Incenter Circles, Chromogeometry, and the Omega Triangle

1 Introduction

This paper investigates a surprising connection between three closely related Incenter hierarchies of a fixed planar triangle. The framework here is that of Rational Trigonometry ([7], [8]) which allows a consistent universal triangle geometry valid for any symmetric bilinear form, as described in [5], together with the three-fold symmetry of chromogeometry ([9], [10]), which connects the familiar Euclidean (blue) geometry based on the symmetric bilinear form \( x_1, x_2 + y_1, y_2 \), and two relativistic geometries (red and green) based respectively on the bilinear forms \( x_1, x_2 - y_1, y_2 \) and \( x_1, y_2 + y_1, x_2 \). By working with the rational notions of quadrance and spread instead of the transcendental notions of distance and angle, the main laws of Rational Trigonometry allow metrical geometry, and so triangle geometry, to be developed in each of these three geometries in a parallel fashion, with mostly identical formulas and theorems.
computer. So the Omega triangle formed by the three Orthocenters $\Omega \equiv H_bH_cH_a$, introduced in [9], has an intimate connection with the Incenter hierarchies.

Figure 1: The four blue Incenters of $A_1A_2A_3$ and red and green Incenter Circles

These facts relate also to elegant classical properties of quadrangles. In [1] Haskell showed that if two quadrangles have the same diagonal triangle, then all eight points of these quadrangles lie on a single conic; and in [11] Woods found a synthetic derivation of the same result. Now it is obvious that the four Incenters of a triangle, with respect to any bilinear form, will form a standard quadrangle in this sense, meaning that the diagonal triangle coincides with the original triangle. As a consequence, if blue and red Incenters exist, then they must lie on a conic. Our assertion is that this conic is actually a green circle $C_g = C_f \equiv C_b$ with center $H_b$.

In the case of blue Incenters, the four tangent lines to the red incenter circle $C_b$ at the blue Incenters form a standard quadrilateral, implying that they meet in six points $R_{ij}$, which lie two at a time on the three lines of $A_1A_2A_3$; where they are harmonic conjugates with respect to $A_1, A_2$ and $A_3$; and similarly the four tangent lines to the green incenter circle $C_g$ at the blue Incenters meet in six points $G_{ij}$ on the three lines. This is also seen on the above Figure. Similarly there is a corresponding result when we look at red Incenters, and at green Incenters.

The six lines $A_kR_{ij}$, for $i, j, k$ distinct, are the lines of a complete quadrangle, so they meet three at a time at four quad points $Q_{ij}$. Similarly, the six lines $A_kG_{ij}$ meet three at a time at points $G_{ij}$. Somewhat remarkably, the four star lines $s_k \equiv Q_{ij}^bQ_{ij}^g$ form a standard quadrilateral $s_0s_1s_2s_3$. This paper also illustrates our novel approach to triangle geometry initiated in [5], using standard coordinates to establish universal aspects of the subject which are valid over a general bilinear form. This employs an affine change of coordinates to place an arbitrary triangle into standard position, with vertices at $[0,0]$, $[1,0]$ and $[0,1]$. The various triangle centers and constructions are then expressed in terms of the coefficients $a, b$ and $c$ of the matrix

$$C \equiv \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

of the resulting new bilinear form. This allows a systematic augmentation of Kimberling’s Encyclopedia of Triangle Centers ([12], [3], [4]) to arbitrary quadratic forms and general fields.

Standard coordinates also have the advantage of yielding surprisingly simple equations for the three coloured Incenter Circles, which turn out to be, after pleasant simplifications,

$$C_r : Q_r(X) = b_r(2x + 2y - 1)$$
$$C_s : Q_s(X) = b_s(2x + 2y - 1)$$

However the formulas for the star lines $s_j^b$ become rather formidable, but seem to have interesting algebraic aspects. Some intriguing number theoretical questions arise when we inquire into the existence of triangles, over a given field, which have simultaneously blue, red and green Incenters. Studying concrete examples and using empirical computer investigations of Michael Reynolds [6], we make some tentative conjectures on such triangles, both over the rational numbers and over a finite prime field. Finally we extend the results to Spiiker and Nagel points by suitable central dilations.

In the rest of this introduction we recall basic facts from [7] and [5] to formulate triangle geometry over a general bilinear form. We then specialize to the blue, red and green geometries, and use standard coordinates to develop formulas for points and lines (always one of our key aims), and to provide explicit computational proofs of the theorems.

