HOMOTOPY CHARACTERIZATION OF $G$-ANR'S

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Dedicated to Professor Sibe Mardesic on the occasion of his 80th birthday

Abstract. Let $G$ be a compact Lie group. We prove that if each point $x \in X$ of a $G$-space $X$ admits a $G_x$-invariant neighborhood $U$ which is a $G_x$-ANE then $X$ is a $G$-ANE, where $G_x$ stands for the stabilizer of $x$. This result is further applied to give two equivariant homotopy characterizations of $G$-ANR's. One of them sounds as follows: a metrizable $G$-space $Y$ is a $G$-ANR iff $Y$ is locally $G$-contractible and every metrizable closed $G$-pair $(X, A)$ has the $G$-equivariant homotopy extension property with respect to $Y$. In the same terms we also characterize $G$-ANR subsets of a given $G$-ANR space.

1. Introduction

This paper is devoted to homotopy characterization of equivariant absolute neighborhood retracts or $G$-ANR's under the assumption that the acting group $G$ is compact Lie. The non-equivariant analogs of the results presented here are well known (see Borsuk [6], Hu [10] and van Mill [12]).

It was proved in Jaworowski [11] that a finite-dimensional metrizable $G$-space is a $G$-ANR iff it is locally $G$-contractible. Local $G$-contractibility alone is not sufficient to characterize the $G$-ANR's of arbitrary dimension even if $G$ is the trivial group (see [6, Chapter V, §11] for a counterexample). It turns out (see Theorem 5.3(b)) that local $G$-contractibility together with the

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$G$-homotopy extension property (short: $G$-HEP) characterizes the $G$-ANR's among metrizable $G$-spaces of arbitrary dimension. We prove in Theorem 5.1 a "controlled" equivariant version of Borsuk's homotopy extension theorem. In Section 4 we define the property $P(G, V)$ - a stronger property than the $G$-HEP, which alone characterizes the $G$-ANR's among all metrizable $G$-spaces (Theorem 4.4). We should mention here that all these characterizations are based on the following local characterization of $G$-ANE's obtained in Theorem 3.2: a $G$-space $X$ is a $G$-ANE if and only if each point $x \in X$ admits a $G_x$-invariant neighborhood $U$ which is a $G_x$-ANE, where $G_x$ stands for the stabilizer of $x$. In last Theorem 5.4 we prove that a closed invariant subset $A$ of a $G$-ANR space $X$ is a $G$-ANR iff the pair $(X, A)$ satisfies the $G$-HEP with respect to any $G$-space.

2. Preliminaries

Throughout the paper the letter "$G$" will always denote a compact Lie group (though some of the results presented here are valid also in the case of an arbitrary compact acting group $G$).

"A space" will mean a completely regular Hausdorff topological space.

The monographs [7, 13] are our main references for the basic notions of the theory of transformation groups. For the equivariant theory of retracts the reader can see, for instance, [1, 2, 4].

For the convenience of the reader we recall, however, some more special definitions and facts below.

By an action of the group $G$ on a space $X$ we mean a continuous map $(g, x) \mapsto gx$ of the product $G \times X$ into $X$ such that $(gh)x = g(hx)$ and $ex = x$ whenever $x \in X$, $g, h \in G$ and $e$ is the unity of $G$. A space $X$ together with a fixed action of the group $G$ is called a $G$-space.

By a normed linear $G$-space we shall mean a real normed linear space $L$ on which $G$ acts by means of linear isometries, i.e., $g(\lambda x + \mu y) = \lambda gx + \mu gy$ and $\|gx\| = \|x\|$ for all $g \in G$, $x, y \in L$ and $\lambda, \mu \in \mathbb{R}$.

A continuous map $f : X \to Y$ of $G$-spaces is called an equivariant map or, for short, a $G$-map, if $f(gx) = gf(x)$ for every $x \in X$ and $g \in G$. If $G$ acts trivially on $Y$ then we use the term "invariant map" instead of "equivariant map". By a $G$-embedding we shall mean a topological embedding $X \hookrightarrow Y$ which is a $G$-map.

Let $X$ be a $G$-space. For any $x \in X$, we denote by $G_x$ the stabilizer of $x$ defined by $G_x = \{ g \in G \mid gx = x \}$. A $G$-fixed point is a point $x \in X$ with $G_x = G$.

For a subset $S \subseteq X$ and for a subgroup $H \subseteq G$, the $H$-hull (or $H$-saturation) of $S$ is defined as follows: $H(S) = \{ hs \mid h \in H, s \in S \}$. If $S$ is the one point set $\{ x \}$, then the $H$-hull $H(S)$ is usually denoted by $H(x)$ and called the $H$-orbit of $x$. The set $X/H$ of all $H$-orbits endowed with the quotient
topology is called the $H$-orbit space. A subset $A \subset X$ is called $H$-invariant, or simply, an $H$-subset if it coincides with its $H$-hull, i.e., $A = H(A)$. We shall often use the term “invariant subset” for a “$G$-invariant subset”.

