{R}^n$ and the family $C^0_n$ of compact bodies (a member $A$ of $C^n$ is a body whenever $A = \text{cl int} A$). Let $\varrho^B_H$ be the Hausdorff metric in $C^n$ associated with the metric $\| \cdot \|$ (compare [14]):

\begin{align}
(1.1) \quad \varrho^B_H(A_1, A_2) := \max\{ \varrho^B_H(A_1, A_2), \varrho^B_H(A_2, A_1) \},
\end{align}

where the oriented Hausdorff metric $\varrho^B_H$ is defined by the formula

\begin{align}
(1.2) \quad \varrho^B_H(A_1, A_2) := \inf\{ \varepsilon > 0 \mid A_1 \subseteq A_2 + \varepsilon B \}
\end{align}

for every $A_1, A_2 \in C^n$.

Since

\begin{align}
(1.3) \quad \varrho^B_H(A_1, A_2) = \sup_{x \in A_1} \varrho^B(x, A_2),
\end{align}

it follows that

\begin{align}
\varrho^B_H(A_1, A_2) = \max\{ \sup_{x_1 \in A_1} \varrho^B(x_1, A_2), \sup_{x_2 \in A_2} \varrho^B(x_2, A_1) \}.
\end{align}

Proof of (1.3) is the same as for the Euclidean case (see [8, 1.2.2]).

In what follows, $C^B, C^B_0, K^B, K^B_0$ and $O^B$ are the families of nonempty compact subsets of $\mathbb{R}^n$, compact bodies, nonempty compact convex subsets, convex bodies, and strictly convex compact sets respectively, in each case endowed with $\varrho^B_H$.

2. INVARIANT ČEBYŠEV SETS IN $K^B_0$ AND IN $K^B$

Let us recall the notion of the minimal ring of a convex body (see [1, 7]). Let $A \in K^B_0$. For any $x \in A$, let $R_A(x)$ and $r_A(x)$ be, respectively, the radius of the smallest ball with centre $x$ containing $A$ and the radius of the biggest ball with centre $x$ contained in $A$. By a theorem of Bárány (proved much earlier by Bonnesen [3] for $n = 2$), the function $f_A : A \to \mathbb{R}_+$ defined by

\begin{align}
f_A(x) := R_A(x) - r_A(x)
\end{align}

has a unique minimizer $x_0$, which belongs to $\text{int} A$. This point $x_0$ is called the centre of the minimal ring of $A$; we shall denote it by $c(A)$.

Let

\begin{align}
R(A) := R_A(c(A)) \quad \text{and} \quad r(A) := r_A(c(A)).
\end{align}

Recall that for any two nonempty subsets $A_0, A_1$ of $\mathbb{R}^n$ the affine segment $\Delta(A_0, A_1)$ is defined by

\begin{align}
\Delta(A_0, A_1) := \{(1 - t)A_0 + tA_1 \mid t \in [0, 1]\};
\end{align}

a family $X \subset K^n$ is affine convex provided that $\Delta(A_0, A_1) \subset X$ whenever $A_0, A_1 \in X$.

According to [2, Theorem 2.2],
The family $B^n$ of Euclidean balls in $\mathbb{R}^n$ is an affine convex Čebyšev set in $K^n_0$; for any $A \in K^n_0$ the nearest ball has centre $c(A)$ and radius $\frac{1}{2}(R(A) + r(A))$. The metric projection $\xi_{B^n}$ is continuous.

The family $B^n$ is invariant under the group $\text{Sim}$ of similarities of $\mathbb{R}^n$.

Let now $\text{Iso}_B$ be the group of isometries of $(\mathbb{R}^n, \| \cdot \|_B)$. This group consists of all the affine transformations of $\mathbb{R}^n$ which map the unit ball $B$ onto its translate.

Let, further, $\text{Sim}_B$ be the group of similarities of $(\mathbb{R}^n, \| \cdot \|_B)$:

$$h \in \text{Sim}_B \iff \exists f \in \text{Iso}_B \exists t > 0 \quad h = tf.$$ 

For any subset $A$ of $\mathbb{R}^n$, let $I_B(A)$ be the group of Minkowski self-isometries of $A$:

$$I_B(A) := \{ f \in \text{Iso}_B \mid f(A) = A \}.$$ 

We shall consider the family $B$ of the Minkowski balls in $(\mathbb{R}^n, \| \cdot \|_B)$:

$$B := \{ x + tB \mid x \in \mathbb{R}^n, \ t > 0 \}.$$ 

The following is evident.