1.1 Quadrilaterals and quadrangles

We begin by reminding the reader of some basic facts from the projective geometry of a quadrangle (four points) or quadrilateral (four lines), using a visual presentation to avoid the need to introduce notation.

In Figure 2 we see four blue lines forming a quadrilateral [in this figure colours are not used in a metrical sense, but only as an aide for explanation]. These four blue lines meet in six points, also in blue. These six blue points determine a further three green diagonal lines, forming the diagonal triangle, in yellow, of the original quadrilateral, whose vertices are three green points. Each green point may be joined via a red line to the two blue points not on either of the two green lines it lies on. This produces six red
lines, which somewhat remarkably meet three at a time at four red points, giving the opposite quadrangle from the original blue quadrilateral. Note that there is a natural correspondence between the four original blue lines and the four red points.

Figure 2: A quadrilateral and its opposite quadrangle

The situation is completely symmetric with regard to points and lines. If we had started out with a quadrilateral of four red points, we would join them to form six red lines. These six red lines determine a further three green diagonal points, forming the diagonal triangle of the original quadrilateral, whose sides form three green lines. Each green line meets two of the red lines in two new blue points. These six new blue points lie three at a time on four blue lines, giving the opposite quadrilateral from the original red quadrangle.

The diagonal green points on a green line are harmonic conjugates with respect to the two red lines through the same point.

There is another more subtle remark to be made here concerning symmetry: each of the three diagonal points is canonically associated to a subdivision of the four original blue lines into two subsets of two: namely those subsets whose joins meet at that diagonal point. If we start with a triangle, say the yellow triangle in the Figure formed by three green points and three green lines, then any quadrilateral or quadrangle which has that triangle as its diagonal triangle is called standard.

1.2 Quadrance, spread and standard coordinates

In this section we briefly summarize the main facts needed from rational trigonometry in the general affine setting (see [7], [8]). We work in the standard two-dimensional vector space \( V \), consisting of row vectors \( v = [x, y] \), over a field \( \mathbb{F} \). A line \( l \) is given by an equation of the form \( ax + by + c = 0 \), or equivalently the proportion \( l = \{a : b : c\} \).

We assume a metrical structure determined by a non-degenerate symmetric \( 2 \times 2 \) matrix \( C \): this gives a symmetric bilinear form on vectors:

\[ v \cdot u \equiv v^T C u. \]

Non-degenerate means \( \det C \neq 0 \), and implies that if \( v \cdot u = 0 \) for all vectors \( u \), then \( v = 0 \).

Two vectors \( v \) and \( u \) are then perpendicular precisely when \( v \cdot u = 0 \). Since the matrix \( C \) is non-degenerate, for any vector \( v \) there is, up to a scalar, exactly one vector \( u \) which is perpendicular to \( v \). Two lines \( l \) and \( m \) are perpendicular precisely when they have perpendicular direction vectors.

The bilinear form determines the main metrical quantity: the quadrance of a vector \( v \) is the number

\[ Q_v \equiv v \cdot v. \]

The quadrance between the points \( A \) and \( B \) is \( Q(A, B) \equiv Q_{AB} \). A vector \( v \) is null precisely when \( Q_v = v \cdot v = 0 \), in other words precisely when \( v \) is perpendicular to itself. A line is null precisely when it has a null direction vector. The following basic fact appears in [5].

**Theorem 1 (Parallel vectors)** Vectors \( v \) and \( u \) are parallel precisely when

\[ Q_v Q_u = (v \cdot u)^2. \]

This motivates the following measure of the non-parallelism of two vectors; the spread between non-null vectors \( v \) and \( u \) is the number

\[ s(v, u) \equiv 1 - \frac{(v \cdot u)^2}{Q_v Q_u} = 1 - \frac{(v \cdot u)^2}{(v \cdot v)(u \cdot u)}. \]

The spread \( s(v, u) \) is unchanged if either \( v \) or \( u \) are multiplied by a non-zero number. We define the spread between any non-null lines \( l \) and \( m \) with direction vectors \( v \) and \( u \) to be \( s(l, m) \equiv s(v, u) \). From Theorem 1, the spread between parallel lines is 0. Two non-null lines \( l \) and \( m \) are perpendicular precisely when the spread between them is 1.

A circle is given by an equation of the form \( Q(A, X) = K \) for some fixed point \( A \) called the center, and a number \( K \) called the quadrance. Note that it is not required that a circle have any points \( X \) lying on it: in this case by enlarging the field to a quadratic extension we can guarantee that it does.