A subset $S \subset X$ is called an $H$-slice in $X$, if: (1) $S$ is $H$-invariant, (2) the $G$-hull $G(S)$ is open in $X$, (3) if $g \in G \setminus H$, then $gS \cap S = \emptyset$, (4) $S$ is closed in $G(S)$.

If, in addition, $G(S) = X$, then $S$ is called a global $H$-slice in $X$.

For each $H$-slice $S$, the $G$-hull $G(S)$ is $G$-homeomorphic to the twisted product $G \times_H S$ (see [7, Chapter II, Theorem 4.2]); we will use this fact in what follows without a specific reference.

Recall that, for an $H$-space $Y$, the twisted product $G \times_H Y$ is defined to be the $H$-orbit space of the $H$-space $G \times Y$, where $H$ acts on $G \times Y$ by $h(g,y) = (gh^{-1}, hy)$. Furthermore, there is a natural action of $G$ on $G \times_H Y$ given by $g'[g, y] = [g'g, y]$, where $[g,y]$ denotes the $H$-orbit of $(g,y) \in G \times Y$ and $g' \in G$. We shall identify $Y$, as an $H$-space, with the $H$-invariant subset $\{[e, y] \mid y \in Y\}$ of $G \times_H Y$.

The following result plays a central rule in the theory of topological transformation groups (see [7, Chapter II, Theorem 5.4]):

THEOREM 2.1 (Slice theorem). Let $X$ be a $G$-space, $x \in X$ and $U$ a neighborhood of $x$. Then there exists a $G_x$-slice $S_x \subset X$ such that $x \in S_x \subset U$.

A $G$-space $Y$ is called an absolute neighborhood $G$-extensor (notation: $Y \in G$-ANE) if, for any closed invariant subset $A$ of a metrizable $G$-space $X$ and any $G$-map $f : A \to Y$, there exist an invariant neighborhood $U$ of $A$ in $X$ and a $G$-map $\psi : U \to Y$ that extends $f$. If, in addition, one can always take $U = X$, then we say that $Y$ is an absolute $G$-extensor (notation: $Y \in G$-AE). The map $\psi$ is called a $G$-extension of $f$.

A metrizable $G$-space $Y$ is called an absolute neighborhood $G$-retract (notation: $Y \in G$-ANR), provided that for any closed $G$-embedding of $Y$ in a metrizable $G$-space $X$, there exists a $G$-retraction $r : U \to Y$, where $U$ is an invariant neighborhood of $Y$ in $X$. If, in addition, one can always take $U = X$, then we say that $Y$ is an absolute $G$-retract (notation: $Y \in G$-AR).

It is known [2] that a metrizable $G$-space is a $G$-ANR (resp., a $G$-AR) iff it is a $G$-ANE (resp., a $G$-AE); we shall often use this fact throughout the paper without an additional reference.

As usual, the letter $I$ will stand for the closed interval $[0,1]$.

Let $X$ and $Y$ be $G$-spaces. A homotopy $F_t : X \to Y$, $t \in I$, is called a $G$-homotopy, if $F_t(gx) = gF_t(x)$ for every $x \in X$, $g \in G$ and $t \in I$. Two $G$-maps $f, \varphi : X \to Y$ are $G$-homotopic, if there exists a $G$-homotopy $F_t : X \to Y$ such that $F_0 = f$ and $F_1 = \varphi$.

Let $\gamma$ be an open covering of the $G$-space $Y$. Then a $G$-homotopy $F_t : X \to Y$, $t \in I$, is said to be limited by $\gamma$, or simply, a $\gamma$-$G$-homotopy provided
for any \( x \in X \), there exists \( \Gamma \in \gamma \) such that \( F_t(x) \in \Gamma \) for all \( t \in I \). In such a case \( F_0 \) and \( F_1 \) are called \( \gamma \)-G-homotopic G-maps.

A G-subset \( A \) of a G-space \( X \) is called G-contractible in \( X \) if the identity inclusion \( A \hookrightarrow X \) is G-homotopic to a constant map \( A \to \{ x_0 \} \), where \( x_0 \in X \) is a G-fixed point. Respectively, \( X \) is called locally G-contractible at the point \( x \in X \) if for every \( G_x \)-invariant neighborhood \( U \) of \( x \) there exists a \( G_x \)-invariant neighborhood \( V \) of \( x \) such that \( V \) is \( G_x \)-contractible in \( U \). A G-space \( X \) is called locally G-contractible if it is locally \( G_x \)-contractible at each point \( x \in X \).

In the sequel we will need the following known results:

**Proposition 2.2.** Let \( K \) be a closed subgroup of \( G \), and \( S \) a \( K \)-space. Then \( S \) is a neighborhood \( K \)-retract of the twisted product \( G \times_K S \).

**Proof.** See [5, Proposition 4.1].

**Proposition 2.3.** If a G-space \( Y \) is the union of a family of invariant open G-ANE subsets \( Y_\mu \subset Y \), \( \mu \in M \), then \( Y \) is a G-ANE as well.

