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ON THE FUNDAMENTAL GROUP OF $\mathbb{R}^3$ MODULO THE CASE-CHAMBERLIN CONTINUUM

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Dedicated to Professor Sibe Mardešić on the occasion of his 80th birthday

Abstract. It has been known for a long time that the fundamental group of the quotient of $\mathbb{R}^3$ by the Case-Chamberlin continuum is nontrivial. In the present paper we prove that this group is in fact, uncountable.

1. Introduction

In the 1960's, during the early days of the decomposition theory, the quotient space X^3 of the Euclidean 3-space $\mathbb{R}^3$ by the classical Case-Chamberlin continuum C (see [3]) was one of the most interesting examples. One of the most important questions was whether X^3 is simply connected. It was settled – in the negative – by Armentrout [1] and Shrikhande [10]. However, it remained an open problem until present day to determine how big is the fundamental group of X^3. In this paper we give the solution for this problem – namely, we show that the fundamental group $\pi_1(\mathbb{R}^3/C)$ is *uncountable*.

Consider the Case-Chamberlin inverse sequence $\mathcal{P}$ (see [3], [5, p. 628]):

$$P_0 \xleftarrow{f_0} P_1 \xleftarrow{f_1} P_2 \xleftarrow{f_2} \cdots$$

where $P_0 = \{p_0\}$ is a singleton, P_i is a bouquet of two circles $S^1_{a_i} \bigvee S^1_{b_i}$, and p_i is the base point of the bouquet $S^1_{a_i} \bigvee S^1_{b_i}$, for every $i > 0$.

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Fix an orientation on each of the circles of the bouquet. Let

$$f_i : S^1_{a_{i+1}} \bigvee S^1_{b_{i+1}} \to S^1_{a_i} \bigvee S^1_{b_i}$$

be a piecewise linear mapping which maps the base point p_{i+1} to the base point p_i and maps the natural generators a_{i+1} and b_{i+1} of $\pi_1(S^1_{a_{i+1}} \bigvee S^1_{b_{i+1}})$ to the commutators $[a_i, b_i]$ and $[a_i^2, b_i^2]$ of $\pi_1(S^1_{a_i} \bigvee S^1_{b_i})$, respectively.

The *Case-Chamberlin continuum* C is then defined as the inverse limit $\lim_{\leftarrow} \mathcal{P}$ of the Case-Chamberlin inverse sequence $\mathcal{P}$ (see [3]). Obviously, C is a 1-dimensional continuum and therefore it is embeddable in $\mathbb{R}^3$ (see [4]). It is well-known that the homotopy types of the quotient space $\mathbb{R}^3/f(C)$ are the same for all embeddings f of C into $\mathbb{R}^3$ (see [2]). The main result of our paper is the following theorem:

THEOREM 1.1. *Let C be the Case-Chamberlin continuum embedded in $\mathbb{R}^3$. Then the fundamental group $\pi_1(\mathbb{R}^3/C)$ of the quotient space $\mathbb{R}^3/C$ is uncountable.*

2. Preliminaries

Let G be a group. By the *commutator* of the elements a an b of G we mean the element $[a, b] = a^{-1}b^{-1}ab$ of G. Let G_n be the *lower central series* which is defined inductively (see [9]):

$$G_1 = G, \qquad G_{n+1} = [G_n, G],$$

where $[G_n, G]$ is the group generated by the set $\{[a, b] : a \in G_n, b \in G\}$.

Obviously, $G_n \supseteq G_{n+1}$, for every n. By the *weight* $w(g)$ of an element $g \in G$ we mean the maximal number n such that $g \in G_n$ if such a number exists, and ∞ otherwise. So the weight of any element of a perfect group is equal to ∞. We shall need the following result from [8, Chapter I, Proposition 10.2]:

PROPOSITION 2.1. *For any free group F the lower central series F_n has trivial intersection, i.e., $\bigcap_{n=1}^{\infty} F_n = \{e\}$.*

That is, in any free group the weight of an element x is finite if and only if $x \neq e$. Let

$$C(f_0, f_1, f_2, \dots)$$

be the infinite mapping cylinder of $\mathcal{P}$ (see e.g. [7, 11]) and let $\widetilde{\mathcal{P}}$ be its natural compactification by the Case-Chamberlin continuum C. Let $\mathcal{P}^*$ be the quotient space of $\widetilde{\mathcal{P}}$ by C.