The three particular planar geometries we are most interested in come from the blue, red and green bilinear forms given by the respective matrices

\[ C_b \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_r \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad C_g \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

The corresponding formulas for the blue, red and green quadrances between points \( A_1 \equiv [x_1, y_1] \) and \( A_2 \equiv [x_2, y_2] \)
are
\[
Q_b(A_1, A_2) = (x_2 - x_1)^2 + (y_2 - y_1)^2
\]
\[
Q_c(A_1, A_2) = (x_2 - x_1)^2 - (y_2 - y_1)^2
\]
\[
Q_s(A_1, A_2) = 2(x_2 - x_1)(y_2 - y_1).
\]

It will be useful to discuss triangle geometry then in a general setting: suppose \( v_1 \circ v_2 \equiv v_1 Bv_2^T \) is a symmetric bilinear form, with \( B \) a symmetric \( 2 \times 2 \) matrix. Suppose \( \phi : V \to V \) is a linear transformation given by an invertible \( 2 \times 2 \) matrix \( M \), so that \( \phi(v) = vM = w \), with inverse matrix \( N \), so that \( wN = v \). The new bilinear form \( w_1 \cdot w_2 \equiv (v_1 N) \circ (v_2 N) \) then has matrix \( D = NBN^T \). Suppose that \( \mathbf{X}_1 \mathbf{X}_2 \mathbf{X}_3 \) is a triangle in the vector space \( V \) which has a distinguished symmetric bilinear form \( \circ \). We may move this triangle by a combination of a translation (which does not effect the bilinear form), and a linear transformation \( \phi \), so that the triangle is in what we call standard form, with points
\[
A_1 \equiv [0,0], \quad A_2 \equiv [1,0] \quad \text{and} \quad A_3 \equiv [0,1]
\]
and lines
\[
l_1 \equiv A_2 A_3 = \langle 1 : 1 : -1 \rangle
\]
\[
l_2 \equiv A_1 A_3 = \langle 1 : 0 : 0 \rangle
\]
\[
l_3 \equiv A_2 A_1 = \langle 0 : 1 : 0 \rangle.
\]
Whatever the initial matrix \( B \), the new bilinear form \( \cdot \) is given by
\[
v \cdot u \equiv vDu^T \quad \text{where} \quad D \equiv NBN^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix}
\]
for some numbers \( a, b, \) and \( c \). We may then replace arguments involving the general triangle \( \mathbf{X}_1 \mathbf{X}_2 \mathbf{X}_3 \) and the bilinear form \( \circ \) with ones involving the simpler triangle \( A_1 A_2 A_3 \). What we prove for the standard triangle \( A_1 A_2 A_3 \) with bilinear form given by the matrix \( D \) will be true for the original triangle with bilinear form given by the original matrix \( B \).

We will assume that \( D \) is invertible, so that
\[
\Delta \equiv \det D = ac - b^2
\]
is non-zero. Another important quantity is the mixed trace
\[
d \equiv a + c - 2b
\]
that appears in many formulas. With these notations, we have the following result from [5].

**Theorem 2 (Standard quadrances and spreads)** The quadrances and spreads of \( A_1 A_2 A_3 \) are
\[
Q_1 \equiv Q(A_2, A_3) = d
\]
\[
Q_2 \equiv Q(A_1, A_3) = c
\]
\[
Q_3 \equiv Q(A_1, A_2) = a
\]
and
\[
s_1 \equiv s(A_1 A_2, A_1 A_3) = \frac{\Delta}{ac}
\]
\[
s_2 \equiv s(A_2 A_3, A_2 A_1) = \frac{\Delta}{ad}
\]
\[
s_3 \equiv s(A_3 A_1, A_3 A_2) = \frac{\Delta}{cd}.
\]

Furthermore
\[
1 - s_1 = \frac{b^2}{ac}, \quad 1 - s_2 = \frac{(a - b)^2}{ad}, \quad 1 - s_3 = \frac{(c - b)^2}{cd}.
\]

Note that the centroid of \( A_1 A_2 A_3 \) is
\[
G = \left[ \frac{1}{3} \frac{1}{3} \frac{1}{3} \right].
\]

### 1.3 Bilinees, Incenters and some other triangle centers

A **biline** of the non-null vertex \( l_1 l_2 \) is a line \( b \) which passes through \( l_1 l_2 \) and satisfies \( s(l_1, b) = s(b, l_2) \). The existence of bilines depends on number theoretical considerations of a particularly simple kind.