**Proposition 2.1.** The family $B$ is invariant under $\text{Sim}_B$.

Let us note

**Proposition 2.2.** The family $B$ is affine convex.

**Proof.** For every $t \in [0, 1]$ and every two balls $B_1, B_2$ in $(\mathbb{R}^n, \| \cdot \|_B)$, there exist $a_1, a_2 \in \mathbb{R}^n$ and $\alpha_1, \alpha_2 > 0$ such that

$$(1 - t)B_1 + tB_2 = (1 - t)(a_1 + \alpha_1 B) + t(a_2 + \alpha_2 B) = a + \alpha B,$$

where $a = (1 - t)a_1 + t\alpha_2$ and $\alpha = (1 - t)\alpha_1 + t\alpha_2$. Thus, the affine segment $\Delta(B_1, B_2)$ is a subset of $B$. □

We shall first consider hyperspace $K^n_0$ of convex bodies, over a Minkowski space $(\mathbb{R}^n, \| \cdot \|_B)$.

Carla Peri in [9] extended in the natural way the notion of minimal ring to arbitrary Minkowski spaces.\(^2\) In [10] among other results she obtained the following:

*If the unit ball $B$ in a Minkowski space is strictly convex, then every convex body $A$ has a unique minimal ring with respect to $B$.*

We refer to this unique minimal ring as $B$-minimal ring of $A$ and use the symbols $c^B(A), R^B(A)$, and $r^B(A)$ to denote centre, outer radius, and inner radius of the $B$-minimal ring of $A$.

\(^{1}\)It may happen that the group $\text{Iso}_B \cap GL(n)$ of linear Minkowski isometries consists of only two elements: identity and reflection at 0 (see [14, p. 14-17]).

\(^{2}\)She uses the name “minimal shell” for minimal ring.
The following theorem combined with Proposition 2.2 is a counterpart of [2, Theorem 2.2] (for continuity of metric projection, see Remark 4.4 and Proposition 4.5 below).

**Theorem 2.3.** Let \((\mathbb{R}^n, \| \cdot \|_B)\) be a Minkowski space with strictly convex unit ball \(B\). Then

(i) the family \(B\) is an affine convex Čebyšev set in \(K^B_0\). For every convex body \(A\), the ball nearest to \(A\) in the sense of \(\mathcal{P}^B_H\) has centre \(c^B(A)\) and radius \(\frac{1}{2}(r^B(A) + R^B(A))\).

(ii) for every convex body \(A\),

\[
\mathcal{P}^B_H(A, \xi_B(A)) = \frac{1}{2}(R^B(A) - r^B(A)).
\]

**Proof.** The proof is analogous to that of [2, Theorem 2.2].

We are now looking for a counterpart of [2, Theorem 2.3], which states that every Čebyšev set in \(K^B_0\) invariant under \(\text{Sim}\) contains \(B^n\).

Notice that the proof of that theorem is based on the following characterization of Euclidean balls:

*If a convex body \(A\) in \(\mathbb{R}^n\) is invariant under all linear isometries of \(\mathbb{R}^n\), then \(A\) is a ball with centre 0.*

Generally, a Minkowski ball \(tB\) cannot be characterized as a convex body in \(\mathbb{R}^n\) invariant under the linear Minkowski isometries. For instance, if \(B\) is a cube, the group of linear isometries is a discrete group and there exist many centrally symmetric convex bodies invariant under these isometries. Thus a Minkowski space counterpart of [2, Theorem 2.3] must be proved differently.

For a Minkowski space \((\mathbb{R}^n, \| \cdot \|_B)\), let

\[
C_B := \{ C \in K_0^n \mid \exists x \in \mathbb{R}^n \ I_B(C) = I_B(x + B) \}.
\]

So, \(C_B\) consists of all the convex bodies with the same group of Minkowski self-isometries as the balls.

**Theorem 2.4.** Every Čebyšev set \(\mathcal{X}\) in \(K^B_0\) invariant under \(\text{Sim}_B\) contains the orbit \(\text{Sim}_B(C)\) for some \(C \in C_B\).