Obviously, $\mathcal{P}^*$ is homeomorphic to the one-point compactification of an infinite 2-dimensional polyhedron $C(f_0, f_1, f_2, \dots)$. Let

$$C(f_k, f_{k+1}, f_{k+2}, \dots)$$

be the mapping cylinder of the inverse sequence:

$$P_k \xleftarrow{f_k} P_{k+1} \xleftarrow{f_{k+1}} P_{k+2} \xleftarrow{f_{k+2}} \cdots .$$

We shall denote the corresponding one–point compactification by

$$C(f_k, f_{k+1}, f_{k+2}, \dots)^*.$$

We shall consider $C(f_k, f_{k+1}, f_{k+2}, \dots)^*$ as a subspace of $\mathcal{P}^*$ and we shall denote the compactification point by p^*.

We consider P_i, for $i \geq 0$, as a subspace of $C(f_0, f_1, \dots)$ and we consider $C(f_k, f_{k+1}, f_{k+2}, \dots)$, for $k \geq 0$, as a subspace of $\widetilde{\mathcal{P}}$. Obviously, P_1 is a strong deformation retract of $C(f_1, f_2, \dots)$. We have the following homomorphism

$$\varphi_{i+1} = (f_1 \cdots f_i)_\sharp : \pi_1(P_{i+1}) \to \pi_1(P_1)$$

which is a monomorphism, since it is the composition of monomorphisms $(f_i)_\sharp : \pi_1(P_{i+1}) \to \pi_1(P_i)$. Note that for a fixed i, the elements $[a_i, b_i]$ and $[a_i^2, b_i^2]$ are free generators of a subgroup $(f_i)_\sharp(\pi_1(P_{i+1}))$ of $\pi_1(P_i)$ (see [9, p. 119, Exercise 12]).

Since φ_i is a monomorphism, we can consider the group $\pi_1(P_i)$ as a subgroup of $\pi_1(P_1) = F$, where F is a free group on two generators a_1 and b_1. In particular, by identification, we have

$$a_2 = [a_1, b_1], \quad a_3 = [a_2, b_2] = [[a_1, b_1], [a_1^2, b_1^2]], \quad \text{etc.}$$

Since $a_i \neq e$, the weight $w(a_i)$ is a finite number (cf. Proposition 2.1 above). It follows by definition of a_i that $w(a_i) \geq i$, for every i.

Choose an increasing sequence of natural numbers $\{n_i\}$ as follows: Let $n_0 = 1$ and $n_1 = 2$. If n_k is already defined, then let n_{k+1} be any natural number such that $n_{k+1} > w(a_{n_k})$ for $k \geq 1$. Then we have $a_{n_k} \notin F_{n_{k+1}}$.

Let I_i be the unit segment which connects the points p_{i+1} and p_i and which corresponds to the mapping cylinder of the mapping $f_i|_{\{p_{i+1}\}}$ of the one-point set $\{p_{i+1}\}$ to the one-point set $\{p_i\}$, for $i \geq 0$.

To define a certain kind of loops we need a new notion. For two paths $f, g : \mathbb{I} \to X$ satisfying $f(1) = g(0)$, let $fg : \mathbb{I} \to X$ be the path defined by:

$$fg(s) = \begin{cases} f(2s) & \text{if } 0 \leq s \leq 1/2, \\ g(2s-1) & \text{if } 1/2 \leq s \leq 1. \end{cases}$$

We also let

$$\overline{f}(s) = f(1-s) \text{ for } 0 \leq s \leq 1.$$

Two paths are simply said to be *homotopic*, if they are homotopic relative to the end points. A *loop* in X is a path $f : \mathbb{I} \to X$, satisfying $f(0) = f(1)$. For a sequence of units and zeros

$$\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots), \qquad \varepsilon_i \in \{0, 1\}$$

define a path $g_\varepsilon : \mathbb{I} \to \mathcal{P}^*$ so that the following properties hold:

(1) $g_\varepsilon(0) = p_1$ and $g_\varepsilon(1) = p^*$,

(2) g_ε maps $[(2k-2)/(2k-1),(2k-1)/2k]$ homeomorphically onto $\bigcup_{i=n_{k-1}}^{n_k-1} I_i$ starting from $p_{n_{k-1}}$ to p_{n_k} for $k\geq 1$, and

(3) g_ε maps $[(2k-1)/2k, 2k/(2k+1)]$ onto $S^1_{a_{n_k}}$ as a winding in the positive direction, if $\varepsilon_k = 1$, and g_ε maps $[(2k-1)/2k, 2k/(2k+1)]$ to the point set $\{p_{n_k}\}$ constantly otherwise, for $k\geq 1$.