**Theorem 3 (Existence of Triangle bilines)** The Triangle \( A_1 A_2 A_3 \) has bilines at each vertex precisely when we can find numbers \( u, v, w \) in the field satisfying
\[
a = u^2, \quad ad = v^2, \quad cd = w^2. \tag{2}
\]

In this case we can choose \( u, v, w \) so that \( acd = uvw \) and \( du = vw, \quad cv = uv \) and \( aw = uv \). \tag{3}

We now summarize some basic triangle centers of the standard triangle \( A_1 A_2 A_3 \), assuming the existence of bilines. These formulas involve the entries \( a, b, c \) of \( D \) from (1), as well as the secondary quantities \( u, v \) and \( w \) from (2), satisfying (3). The formulas and proofs are found in [5].

The four Incenters are
\[
I_0 = \frac{1}{d + v - w}[-w, v], \quad I_1 = \frac{1}{d - v + w}[w, -v],
\]
\[
I_2 = \frac{1}{d + v + w}[w, v], \quad I_3 = \frac{1}{d - v - w}[-w, -v].
\]

Notice that \( I_1, I_2 \) and \( I_3 \) may be obtained from \( I_0 \) by changing signs of: both \( v \) and \( w \); just \( w \), and just \( v \) respectively. This four-fold symmetry will hold more generally and note that it means that we can generally just record the values of \( I_0 \). The Orthocenter \( H \), Circumcenter \( C \) and De Longchamps point \( X_{20} \) (the orthocenter of the double triangle) are
\[
H = \frac{b}{\Delta} [c - b, a - b] \tag{4}
\]
\[
C = \frac{1}{2\Delta} [c(a - b), a(c - b)]
\]
\[
X_{20} = \frac{1}{\Delta} [b^2 - 2bc + ac, b^2 - 2ab + ac].
\]
2 The Incenter Circle theorem

Here is the main theorem of the paper, illustrated for green Inccenters of the triangle $A_1A_2A_3$ in Figure 3. The situation is completely symmetric between the three geometries blue, red and green.

**Theorem 4 (Incenter Circles)** If a triangle $A_1A_2A_3$ has four blue Inccenters $I_{b_1}^1, I_{b_2}^1, I_{b_2}^2$ and $I_{b_3}^1$, then they all lie both on a red circle $C_r^b$ with center the red Orthocenter $H_r$, and on a green circle $C_g^b$ with center the green Orthocenter $H_g$, and similarly for the other colours. Furthermore, if both red and green Inccenters exist, then they lie on the same blue circle, so that $C_b^b = C_r^b = C_g^b$, and similarly for the other colours.

**Proof.** To prove that the four blue Inccenters $I_{b_1}^1, I_{b_2}^1, I_{b_2}^2$ and $I_{b_3}^1$ lie on a red circle $C_r^b$ with center $H_r$, we need show that

$$Q_r(H_r,I_b^1) = Q_r(H_r,I_b^1) = Q_r(H_r,I_b^2) = Q_r(H_r,I_b^3).$$

First we find the bilinear forms for the blue, red and green geometries. After translating, and then applying a linear transformation with the matrix $M$, we send the original triangle to the standard triangle $A_1A_2A_3$. If $M^{-1} = N = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then the bilinear forms for the blue, red and green geometries become respectively the matrices

$$D_b = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^T = (\alpha, \beta),$$

$$D_r = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^T = (a_r, b_r),$$

$$D_g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^T = \left(\frac{a_g}{b_g}, \frac{c_g}{b_g}\right).$$

There are interesting relations between the introduced quantities; for example

$$a_r^2 = a_g^2 + a_r^2, \quad a_b c_b = b_r^2 + b_r^2,$$

$$a_r c_r = b_r^2 - b_r^2, \quad a_g c_g = b_g^2 - b_g^2, \quad c_r^2 = c_g^2 + c_r^2$$

and

$$a_b c_g - 2b_r b_g + c_b a_g = 0, \quad a_b c_r - 2b_g b_r + c_b a_r = 0,$$

$$a_g c_r - 2b_g b_r + c_g a_r = 0.$$

The determinants of $D_b, D_r$ and $D_g$ are respectively

$$\Delta_b = (\alpha \delta - \beta \gamma)^2, \quad \Delta_r = \Delta_g = - (\alpha \delta - \beta \gamma)^2 = -\Delta_b$$

and the mixed traces are

$$d_b = (\alpha - \gamma)^2 + (\beta - \delta)^2, \quad d_r = (\alpha - \gamma)^2 - (\beta - \delta)^2,$$

$$d_g = 2(\alpha - \gamma)(\beta - \delta).$$

Note also the relation $d_b^2 = d_r^2 + d_g^2.$ If the original triangle has four blue Inccenters, then the Existence of Triangle bilines theorem shows that we may choose numbers $u_b, v_b, w_b$ satisfying (2) and (3), so that