**Proof.** It is easy to see that \(C_B\) is invariant under \(\text{Sim}_B\). Thus it suffices to prove that

\[C_B \cap \mathcal{X} \neq \emptyset.\]

If \(I_B(A) = I_B(x + B)\) for some \(A \in \mathcal{X}\) and some \(x \in \mathbb{R}^n\), then \(A \in C_B \cap \mathcal{X} \neq \emptyset\). Assume that for every \(A \in \mathcal{X}\) and every \(x\)

\[
I_B(A) \subset I_B(x + B),
\]

and suppose, to the contrary, that \(C_B \cap \mathcal{X} = \emptyset\). Take \(C \in C_B\) and let \(A\) be the element of \(\mathcal{X}\) nearest to \(C\) with respect to \(\mathcal{P}^B_H\). Then by (2.2), there exists \(f \in I_B(x + B)\) for some \(x\) such that \(f(A) \neq A\).
Since $I_B(x + B) \subseteq \text{Iso}_B$ for every $x$, it follows that
\[ \varphi_B^0(C, A) = \varphi_B^0(f(C), f(A)) = \varphi_B^0(C, f(A)), \]
contrary to the assumption that $X$ is a Čebyšev set in $\mathcal{K}_B^n$.

We shall now consider the hyperspace $\mathcal{K}^B$ of nonempty compact convex sets over a Minkowski space $(\mathbb{R}^n, \| \cdot \|_B)$. For any nonempty compact convex set $A$, define $B$-Čebyšev centre of $A$ to be the centre $x$ of a minimal ball $x + B$ (i.e., a ball with minimal $B$-radius) containing $A$. If $B$ is strictly convex, then such a point is unique ([6]); we denote it by $\hat{c}^B(A)$. Generally, the point $\hat{c}^B(A)$ need not belong to $A$; as is well known, $\hat{c}^B(A)$ belongs to $A$ for every $A \in \mathcal{K}^n$ if and only if either $n = 2$ or $B = B^n$ (compare [6, p. 139]). We denote by $\hat{R}^B(A)$ the $B$-radius of the minimal ball with centre $\hat{c}^B(A)$ containing $A$.

The following theorem is a counterpart of [2, Theorem 3.3].

**Theorem 2.5.** Let $(\mathbb{R}^n, \| \cdot \|_B)$ be a Minkowski space with strictly convex unit ball $B$. Then

(i) $[\mathbb{R}^n]$ and $B[\mathbb{R}^n]$ are affine convex Čebyšev sets in $\mathcal{K}^B$, invariant under $\text{Sim}_B((\mathbb{R}^n, \| \cdot \|_B))$; the metric projections are defined by the formulae
\begin{equation}
\xi_{[\mathbb{R}^n]}(A) := \{ \hat{c}^B(A) \}
\end{equation}
and
\begin{equation}
\xi_{B[\mathbb{R}^n]}(A) := \begin{cases} 
\hat{c}^B(A) + \frac{1}{2}(r^B(A) + R^B(A))B & \text{if } \dim A = n, \\
\hat{c}^B(A) + \frac{1}{2}\hat{R}^B(A)B & \text{if } 0 < \dim A < n, \\
\{a\} & \text{if } A = \{a\}
\end{cases}
\end{equation}

(ii) both metric projections are continuous.

**Proof.** The proof of (i) is analogous to those of [2, Theorems 3.2 and 3.3]. For (2.4) we apply Theorem 2.3 above.

(ii): Since for every two Minkowski spaces of the same dimension the associated Hausdorff metrics are uniformly topologically equivalent (see [14, p. 61]), it follows that for every Minkowski space $(\mathbb{R}^n, \| \cdot \|_B)$ the space $\mathcal{K}^B$ is finitely compact, as it is for $\mathcal{K}^n$. Thus, metric projection on any Čebyšev set in $\mathcal{K}^B$ is continuous (compare [2, Proposition 1.6]).

As a counterpart of [2, Theorem 3.4] we obtain the following analogue of Theorem 2.4 above.

**Theorem 2.6.** Let $\mathcal{C}_B$ be defined by (2.1). Then every Čebyšev set $X$ in $\mathcal{K}^B$ invariant under $\text{Sim}_B$ contains $[\mathbb{R}^n] \cup \text{Sim}_B(C)$ for some $C \in \mathcal{C}_B$.

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3A Čebyšev centre is sometimes referred to as Čebyšev point; see [2, 8].
3. Families of translates in $K^B$ and $O^B$

The family of singletons, $[\mathbb{R}^n]$, which is an example of Čebyšev set in $K^B$ when the unit ball $B$ is strictly convex (see Theorem 2.5(i)), is the simplest example of a family of translates in $K^B$. As was proved in [5] (see Proposition 3.5 and Remark 3.6), in the Euclidean case, this is the only possible example of a family $f A + x_j \in \mathbb{R}^n$ which is a Čebyšev set in $K^n$; if the set $A$ is not a singleton, the family of its translates is a Čebyšev set in $O^n$ but generally not in $K^n$.