Let $h:\mathbb{I}\to\mathcal{P}^*$ be a path from p^* to p_1 which maps $\mathbb{I}$ homeomorphically onto $\bigcup_{i=1}^{\infty} I_i\cup\{p^*\}$. Finally, let $f_\varepsilon = g_\varepsilon h$. Then f_ε is a loop with base point p_1 corresponding to

$$a_\varepsilon = {a_{n_1}}^{\varepsilon_1}{a_{n_2}}^{\varepsilon_2}{a_{n_3}}^{\varepsilon_3}\cdots .$$

3. Proof of Theorem 1.1

For our proof of Theorem 1.1 we shall need the following two lemmata:

Lemma 3.1. *Let C be the Case-Chamberlin continuum embedded in $\mathbb{R}^3$. Then the quotient space $\mathbb{R}^3/C$ is homotopy equivalent to the 2-dimensional compactum $\mathcal{P}^*$.*

Proof. The proof is completely analogous to the proof of the first assertion of [6, Theorem 1.1] and therefore we shall omit it. □

Lemma 3.2. *Let p_0, p_1, p^* be distinct points in a Hausdorff space X and let f be a loop with base point p_1 such that $f^{-1}(\{p_0\})$ is empty and $f^{-1}(\{p^*\})$ is a singleton. If f is null-homotopic, then there exists a loop f' in $X\setminus\{p_0,p^*\}$ such that f and f' are homotopic in $X\setminus\{p_0\}$.*

Proof. Since f is null-homotopic, we have a homotopy $F:\mathbb{I}\times\mathbb{I}\to X$ from f to the constant mapping to p_1, i.e.,

$$F(s,0)=f(s),\quad F(s,1)=F(0,t)=F(1,t)=p_1 \qquad \text{for } s,t\in\mathbb{I}.$$

Let $\{s_0\}$ be the singleton $f^{-1}(\{p^*\})$. Let M be the connectedness component of $F^{-1}(\{p^*\})$ containing $(s_0,0)$, and O the connectedness component of $\mathbb{I}\times\mathbb{I}\setminus M$ containing $\mathbb{I}\times\{1\}$. Define $G:\mathbb{I}\times\mathbb{I}\to X$ by:

$$G(s,t)=\begin{cases} F(s,t) & \text{if } (s,t)\in O,\\ p^* & \text{otherwise.}\end{cases}$$

Then G is also a homotopy from f to the constant mapping to p_1 and $G^{-1}(\{p_0\})$ is contained in O.

Consider $G^{-1}(\{p^*,p_0\})\cap O$ and $\mathbb{I}\times\mathbb{I}\setminus O$. By definition of M, $G^{-1}(\{p^*,p_0\})\cap O$ is compact and disjoint from $(\mathbb{I}\times\mathbb{I}\setminus O)\cup\mathbb{I}\times\{0\}$. Using a polygonal neighborhood of $(\mathbb{I}\times\mathbb{I}\setminus O)\cup\mathbb{I}\times\{0\}$ whose closure is disjoint from $G^{-1}(\{p^*,p_0\})\cap O$, we get a piecewise linear injective path $g:\mathbb{I}\to\mathbb{I}\times\mathbb{I}$ such that

$$\mathrm{Im}(G\circ g)\subseteq X\setminus\{p_0,p^*\},\ \ g(0)\in\{0\}\times\mathbb{I} \text{ and } g(1)\in\{1\}\times\mathbb{I}$$

and $\mathrm{Im}(g)$ divides $\mathbb{I}\times\mathbb{I}$ into two components, one of which contains $G^{-1}(\{p_0\})$ and the other contains $M \cup \mathbb{I} \times \{0\}$. We now see that $G \circ g$ is the desired loop f'. ∎

Proof of Theorem 1.1. By Lemma 3.1, it clearly suffices to consider $\pi_1(\mathcal{P}^*)$ instead of $\pi_1(\mathbb{R}^3/C)$. Suppose therefore, that the group $\pi_1(\mathcal{P}^*)$ was at most countable. We can assume that p_1 is the base point of the space $\mathcal{P}^*$ and all of its subspaces considered below. Since the set of all sequences of units and zeros is uncountable, then there would exist an uncountable set E, such that for every $\varepsilon, \varepsilon'$ from E, the loops f_ε and $f_{\varepsilon'}$ with the base point p_1 would be homotopy equivalent. Fix a loop f_{ε_0} ($\varepsilon_0 \in E$).