**Proof.** See [4, Corollary 5.7].

**Proposition 2.4.** Let \( K \) be a closed subgroup of \( G \), and \( S \) a \( K \)-space. Then every \( K \)-map \( f : S \to Y \) in a G-space \( Y \) induces a G-map \( f^0 : G \times_K S \to Y \) according to the formula: \( f^0([g,s]) = gf(s) \) for any \([g,s] \in G \times_K S \).

**Proof.** See [8, Chapter I, Proposition 4.3].

**Proposition 2.5.** Let \( K \) be a closed subgroup of \( G \), and \( S \) a global \( K \)-slice of the G-space \( X \). If \( S \) is a \( K \)-ANE then \( X \) is a G-ANE.

**Proof.** See [13, Corollary 1.7.16].

### 3. Local G-ANE’s

**Definition 3.1.** A G-space \( X \) is called a local G-ANE if each point \( x \in X \) admits a \( G_x \)-invariant neighborhood \( U \) which is a \( G_x \)-ANE.

The following local characterization of G-ANE’s plays a fundamental role in the paper:

**Theorem 3.2.** A G-space \( X \) is a G-ANE if and only if \( X \) is a local G-ANE.

**Proof.** If \( X \) is a G-ANE then \( X \) is also an \( H \)-ANE for any closed subgroup \( H \subset G \) (see [14, Corollary 4.5]). In particular, \( X \) is a \( G_x \)-ANE for any \( x \in X \).

Now assume that \( X \) is a local G-ANE. For any \( x \in X \), let \( U \) be a \( G_x \)-invariant neighborhood of \( x \) which is a \( G_x \)-ANE. By Theorem 2.1, one can choose a \( G_x \)-slice \( S_x \) such that \( x \in S_x \subset U \). Since the G-hull \( G(S_x) \) is G-homeomorphic to the twisted product \( G \times_{G_x} S_x \), by Proposition 2.2, \( S_x \) is
4. Homotopy characterization of G-ANR's

Recall that a covering $U$ of a G-space $Y$ is called a $G$-covering if $gU \in U$ for every $U \in \mathcal{U}$ and $g \in G$. Two continuous maps $f, \varphi : X \to Y$ are called $U$-near, if for every $x \in X$ there exists $U \in \mathcal{U}$ such that $\{f(x), \varphi(x)\} \subseteq U$.

**Definition 4.1.** Let $Y$ be a G-space and let $\mathcal{U}$ and $\mathcal{V}$ be open G-coverings of $Y$ such that $\mathcal{V}$ is a refinement of $\mathcal{U}$. We say that $Y$ satisfies the property $\mathcal{P}(G, \mathcal{U}, \mathcal{V})$ if for any two $\mathcal{V}$-near $G$-maps $f, \varphi : X \to Y$ defined on a metrizable G-space $X$ and any $V$-G-homotopy $j_t : A \to Y$, $t \in I$, defined on a closed $G$-subset $A$ of $X$ with $j_0 = f|_A$ and $j_1 = \varphi|_A$, there exists a $U$-G-homotopy $J_t : X \to Y$, $t \in I$, with $J_0 = f$, $J_1 = \varphi$ and $J_t|_A = j_t$ for every $t \in I$.

If $\mathcal{U} = \{Y\}$ is the one element covering, then we shall write $\mathcal{P}(G, \mathcal{V})$ instead of $\mathcal{P}(G, \mathcal{U}, \mathcal{V})$.

**Theorem 4.2.** If $Y$ is a G-ANR and $\mathcal{U}$ a given open $G$-covering of $Y$, then there exists an open $G$-covering $\mathcal{V}$ of $Y$ which is a refinement of $\mathcal{U}$ such that $Y$ satisfies the property $\mathcal{P}(G, \mathcal{U}, \mathcal{V})$ from Definition 4.1.

**Proof.** By [2, Corollary 5], we can assume that $Y$ is an invariant closed subset of a normed linear $G$-space $L$. Since $Y$ is a G-ANR, there exists an invariant neighborhood $M$ of $Y$ in $L$ and an equivariant retraction $r : M \to Y$. Consider the open covering $r^{-1}(\mathcal{U}) = \{r^{-1}(U) \mid U \in \mathcal{U}\}$ of $M$. Let $W$ consist of all open balls of $L$ each of which is contained in an element of $r^{-1}(\mathcal{U})$. Clearly, $W$ is an open $G$-covering of $M$ which refines $r^{-1}(\mathcal{U})$. Put $\mathcal{V} = \{W \cap Y \mid W \in W\}$. We claim that $\mathcal{V}$ is the required $G$-covering of $Y$.

Indeed, let $X$ be a metrizable G-space and $A$ a closed G-subset of $X$. Assume further that $f, \varphi : X \to Y$ are any two $V$-near $G$-maps defined on $X$ and $j_t : A \to Y$, $t \in I$, is a given $V$-G-homotopy defined on $A$ with $j_0 = f|_A$ and $j_1 = \varphi|_A$.

We construct a $W$-G-homotopy $\psi_t : X \to M$, $t \in I$, by putting

$$\psi_t(x) = (1 - t)f(x) + t\varphi(x)$$

for every $x \in X$ and every $t \in I$.