$$u_b^2 = (\alpha^2 + \beta^2) (\gamma^2 + \delta^2),$$

$$v_b^2 = (\alpha^2 + \beta^2) (\alpha - \gamma)^2 + (\beta - \delta)^2,$$

$$w_b^2 = (\gamma^2 + \delta^2) (\alpha - \gamma)^2 + (\beta - \delta)^2.$$
If we set \( e_b \equiv d_b + v_b - w_b \) then
\[
\frac{1}{H_t} = -\left( \frac{b_r(c_r - b_r) + w_b}{\Delta_r} \right) + \frac{w_b}{\Delta_r}
\]
so that
\[
Q_r \left( H_t, t_0^r \right) = \left( \frac{1}{H_t} \right) D_r \left( \frac{1}{H_t} \right)
\]
\[
\begin{align*}
&= \frac{b_r(c_r - b_r) + w_b}{\Delta_r} + \frac{w_b}{\Delta_r} \left( b_r \frac{b_r(c_r - b_r)}{\Delta_r} - \frac{w_b}{\Delta_r} \right) \\
&= \frac{1}{\Delta_r} \left( a_r(b_r(c_r - b_r)) + \Delta_r w_b \right) + c_r(b_r(a_r - b_r) - \Delta_r v_b)
\end{align*}
\]

Use the relation \( \Delta_r = a_r c_r - b_r^2 \) to get
\[
Q_r \left( H_t, t_0^r \right) = \frac{b_r(c_r - b_r) + w_b}{\Delta_r} \left( \frac{b_r(c_r - b_r) + w_b}{\Delta_r} \right) - 2a_r c_r(b_r(c_r - b_r)) + b_r(a_r - b_r)^2 w_b + b_r^2 d_b^2
\]
where we have collected \( v_b^2, w_b^2 \) and \( d_b^2 \) of the numerator of (5), to rewrite it.
Replace \( v_b^2 = a_o d_b, w_b^2 = c_d d_b \) and \( v_d w_b = a_b d_b \) and the values of \( a_o, c_o, d_o, a_r, c_r, b_r \) in terms of \( \alpha, \beta, \gamma, \delta \) to get the factorization
\[
\begin{align*}
2b_r(b_r(c_r - b_r)) &+ a_r(b_r - c_r)^2 a_b d_b + c_r(a_r - b_r)^2 c_b d_b + b_r^2 (a_r - c_r - 2b_r) d_b^2 \\
&= d_b \left[ \frac{2b_r(b_r(c_r - b_r)) + a_r(b_r - c_r)^2 a_b + c_r(a_r - b_r)^2 c_b + b_r^2 (a_r - c_r - 2b_r) d_b^2}{\Delta_r} \right] \\
&= 2d_b (\alpha \gamma - \beta \delta)(\alpha^2 - \alpha \gamma + \gamma^2 + \beta^2 - \beta \delta)\delta - \beta^2 - \alpha \gamma + \gamma^2 + \beta^2 - \beta \delta) \delta
\end{align*}
\]

and also note that
\[
\left( a_b d_b - v_b - w_b \right)^2 = d_b \left[ \frac{2b_r(b_r(c_r - b_r)) + a_r(b_r - c_r)^2 a_b + c_r(a_r - b_r)^2 c_b + b_r^2 (a_r - c_r - 2b_r) d_b^2}{\Delta_r} \right]
\]
Combine (6) and (7), to get the surprisingly simple formula
\[
Q_r \left( H_t, t_0^r \right) = \frac{\alpha \gamma - \beta \delta}{\Delta_r} \Delta_r \\
\]

We may now repeat the calculation to see that \( Q_r \left( H_t, t_1^r \right) = Q_r \left( H_t, t_2^r \right) = Q_r \left( H_t, t_3^r \right) = K_r \), showing that indeed the four blue Incircles lie on the red circle \( C_b^x \) with quadran ce \( K_r \) and center \( H_r \). Note that the expression for \( K_r \) depends only on the matrix \( D_r \). Now a similar derivation shows that
\[
Q_r \left( H_t, t_0^r \right) = \frac{b_r(a_r - b_r)(b_r - c_r) \equiv K_r, i = 0, 1, 2, 3.
\]