The following theorem is a “Minkowski counterpart” of [2, Theorem 4.5], which concerns possible Čebyšev subsets of $[\mathbb{R}^n]$.

**Theorem 3.1.** For a Minkowski space $(\mathbb{R}^n, \| \cdot \|)$ with the unit ball $B$ the following are equivalent:

(i) the ball $B$ is strictly convex;

(ii) for every convex, closed subset $T$ of $\mathbb{R}^n$ with nonempty interior, the set $[T]$ of singletons is a Čebyšev set in $K^B$;

(iii) there exists a convex, closed subset $T$ of $\mathbb{R}^n$ with nonempty interior such that $[T]$ is a Čebyšev set in $K^B$.

**Proof.** (i) $\implies$ (ii). We can follow the proof of [2, Theorem 4.5], because if $B$ is strictly convex, then in view of Lemma 1.1, for two balls $x_1 + \alpha B$ and $x_2 + \alpha B$ with nonempty intersection, the ball with centre $\frac{1}{2}(x_1 + x_2)$, circumscribed over the intersection, has radius smaller than $\alpha$.

(ii) $\implies$ (iii) is evident.

(iii) $\implies$ (i). Suppose, to the contrary, that $B$ is not strictly convex and let $T$ be as in (iii). Take an $x \in \text{int}T$. There exists an $\alpha > 0$ such that $B' := x + \alpha B \subseteq T$. Since $B'$, as a homothet of $B$, is not strictly convex, its boundary contains a segment $\Delta(b_1, b_2)$. Since $x$ is the centre of $B'$, also $\Delta(2x - b_1, 2x - b_2) \subseteq \text{bd}B'$.

Let $b_0 := \frac{1}{2}(b_1 + b_2)$ and $v := b_1 - b_0$. Take a test set $X := \Delta(b_0, 2x - b_0)$ and let

$$x_1 := x + v \text{ and } x_2 := x - v.$$  

It is easy to see that for every $t \in [0, 1]$

$$\varphi_H^B((1 - t)x_1 + tx_2), X) \leq \alpha$$

and

$$\varphi_H^B(X, [(1 - t)x_1 + tx_2]) = \alpha.$$  

Thus

$$\varphi_H^B(X, [(1 - t)x_1 + tx_2]) = \alpha$$

for every $t \in [0, 1]$.

On the other hand, for every $y \in T$

$$\varphi_H^B(X, \{y\}) \geq \alpha,$$
whence all the elements of $\Delta(\{x_1\}, \{x_2\})$ are nearest to $X$, i.e., $[T]$ is not a Čebyšev set.

The following example shows that the assumption $\text{int}T \neq \emptyset$ is essential for the implication (iii) $\implies$ (i) above.

**Example 3.2.** Let $T := \Delta(a, -a)$ for $a = (\frac{1}{2}, 0, \ldots, 0)$ and let $B := B^n \cap \{x = (x_1, \ldots, x_n) \mid x_1 \in [-\frac{1}{2}, \frac{1}{2}]\}$. Take a test set $X \in K^n$ and let $g_H(X, \{b\}) = \alpha > 0$ for some $b, b' \in T$. Then $X \subseteq (b + \alpha B) \cap (b' + aB)$, whence by Lemma 1.1 there exists $\alpha_0 < \alpha$ such that $b_0 + \alpha_0 B \supset X$ for $b_0 = \frac{1}{2}(b + b')$. Thus $g_H(X, \{b_0\}) \leq \alpha_0 < \alpha$. Hence $[T]$ is a Čebyšev set in $K^n$, though $B$ is not strictly convex.

We now pass to families of translates in $O^n$ (see [5]). We will need the following well known result:

**Lemma 3.3.** If $A_1, A_2 \in O^n$, then $A_1 + A_2 \in O^n$.

**Theorem 3.4.** For a Minkowski space $(\mathbb{R}^n, \|\cdot\|_B)$ the following are equivalent:

(i) the ball $B$ is strictly convex;

(ii) for every $A \in O^n$ the set $A = \{A + x \mid x \in \mathbb{R}^n\}$ is a Čebyšev set in $O^B$;

(iii) there exists $A \in O^n$ such that the set $A = \{A + x \mid x \in \mathbb{R}^n\}$ is a Čebyšev set in $O^B$.

**Proof.** The Euclidean version of the implication (i) $\implies$ (ii) coincides with [5, Theorem 3.3]. The only property of the ball $B^n$ used in the proof of that theorem is strict convexity of $A + \alpha B^n$ for every $A$ strictly convex ([5, Proposition 1.3]). In view of Lemma 3.3, the Minkowski sum of two strictly convex sets is strictly convex. Thus (i) $\implies$ (ii).