Then every loop $f_\varepsilon\overline{f_{\varepsilon_0}}$ is null-homotopic for every $\varepsilon \in E$. Since $\{s : g_\varepsilon\overline{g_{\varepsilon_0}}(s) = p^*\}$ is a singleton, we can apply Lemma 3.2 to $g_\varepsilon\overline{g_{\varepsilon_0}}$. Since $f_\varepsilon\overline{f_{\varepsilon_0}}$ is homotopic to $g_\varepsilon\overline{g_{\varepsilon_0}}$ in $\mathcal{P}^* \setminus \{p_0\}$, we conclude that $f_\varepsilon\overline{f_{\varepsilon_0}}$ is homotopic to a loop f'_ε in $\mathcal{P}^* \setminus \{p_0, p^*\}$, where the homotopy is in $\mathcal{P}^* \setminus P_0$.

Since E is uncountable and $\mathcal{P}^* \setminus \{p_0, p^*\}$ is homotopy equivalent to the bouquet of two circles $S^1_{a_1} \bigvee S^1_{b_1}$, that is, $\pi_1(\mathcal{P}^* \setminus \{p_0, p^*\})$ is countable, there exist distinct ε and ε' in E such that f'_ε is homotopic to $f'_{\varepsilon'}$ in $\mathcal{P}^* \setminus \{p_0, p^*\}$ and hence in $\mathcal{P}^* \setminus P_0$. It follows that $f_\varepsilon\overline{f_{\varepsilon_0}}$ is homotopic to $f_{\varepsilon'}\overline{f_{\varepsilon_0}}$ and hence f_ε is homotopic to $f_{\varepsilon'}$ in $\mathcal{P}^* \setminus P_0$. Let k be the minimal number such that $\varepsilon_k \neq \varepsilon'_k$, say $\varepsilon_k = 1$ and $\varepsilon'_k = 0$. Let Y_k be the quotient space of $\mathcal{P}^* \setminus P_0$ by the closed subspace $C(f_{k+1}, f_{k+2}, f_{k+3}, \dots)^*$. Consider the projection

$$q : \pi_1(\mathcal{P}^* \setminus P_0) \to \pi_1(Y_{n_{k+1}})$$

and let $[f_\varepsilon]$ and $[f_{\varepsilon'}]$ be the homotopy classes containing f_ε and $f_{\varepsilon'}$ respectively. Since $a_{n_{k+1}}, b_{n_{k+1}} \in F_{n_{k+1}}$, $F/F_{n_{k+1}}$ is a quotient group of $\pi_1(Y_{n_{k+1}})$. Then, $q([f_\varepsilon]) = q(a_{n_1}^{\varepsilon_1}) \cdots q(a_{n_{k-1}}^{\varepsilon_{k-1}}) q(a_{n_k})$ and $q([f_{\varepsilon'}]) = q(a_{n_1}^{\varepsilon_1}) \cdots q(a_{n_{k-1}}^{\varepsilon_{k-1}})$. Since $a_{n_k} \notin F_{n_{k+1}}$, it follows that $q(a_{n_k})$ is non-trivial and hence f_ε is not homotopic to $f_{\varepsilon'}$ in $\mathcal{P}^* \setminus P_0$. This contradiction shows that our initial assumption was false and therefore $\pi_1(\mathcal{P}^*) \cong \pi_1(\mathbb{R}^3/C)$ is indeed an uncountable group, as asserted. ∎

Question 3.3. *Let C be the Case-Chamberlin continuum embedded in $\mathbb{R}^3$. Is the first singular homology group with integer coefficients $H_1(\mathbb{R}^3/C; \mathbb{Z})$ of the quotient space $\mathbb{R}^3/C$ also uncountable?*

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