Hence the four blue Incircles also lie on a green circle \( C_g^y \) with quadran ce \( K_r \) and center \( H_g \). Similarly we find that if a triangle has four red Incircles, then they lie on a blue circle \( C_b^x \) with center \( H_b \) and quadran ce
\[
Q_b \left( H_t, t_0^r \right) = Q_b \left( H_t, t_1^r \right) = \frac{b_r(a_r - b_r)(b_r - c_r) \equiv K_b \}
\]
as well as on a green circle \( C_g^y \) with center \( H_g \) and quadran ce \( K_g \). Similarly a triangle has four green Incircles, then they lie on a blue circle \( C_b^x \) with center \( H_b \) and quadran ce \( K_b \). The proof is complete.\( \square \)

Figure 4: Three Incenter Circles \( C_b, C_r, C_g \). We now call \( C_b = C_b^x, C_r = C_b^y, C_g = C_g^y \) and \( C_b = C_b^y \) the blue, red and green Incenter Circles respectively. In Figure 4 we see a (small) triangle \( A_1 A_2 A_3 \) with its Omega triangle \( \overline{H_t H_t H_t} \) and the three Incircles, whose respective meets give the twelve blue, red and green Incenters.

2.1 Equations of Incenter Circles

Theorem 5 (Incenter Circles equations) In standard coordinates with \( X = [x, y] \), the blue, red and green Incenter circles, when they exist, have respective equations
\[
\begin{align*}
C_b : Q_b(X) &= b_b(2x + 2y - 1) \\
C_r : Q_r(X) &= b_r(2x + 2y - 1) \\
C_g : Q_g(X) &= b_g(2x + 2y - 1). \\
\end{align*}
\]
Proof. The derivation of these equations, using the formulas established above for the orthocenters $H_r$ and coloured Incenters, is somewhat involved algebraically, although the basic idea is simple. We show how to find the equation of the red Incenter Circle $C_r$, with center $H_r$, which four blue Incenters and four green Incenters lie on if they exist. From the definition of a red circle, we get the equation $Q_r(H_r,X) = K_r$, and then substitute the values of $H_r$ and $K_r$ to get

$$\begin{align*}
\frac{b_r(c_r-b_r) - x}{\Delta_r} - y &= \frac{b_r(a_r-b_r) - x}{\Delta_r} - y \left(\frac{b_r(c_r-b_r) - x}{\Delta_r} - y \right) \\
&= b_r(a_r-b_r)(b_r-c_r) \\
&=\frac{b_r(a_r-b_r)(b_r-c_r)}{\Delta_r},
\end{align*}$$

or after expansion

$$\begin{align*}
\frac{1}{\Delta_r^2}
\begin{bmatrix}
2b_r^2x + 2\Delta_r^2 b_r bx + \Delta_r^2 c_y^2 + \Delta_r^2 b_r^2 (a_r - b_r + c_r) \\
- 2\Delta_r^2 b_r x - 2\Delta_r^2 b_r y
\end{bmatrix}
&= b_r(a_r-b_r)(b_r-c_r).
\end{align*}$$

This may be rewritten, using $\Delta_r = a_r c_r - b_r^2$, in the form

$$\begin{align*}
\Delta_r a_r x^2 + 2\Delta_r b_r x y + \Delta_r c_r y^2 - 2\Delta_r b_r x - 2\Delta_r b_r y + b_r (a_r - b_r) (b_r - c_r) = 0.
\end{align*}$$

Now cancel $\Delta_r$, and rearrange to get

$$\begin{align*}
a_r x^2 + 2b_r x y + c_r y^2 - 2b_r x - 2b_r y + b_r = 0.
\end{align*}$$

which has the form stated in the theorem. The same kind of calculation establishes the formulas for $C_b$ and $C_g$. \(\square\)

Note that the equations for the Incenter Circles $C_b$, $C_r$ and $C_g$ allow them to be defined whether or not the corresponding Incenters exist! Incenters then exist precisely as meets of these Incenter Circles: for example the red Incenters $I_{11}^r$, $I_{12}^r$, $I_{21}^r$, $I_{22}^r$ are just the meets of $C_r$ and $C_g$, if these exist in the field in which we work.

### 2.2 Tangent lines of Incenter Circles

Now we consider tangent lines to Incenter circles. Figure 5 shows the four blue Incenters of $A_1 A_2 A_3$, together with the red and green Incenter Circles passing through them, namely $C_r$ and $C_g$. At each of the four Incenters $I_i^b$, $i = 1, 2, 3, 4$ we have the tangent lines $t_{i1}^b$ and $t_{i2}^b$ to the red and green Incenter Circles $C_r$ and $C_g$ respectively.