(ii) $\implies$ (iii) is evident.

(iii) $\implies$ (i). Suppose, to the contrary, that (iii) holds and $B$ is not strictly convex. In view of the implication (iii) $\implies$ (i) in Theorem 3.1 we may assume that $A$ is not a singleton.

Let $\Delta(b, b') \subset \text{bd}B$ and so $\Delta(-b', -b) \subset \text{bd}B$. Let $b_1 = \frac{1}{2}(b + b'), b_2 = -b_1$, and $u = \frac{b - b'}{\|b - b'\|}$.

We shall construct a strictly convex body $C \subset B$ such that

(a) $C$ is not contained in any ball $tB$ for $t < 1$,

(b) there exists $t_0 > 0$ such that $0 \in C + tu \subseteq B$ for every $t \leq t_0$.

Let $B_0$ be the Euclidean ball with centre 0 and radius $r = \frac{1}{2}\|b - b'\|$ and let $H$ be a linear hyperplane orthogonal to $b_1 - b_2$. For every $c \in H \cap \text{bd}B_0$
there exists a unique circle passing through $b_1, b_2, c$. Let $L_c$ be the arc of this circle with endpoints $b_1, b_2$. We define

$$C := \text{conv} \bigcup \{L_c \mid c \in H \cap \text{bd}B_0\}.$$ 

It is easy to check that $\text{bd}C \setminus \{b_1, b_2\}$ consists of elliptic points (i.e., points with positive Gauss curvature), whence $C$ is strictly convex. Evidently conditions (a) and (b) are satisfied.

Let now $X := A + C$. This test body is strictly convex because both $A$ and $C$ are. To prove that there is more than one translate of $A$ nearest to $X$, it suffices to show that there is more than one translate of $X$ nearest to $A$.

Let $t_0$ be as in (b). Since

$$\varrho_H^B(X + tu, A) = \varrho_H^B(C + tu, \{0\}) = \inf\{\alpha > 0 \mid C + tu \subseteq \alpha B\},$$

by (a) and (b) it follows that

$$\varrho_H^B(X + tu, A) = 1$$

for all $t \leq t_0$.

On the other hand, by (b), the origin belongs to $C + tu$ for sufficiently small $t$, whence there exists $t_1 > 0$ such that for $t \leq t_1$

$$\varrho_H^B(A, X + tu) = \varrho_H^B(\{0\}, C + tu) = 0.$$ 

Hence, for all $t \leq \min\{t_0, t_1\}$,

$$\varrho_H^B(X + tu, A) = 1,$$

a contradiction.

4. Final remarks and open problems

Remark 4.1. One of the main results of [5] concerns strictly nested families in $\mathcal{C}^n$ ([5, Theorem 2.5]). Let us observe that no Euclidean property of the unit ball $B^n$ was used in [5, Section 2]; hence the statements 2.5 - 2.9 in [5] remain valid in arbitrary Minkowski space with a unit ball $B$. In particular,

- Every closed, dense, strongly nested family in $\mathcal{C}^B$ is a Čebysev set relative to $K^B$.
- No nested family is a Čebysev set in $\mathcal{C}^B$ or in $\mathcal{C}^B_0$.

Remark 4.2. Theorem 5.2 in [2] is valid for arbitrary Minkowski space:

Every strictly affine convex subfamily of $K^n$ is a Čebysev set in $K^B$.

Remark 4.3. Proposition 4.7 in [2] can be extended over Minkowski spaces with strictly convex unit ball:

If $B$ is strictly convex, then no ball in $K^B$ is a Čebysev set.
Remark 4.4. Theorem 2.2 in [2] contains information about continuity of the metric projection, while Theorem 2.3 above does not. The reason is that the argument used in proof of [2, Theorem 2.2] is based on some special properties of the Euclidean space. However, the continuity of $\xi_B$ can be easily deduced from the continuity of the metric projection of $K^B$ onto the closure of $B$ (see Theorem 2.5 above).

Proposition 4.5. If $(\mathbb{R}^n, \| \cdot \|_B)$ has strictly convex unit ball $B$, then the metric projection $\xi_B$ is continuous.

Proof. Evidently,
$$\xi_B = \xi_{B,[\mathbb{R}^n]} | K^B_0.$$ Thus the assertion follows directly from Theorem 2.5(ii).

Problem 4.1. Is strict convexity of $B$ necessary for existence of Čebyšev sets in $K^B_0$ and $K^B$ invariant under $\text{Sim}_B$?

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