Consider the closed $G$-subset

$$T = (X \times \{0\}) \cup (A \times I) \cup (X \times \{1\})$$
of the topological product $P = X \times I$ endowed with the $G$-action: $g(x, t) = (gx, t)$. Define a $G$-map $\Phi : T \times I \to Y$ by the rule:

$$\Phi(x, t) = \begin{cases} f(x), & \text{if } x \in X \text{ and } t = 0 \\ j_t(x), & \text{if } x \in A \text{ and } t \in I \\ \varphi(x), & \text{if } x \in X \text{ and } t = 1 \end{cases}$$

Since $Y$ is a $G$-ANR, it follows that $\Phi$ has a $G$-extension $\Psi : N \to Y$ over a $G$-neighborhood $N$ of $T$ in $P$.

By means of compactness of the unit interval $I$, one can easily prove the existence of an open neighborhood $C_0$ of $A$ in $X$, such that $C_0 \times I$ is contained in $N$ and that the homotopy $\xi_t : C_0 \to Y$, $t \in I$, defined by

$$\xi_t(x) = \Psi(x, t), \quad x \in C_0, \ t \in I$$

is a $V$-homotopy. Since $G$ is compact, one can choose an invariant neighborhood $C$ of $A$ in $X$ such that $C \subset C_0$. Then the restriction $\xi_t = \xi_t|C$, $t \in I$, is a $V$-G-homotopy.

Further, choose an open invariant set $B$ in $X$ such that

$$A \subset B \subset \overline{B} \subset C.$$

Then, by the equivariant Urysohn lemma, there exists an invariant map $s : X \to I$ such that

$$s(x) = \begin{cases} 0, & \text{if } x \in X \setminus B \\ 1, & \text{if } x \in A \end{cases}$$

Define a $G$-homotopy $\theta_t : X \to M$, $t \in I$, by the rule:

$$\theta_t(x) = \begin{cases} (1 - s(x))\psi_t(x) + s(x)\xi_t(x), & \text{if } x \in C \\ \psi_t(x), & \text{if } x \in X \setminus B. \end{cases}$$

Each $\theta_t$ is a $G$-map since $\psi_t$ and $\xi_t$ are so and $G$ acts linearly on $L$.

Let us prove that $\theta_t$ is a $W$-homotopy. For this purpose, let $x$ be an arbitrary point of $X$. We will prove the existence of a $W$-$\mu$ such that $\theta_t(x) \in W_\mu$ for every $t \in I$.

Consider two cases.

**Case I.** $s(x) = 0$. In this case, we have $\theta_t(x) = \psi_t(x)$ for every $t \in I$.

Since $\psi_t$ is a $W$-homotopy, there is a $W_\mu \in W$ such that $\theta_t(x) = \psi_t(x) \in W_\mu$ for every $t \in I$.

**Case II.** $s(x) > 0$. In this case, we have $x \in B \subset C$. Since $\xi_t$ is a $W$-homotopy, there exists a $W_\mu \in W$ such that $\xi_t(x) \in W_\mu$ for every $t \in I$. In particular, $W_\mu$ contains both points $\xi_0(x) = f(x)$ and $\xi_1(x) = \varphi(x)$. 


Since $W_\mu$ is a convex set, it follows that $\psi_t(x) \in W_\mu$ for every $t \in I$. Now, since the convex set $W_\mu$ contains both points $\psi_t(x)$ and $\xi_t(x)$, it must also contain $\theta_t(x)$ for every $t \in I$. Thus, we have proved that

$$\theta_t : X \to M, \quad t \in I$$

is a $W$-homotopy.

Finally, define a $G$-homotopy $J_t : X \to Y$, $t \in I$, by taking

$$J_t(x) = r(\theta_t(x)), \quad x \in X \text{ and } t \in I.$$

Since $\theta_t$ is a $W$-homotopy and $W$ is a refinement of $r^{-1}(U)$, it follows that $J_t$ is a $U$-homotopy. On the other hand, since $r$ is a retraction, it is easy to verify that $J_0 = f$, $J_1 = \varphi$, and $J_t|_A = j_t$ for every $t \in I$.

**Proposition 4.3.** Let $Y$ be a $G$-space and $V$ an open $G$-covering of $Y$. If $Y$ satisfies the property $\mathcal{P}(G, V)$ then it also satisfies the property $\mathcal{P}(K, V)$ for every closed subgroup $K \subset G$.

**Proof.** Let $X$ be a metrizable $K$-space and $A$ a closed $K$-invariant subset of $X$. Assume that $f, \varphi : X \to Y$ are two $V$-near $K$-maps and $j_t : A \to Y$, $t \in I$, is a $V$-$K$-homotopy with $j_0 = f|_A$ and $j_1 = \varphi|_A$. Then the twisted product $X' = G \times_K X$ is a metrizable $G$-space and $A' = G \times_K A$ is a $G$-invariant closed subset of $X'$.

Now, by Proposition 2.4, the $K$-maps $f$, $\varphi$ and the $K$-homotopy $j_t$ induce $G$-maps $f', \varphi' : X' \to Y$ and a $G$-homotopy $j'_t : A' \to Y$, $t \in I$, respectively.