**Theorem 6 (Incenter tangent meets)** The tangent lines $t_{i1}^b$, $t_{i2}^r$, $t_{i2}^g$ to the red Incenter circle $C_r$ at the blue Incenters form a standard quadrilateral, as do the tangent lines $t_{i1}^g$, $t_{i2}^b$, $t_{i2}^r$ at the green Incenter circle $C_g$. The same holds for the red and green Incenters, if they exist.

This implies that the meets $t_{i1}^r \equiv t_{i2}^b$ and $t_{i3}^b \equiv t_{i2}^r$ lie on $l_1 = A_2 A_3$, and are harmonic conjugates with respect to $A_2$ and $A_3$. Similarly $t_{i2}^r \equiv t_{i1}^g$ and $t_{i3}^g \equiv t_{i1}^r$ lie on $l_2 = A_1 A_3$, and are harmonic conjugates with respect to $A_1$ and $A_3$; and $t_{i3}^r \equiv t_{i1}^b$ and $t_{i3}^g \equiv t_{i1}^r$ lie on $l_3 = A_1 A_2$, and are harmonic conjugates with respect to $A_1$ and $A_2$.

The points $G_{10}^b \equiv t_{i2}^b$, $G_{11}^r \equiv t_{i2}^r$, and $G_{12}^g \equiv t_{i2}^g$ lie on $l_1$, and are harmonic conjugates with respect to $A_2$ and $A_3$. Similarly $G_{12}^r \equiv t_{i1}^g$ and $G_{13}^b \equiv t_{i1}^b$ lie on $l_2$, and are harmonic conjugates with respect to $A_1$ and $A_3$, and $G_{13}^g \equiv t_{i1}^r$ and $G_{12}^b \equiv t_{i2}^g$ lie on $l_3$, and are harmonic conjugates with respect to $A_1$ and $A_2$.

**Proof.** We prove the result for the meets $G_{ij}^r$ of the green tangent lines $t_{ij}^r$ associated to the blue Incenters; the other cases are similar. We find the joins of a blue Incenter $I_i^b$ and the green Orthocenter $H_g$ to be

### Figure 5: Incenter tangent meets
The tangent line \( t_{i0}^b \) is the line green perpendicular to \( H_x t_{i0}^b \) passing through \( P_i^b \). These we may calculate to be

\[
\begin{align*}
  t_{i0}^b &= \left\langle (a_x - b_x) u_b + b_y v_a + (a_x - b_x) v_b + b_y w_a + a_x (b_x - c_x) \right\rangle \\
  t_{i1}^b &= \left\langle (a_x - b_x) u_b + b_y v_a - (a_x - b_x) v_b + b_y w_a + a_x (b_x - c_x) \right\rangle \\
  t_{i2}^b &= \left\langle -(a_x - b_x) u_b + b_y v_a - (a_x - b_x) v_b + b_y w_a + a_x (b_x - c_x) \right\rangle \\
  t_{i3}^b &= \left\langle -(a_x - b_x) u_b + b_y v_a + (a_x - b_x) v_b + b_y w_a + a_x (b_x - c_x) \right\rangle.
\end{align*}
\]

We could verify directly that these four lines form a standard quadrilateral. But we prefer to verify that the meets of these four tangent lines agree with the following means with the side lines of \( A_1 A_2 A_3^* \):

\[
\begin{align*}
  G_{00}^b &= \frac{t_{00}^b}{s_{00}^b} = \frac{t_{i0}^b}{s_{i0}^b} \\
  G_{23}^b &= \frac{t_{00}^b}{s_{00}^b} = \frac{t_{i0}^b}{s_{i0}^b} \\
  G_{03}^b &= \frac{t_{00}^b}{s_{00}^b} = \frac{t_{i0}^b}{s_{i0}^b} \\
  G_{12}^b &= \frac{t_{00}^b}{s_{00}^b} = \frac{t_{i0}^b}{s_{i0}^b}
\end{align*}
\]

where

\[
\begin{align*}
  \lambda_{00} &= (c_x - a_x) u_b + (b_y - c_y) v_b + (b_y - c_b) w_b \\
  \lambda_{23} &= (a_y - c_y) u_b + (b_y - c_y) v_b \\
  \lambda_{03} &= (b_y - c_y) u_b + (c_y - 2b_y) v_b + b_y w_b \\
  \lambda_{12} &= (a_y - c_y) u_b + (c_y - 2b_y) v_b + 2b_y w_b
\end{align*}
\]

The fact that \( A_2, A_3, G_{00}^b, G_{23}^b \) form a harmonic range etc. is an immediate consequence of a well known fact about standard quadrilaterals in projective geometry, since we have shown that the points \( A_1, A_2, A_3 \) are diagonal points of the quadrilateral formed by the four green tangent lines.