Let us check first that $f'$ and $\varphi'$ are $V$-near. Indeed, $f'([g, x]) = gf(x)$ and $\varphi'([g, x]) = g\varphi(x)$ for any $[g, x] \in G \times_K X$. Since $f$ and $\varphi$ are $V$-near, then there exists an element $V \in \mathcal{V}$ which contains both points $f(x)$ and $\varphi(x)$. Consequently, $gf(x), g\varphi(x) \in gV$, and since $gV \in \mathcal{V}$ (remember that $\mathcal{V}$ is a $G$-covering), we conclude that the $G$-maps $f'$ and $\varphi'$ are $V$-near.

Next, let us check that $j'_t : A' \to Y$, $t \in I$, is a $V$-homotopy. Indeed, $j'_t([g, a]) = gj_t(a)$ for any $[g, a] \in G \times_K A$ and $t \in I$. Since $j_t : A \to Y$, $t \in I$, is a $V$-homotopy, there exists an element $W \in \mathcal{V}$ such that $j_t(a) \in W$ for all $t \in I$. Consequently, $j'_t([g, a]) = gj_t(a) \in gW$ for all $t \in I$, and since $gW \in \mathcal{V}$, we infer that $j'_t : A' \to Y$, $t \in I$, is a $V$-homotopy.

Now, since the $G$-space $Y$ satisfies the property $\mathcal{P}(G, V)$, there must exist a $G$-homotopy $J'_t : X' \to Y$, $t \in I$, with $J'_0 = f'$, $J'_1 = \varphi'$ and $J'_t|_{A'} = j'_t$ for every $t \in I$. Evidently, the restriction $J_t = J'_t|_X : X \to Y$, $t \in I$, is a $K$-equivariant homotopy with $J_0 = f$, $J_1 = \varphi$ and $J_t|_A = j_t$ for every $t \in I$, as required. This completes the proof.

It turns out that in the class of all metrizable $G$-spaces the property $\mathcal{P}(G, V)$ characterizes the $G$-ANR’s. In fact, we have the following

**Theorem 4.4.** A necessary and sufficient condition for a metrizable $G$-space $Y$ to be a $G$-ANR is the existence of an open $G$-covering $V$ of $Y$ such that $Y$ satisfies the property $\mathcal{P}(G, V)$. 
Proof. The necessity condition follows from Theorem 4.2 by taking $U = \{Y\}$ – the covering consisting of a single open set $Y$.

To prove the sufficiency of the condition $\mathcal{P}(G, \mathcal{V})$, by virtue of Theorem 3.2, it suffices to show that $Y$ is local $G$-ANE.

For, let $y \in Y$ and let $V \in \mathcal{V}$ be an element that contains $y$. By compactness of the group $G_y$, we can and do choose a $G_y$-invariant neighborhood $S$ of $y$ such that $S \subset V$. Define two $G_y$-maps $\phi, \psi : S \to Y$ and a $G_y$-homotopy $\theta_t : \{y\} \to Y$, $t \in I$, by putting

$$
\begin{align*}
\phi(s) &= y, \quad \text{if } s \in S \\
\psi(s) &= s, \quad \text{if } s \in S \\
\theta_t(y) &= y, \quad \text{if } t \in I.
\end{align*}
$$

Obviously, $\phi$ and $\psi$ are $\mathcal{V}$-near $G_y$-maps, and $\theta_t$, $t \in I$, is a $\mathcal{V}$-$G_y$-homotopy. According to Proposition 4.3, $Y$ considered as a $G_y$-space satisfies the condition $\mathcal{P}(G_y, \mathcal{V})$.

Now, since $S$ is a metrizable $G_y$-space and $\{y\}$ is a closed $G_y$-subset of $S$, it follows from $\mathcal{P}(G_y, \mathcal{V})$ that there exists a $G_y$-homotopy $j_t : S \to Y$, $t \in I$, with $j_0 = \phi$, $j_1 = \psi$, and $j_t(y) = y$ for every $t \in I$.

Since the unit interval $I$ is compact and since $j_t(y) = y \in V$ for every $t \in I$, there exists an open $G_y$-invariant neighborhood $U$ of $y$ such that $U \subset S$ and $j_t(U) \subset V$ for every $t \in I$. We will prove that $U$ is a $G_y$-ANE.

To this end, let $f : A \to U$ be any $G_y$-map defined on a closed $G_y$-subspace $A$ of a metrizable $G_y$-space $X$. Define two $G_y$-maps $\xi, \eta : X \to Y$ and a $G_y$-homotopy $J_t : A \to Y$, $t \in I$, by taking

$$
\xi(x) = y = \eta(x), \quad x \in X
$$

and

$$
J_t(x) = \begin{cases} 
  j_{2t}(f(x)), & \text{if } x \in A, \ 0 \leq t \leq \frac{1}{2} \\
  j_{2-2t}(f(x)), & \text{if } x \in A, \ \frac{1}{2} \leq t \leq 1.
\end{cases}
$$

Obviously, $\xi$ and $\eta$ are $\mathcal{V}$-near $G_y$-maps and $J_t$ is a $\mathcal{V}$-$G_y$-homotopy. Hence, by the condition $\mathcal{P}(G_y, \mathcal{V})$, there exists a $G_y$-homotopy $R_t : X \to Y$, $t \in I$, with $R_0 = \xi$, $R_1 = \eta$, and $R_t|_A = J_t$ for every $t \in I$.

Consider the $G_y$-map $r = R_\frac{1}{2} : X \to Y$. By the construction of $r$, one can clearly see that $r|_A = f$. Let $W = r^{-1}(U)$. Then, $W$ is an open $G_y$-neighborhood of $A$ in $X$ and the restriction $r|_W : W \to U$ is a $G_y$-extension of $f$ over $W$. This proves that $U$ is a $G_y$-ANE, and hence, $Y$ is a local $G$-ANE, as required.

5. Equivariant homotopy extension property

By a $G$-pair we shall mean a couple $(X, A)$ where $X$ is a metrizable $G$-space and $A$ a closed $G$-subset of $X$. 

\[\square\]
A $G$-pair $(X, A)$ is said to have the equivariant homotopy extension property (abbreviated: $G$-HEP) with respect to a $G$-space $Y$ iff every partial $G$-homotopy

$$h_t : A \to Y, \ t \in I$$

of an arbitrary $G$-map $f : X \to Y$ has a $G$-extension

$$f_t : X \to Y, \ t \in I \quad \text{such that} \quad f_0 = f.$$  

The $G$-pair $(X, A)$ is said to have the absolute equivariant homotopy extension property (abbreviated: $G$-AHEP) if it has the $G$-HEP with respect to every $G$-space $Y$. In this case one says also that the inclusion $A \hookrightarrow X$ is a $G$-cofibration (see [8, p. 96]).

An immediate consequence of the $G$-HEP of $(X, A)$ with respect to $Y$ is that the equivariant extension problem of a $G$-map $f : A \to Y$ over $X$ depends only on the $G$-homotopy class of $f$. In other words, if $f, \phi : A \to Y$ are $G$-homotopic $G$-maps and if $f$ is $G$-extendable over $X$, then so is $\phi$.

Equivariant version of the well known Borsuk homotopy extension theorem states that if $Y$ is a $G$-ANR, then every $G$-pair $(X, A)$ has the $G$-HEP with respect to $Y$ (see [1, Theorem 5]). Our next theorem establishes a "controlled" version of this result:

**Theorem 5.1.** Let $Y$ be a $G$-ANR and $U$ an open $G$-covering of $Y$. Assume that $A$ is a closed $G$-subset of a metrizable $G$-space $X$, $t \in I$, a partial $U$-$G$-homotopy. If $j_0$ can be extended to a $G$-map $f : X \to Y$, then there exists a $U$-$G$-homotopy $J_t : X \to Y$ such that $J_0 = f$ and $J_t|_A = j_t$ for all $t \in I$.

**Proof.** By the above quoted equivariant Borsuk homotopy extension theorem (see [1, Theorem 5]), there exists a $G$-homotopy $F_t : X \to Y, \ t \in I$ such that $F_0 = f$ and $F_t|_A = j_t$. For each $a \in A$, there exists $U_a \in U$ containing $F_t(a) = j_t(a)$ for all $t \in I$. By means of compactness of the unit interval $I$, there exists a neighborhood $W_a$ of $a$ in $X$ such that

$$F_t(W_a) \subseteq U_a, \quad \text{for all} \quad t \in I.$$  

Put $W = \bigcup_{a \in A} W_a$. Then $W$ is a neighborhood of $A$ in $X$. Due to the compactness of the acting group $G$, there exists a $G$-invariant neighborhood $V$ of $A$ such that $V \subseteq W$.

Next we choose an invariant Urysohn function $\lambda : X \to I$ such that $\lambda|_A = 1$ and $\lambda|_{X \setminus V} = 0$. Define $J_t : X \to Y, \ t \in I$, as follows:

$$J_t(x) = F_{\lambda(x)t}(x), \quad x \in X.$$  

Then, clearly, $J_t(x)$ depends continuously upon the pair $(x, t) \in X \times I$, $J_t$ is equivariant and $J_t|_A = j_t$ for all $t \in I$. In addition,

$$J_0(x) = F_0(x) = f(x)$$
for every \( x \in X \), so \( J_0 = f \). It remains to prove that the \( G \)-homotopy \( J_t \), \( t \in I \), is limited by \( \mathcal{U} \). Indeed, take an arbitrary \( x \in X \). If \( x \in V \) then there exists \( a \in A \) such that \( x \in W_a \). Consequently, by (5.1), for each \( t \in I \) one has:

\[
J_t(x) = F_{\lambda(x)\cdot t}(x) \in U_a.
\]

If \( x \notin V \), then \( \lambda(x) = 0 \), from which it follows that

\[
J_t(x) = F_0(x) = f(x), \quad t \in I.
\]

Since \( \mathcal{U} \) is a covering of \( Y \), there exists an element \( V \in \mathcal{U} \) that contains \( f(x) \), and therefore, \( J_t(x) \) is contained in \( U \) for all \( t \in I \), as required. 

**Proposition 5.2.** Let \( Y \) be a \( G \)-space such that every \( G \)-pair has the \( G \)-HEP with respect to \( Y \). Then for every closed subgroup \( K \subseteq G \), every \( K \)-pair \((X,A)\) has the \( K \)-HEP with respect to \( Y \) considered as a \( K \)-space.

**Proof.** The proof is quite similar to the one of Proposition 4.3.

Local \( G \)-contractibility or \( G \)-HEP alone cannot characterize \( G \)-ANR’s even in the case of the trivial acting group \( G \). Corresponding counterexamples can be found in Borsuk [6, Chapter V, §11] and Hanner [9].

However, we have the following convenient characterization of \( G \)-ANR’s:

**Theorem 5.3.** For a given metrizable \( G \)-space \( Y \), the following three statements are equivalent:

(a) \( Y \) is a \( G \)-ANR.
(b) \( Y \) is locally \( G \)-contractible, and every \( G \)-pair \((X,A)\) has the \( G \)-HEP with respect to \( Y \).
(c) Every point \( y \in Y \) has a \( G_y \)-invariant neighborhood \( V \) such that any \( G_y \)-map \( f : A \to V \) defined on a closed \( G_y \)-subset \( A \) of a metrizable \( G_y \)-space \( X \) has a \( G_y \)-extension \( \phi : X \to Y \).

**Proof.** (a) \( \Rightarrow \) (b). The \( G \)-HEP follows from Theorem 5.1 if we take \( \mathcal{U} = \{Y\} \) – the one element covering. Let us prove that \( Y \) is locally \( G \)-contractible. According to [2, Corollary 5], one can assume that \( Y \) is a closed \( G \)-subset of a normed linear \( G \)-space \( Z \). Since \( Y \) is a \( G \)-ANR, there must exist an open \( G \)-subset \( U \subseteq Z \) and a \( G \)-retraction \( r : U \to Y \). Now, take a point \( y \in Y \) and a \( G_y \)-neighborhood \( W \) of \( y \) in \( Y \). Since \( r^{-1}(W) \) is an open subset of \( Z \), we can choose an open ball \( B(y, \varepsilon) \) centered at \( y \) and having the radius \( \varepsilon > 0 \) such that \( B(y, \varepsilon) \subseteq r^{-1}(W) \). Put \( V = B(y, \varepsilon) \cap Y \). Since \( G \) acts on \( Z \) by means of linear isometries we infer that the ball \( B(y, \varepsilon) \), and hence, also \( V \) is a \( G_y \)-invariant set. Next we define a \( G_y \)-homotopy \( f_t : V \to W \), \( t \in I \), by the formula:

\[
f_t(v) = r(ty + (1-t)v), \quad v \in V.
\]

Clearly \( f_t, t \in I \), is a \( G_y \)-contraction of \( V \) in \( W \) to the \( G_y \)-fixed point \( y \).
(b) \(\implies\) (c). Let \( y \) be an arbitrary point in \( Y \). Since \( Y \) is locally \( G \)-contractible, there exists a \( G_y \)-invariant neighborhood \( V \) of \( y \) which is \( G_y \)-contractible in \( Y \) to a \( G_y \)-fixed point \( z \in Y \) (in general, \( z \) may be different from \( y \)). To prove that \( V \) satisfies (c), let \( f : A \rightarrow V \) be a \( G_y \)-map defined on a closed \( G_y \)-subset \( A \) of a metrizable \( G_y \)-space \( X \). Since \( V \) is \( G_y \)-contractible to the \( G_y \)-fixed point \( z \), it follows that \( f \), considered as a \( G_y \)-map into \( Y \), is \( G_y \)-homotopic to the constant \( G_y \)-map \( c : A \rightarrow Y \) which carries all \( A \) into the point \( z \in Y \). Now, observe that by Proposition 5.2, \((X,A)\) satisfies the \( G_y \)-HEP with respect to \( Y \). Therefore, since \( c \) can be \( G_y \)-extended over \( X \), it then follows that \( f \) has a \( G_y \)-extension \( \phi : X \rightarrow Y \).

(c) \(\implies\) (a). By Theorem 3.2, it suffices to show that \( Y \) is a local \( G \)-ANE. Let \( y \in Y \) be an arbitrary point and let \( V \) be a \( G \)-invariant neighborhood of \( y \) which satisfies (c). We will prove that \( V \) is an \( G_y \)-ANE. For this purpose, let \( f : A \rightarrow V \) be any \( G_y \)-map defined on a closed \( G_y \)-subset \( A \) of a metrizable \( G_y \)-space \( X \). By (c), \( f \) has a \( G_y \)-extension \( \phi : X \rightarrow Y \). The inverse image \( U = \phi^{-1}(V) \) is a \( G_y \)-invariant open set in \( X \) containing \( A \), and the restriction \( \phi|_U : U \rightarrow V \) is a \( G_y \)-extension of \( f : A \rightarrow V \) over \( U \).

Our last result characterizes invariant closed \( G \)-ANR subsets in a \( G \)-ANR space; more precisely, we have the following

**Theorem 5.4.** Let \( X \) be a \( G \)-ANR. Then an invariant closed subset \( A \) of \( X \) is a \( G \)-ANR iff the \( G \)-pair \((X,A)\) has the \( G \)-AHEP.

For the proof we shall need the following two lemmas.

**Lemma 5.5.** A \( G \)-pair \((X,A)\) has the \( G \)-AHEP iff the invariant closed subset

\[ T = (X \times \{0\}) \cup (A \times I) \]

of the \( G \)-space \( P = X \times I \) is a \( G \)-retract of \( P \).

**Proof.** The “only if” part. Let \( f : X \rightarrow T \) denote the \( G \)-map defined by

\[ f(x) = (x,0), \quad x \in X. \]

Define a partial \( G \)-homotopy \( h_t : A \rightarrow T, \quad t \in I, \) of \( f \) by putting

\[ h_t(a) = (a,t) \quad \text{for} \quad a \in A, \quad t \in I. \]

Since \((X,A)\) has the \( G \)-AHEP and \( h_0 = f|_A \), we infer that \( h_t \) has an equivariant extension \( f_t : X \rightarrow T, \quad t \in I, \) such that \( f_0 = f \). Let \( r : P \rightarrow T \) denote the \( G \)-map defined by

\[ r(x,t) = f_t(x), \quad x \in X, \quad t \in I. \]

Then \( r \) is a \( G \)-retraction of \( P \) onto \( T \), and hence, \( T \) is a \( G \)-retract of \( P \).

The “if” part. Assume that \( T \) is a \( G \)-retract of \( P \) with a \( G \)-retraction \( r : P \rightarrow T \). To prove the \( G \)-AHEP of \((X,A)\), let \( f : X \rightarrow Y \) be any \( G \)-map
to a $G$-space $Y$ and $h_t : A \rightarrow Y$, $t \in I$, a partial $G$-homotopy of $f$. Define a $G$-map $H : T \rightarrow Y$ by taking

$$H(x,t) = \begin{cases} f(x), & \text{if } x \in X \text{ and } t = 0 \\ h_t(x), & \text{if } x \in A \text{ and } t \in I. \end{cases}$$

Then $h_t$ has a $G$-extension $f_t : X \rightarrow Y$, $t \in I$, defined by

$$f_t(x) = H(r(x,t)) \quad \text{for every } x \in X, t \in I.$$  

Clearly, $f_0 = f$, and hence, $(X, A)$ has the $G$-AHEP.

**Lemma 5.6.** If $X$ is a $G$-ANR and $A$ is an invariant closed $G$-ANR subset of $X$, then the invariant closed subset

$$T = (X \times \{0\}) \cup (A \times I)$$

of the $G$-space $P = X \times I$ is a $G$-retract of $P$.

**Proof.** Since $X \times \{0\}$ and $A \times I$ are invariant closed $G$-ANR subsets of $T$ and their intersection $A \times \{0\}$ is also a $G$-ANR, it follows from [1, Theorem 4(2)] that $T$ is a $G$-ANR. Hence, the identity $G$-map $i : T \rightarrow T$ has an equivariant extension $j : U \rightarrow T$ over an invariant neighborhood $U$ of $T$ in $P$.

Let us show that then there exists a $G$-retraction $r : P \rightarrow T$. Indeed, due to compactness of the interval $I$, one can find a neighborhood $V$ of $A$ in $X$ such that $V \times I \subset U$. Because of compactness of the acting group $G$ one can assume that $V$ is invariant. Next, since $A$ and $X \setminus V$ are disjoint invariant closed subsets of $X$, using normality of the orbit space $X/G$ (which is in fact even metrizable), one can find an invariant function $\lambda : X \rightarrow I$ such that

$$\lambda(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \in X \setminus V. \end{cases}$$

Define a $G$-map $r : P \rightarrow T$ by putting

$$r(x,t) = j(x,\lambda(x)t)$$

for every $x \in X$ and every $t \in I$. Then $r$ is a $G$-retraction of $P$ onto $T$.  

**Proof of Theorem 5.4.** The “only if” part is a simple combination of Lemmas 5.5 and 5.6.

The “if” part. Assume that the $G$-pair $(X, A)$ has the $G$-AHEP. Then, by Lemma 5.5, $T$ is an equivariant retract of $P$. Since $P = X \times I$ is a $G$-ANR then it follows that $T$ is also a $G$-ANR.

Next, $A$ may be identified, as a $G$-space, with the invariant closed subspace $A \times \{1\}$ of $T$. Evidently, the set

$$V = \{(a,t) \in T \mid a \in A, t > 0\}$$
is an invariant neighborhood of $A$ in $T$. Let $s : V \to A$ denote the $G$-map defined by

$$s(a, t) = (a, 1), \quad a \in A, \quad 0 < t \leq 1.$$  

Since $s$ is clearly a $G$-retraction of $V$ onto $A$, we infer that $A$ is a neighborhood $G$-retract of $T$. Since $T$ is a $G$-ANR, then it follows that $A$ is a $G$-ANR. This completes the proof.

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References