Following the construction of the red lines in the introductory section on Quadrangles and quadrilaterals, we join a point \( G_i^b \) with the triangle point \( A_i^* \) opposite to the triangle line that it lies on; giving six lines \( A_j G_i^b \):

\[
\begin{align*}
  A_0 G_{01}^b &= \left\langle (a_x - b_x) (u_b + w_b - c_b) \right\rangle \equiv 0 \\
  A_1 G_{12}^b &= \left\langle (b_y - c_y) (u_b + w_b - c_b) \right\rangle \equiv 0 \\
  A_2 G_{20}^b &= \left\langle (b_y - c_y) (u_b + w_b - c_b) \right\rangle \equiv 0 \\
  A_3 G_{31}^b &= \left\langle (b_y - c_y) (u_b + w_b - c_b) \right\rangle \equiv 0 \\
  A_4 G_{40}^b &= \left\langle (b_y - c_y) (u_b + w_b - c_b) \right\rangle \equiv 0 \\
  A_5 G_{51}^b &= \left\langle (b_y - c_y) (u_b + w_b - c_b) \right\rangle \equiv 0
\end{align*}
\]

**Theorem 7 (Quad points)** The triples \( \{ A_1 G_{23}^b, A_2 G_{01}^b, A_1 G_{12}^b \} \), \( \{ A_1 G_{23}^b, A_2 G_{02}^b, A_3 G_{03}^b \} \), \( \{ A_1 G_{01}^b, A_2 G_{13}^b, A_3 G_{03}^b \} \) and \( \{ A_1 G_{01}^b, A_2 G_{02}^b, A_3 G_{12}^b \} \) of lines are concurrent in the respective points \( Q_{00}^b, Q_{02}^b, Q_{23}^b \), called the blue/green quad points. The triples \( \{ A_1 R_{23}^b, A_2 R_{01}^b, A_3 R_{12}^b \} \), \( \{ A_1 R_{23}^b, A_2 R_{02}^b, A_3 R_{03}^b \} \), \( \{ A_1 R_{01}^b, A_2 R_{13}^b, A_3 R_{03}^b \} \) and \( \{ A_1 R_{01}^b, A_2 R_{02}^b, A_3 R_{12}^b \} \) are also concurrent in the respective points \( Q_{10}^b, Q_{13}^b, Q_{22}^b \), called the blue/red quad points. Similar results hold for the red and green Incenters, if they exist.

**Proof.** We verify this for the blue/green quad points: this is a consequence of the projective geometry of the complete quadrilateral we mentioned in the first section—if the original four tangent lines are regarded as the blue lines in Figure 6, then the quad points \( Q_{ij}^b \) correspond to the red points. However we want to find explicit formulas and check things directly. The quad point \( Q_{ii}^b \) is naturally associated to the Incenter \( I_i^b \). After some calculation, we find
that these are

\[ Q_{x0}^b = \frac{b_g}{\lambda_0}[(b_g - c_g)(d_gu_b - (b_g - c_b)v_b), \]
\[ (a_g - b_g)((c_b - b_g)v_b + c_gd_b)]] \]
\[ Q_{x1}^b = \frac{b_g}{\lambda_1}[(b_g - c_g)((a_b - b_g)v_b + a_gb_d), \]
\[ (a_g - b_g)(d_gu_b + (a_g - b_g)v_b)]] \]
\[ Q_{x2}^b = \frac{b_g}{\lambda_2}[(c_g - b_g)(a_gb_w - b_gv_b), \]
\[ (b_g - a_g)(b_gw_b - c_gb_v)]] \]
\[ Q_{x3}^b = \frac{b_g}{\lambda_3}[(b_g - c_g)(-d_gb_u + (d_g + b_g)v_b - a_gb_w + a_gb_d), \]
\[ (b_g - a_g)(d_gb_u - c_gb_v + (d_g + b_g)v_b - c_gd_b)]] \]

where

\[ \lambda_0 = (b_g - c_g)(b_gb_d + (b_g - c_b)(a_g - b_g))u_b \]
\[ - (b_g - c_g)(b_gb_d + c_gb_d - c_pb_g)v_b \]
\[ - b_g(a_g - b_g)(b_g - c_g)v_b + c_gb_d(a_g - b_g)d_b \]
\[ \lambda_1 = (a_g - b_g)(b_gb_d + (a_g - b_g)(b_g - c_g))u_b \]
\[ + b_g(b_g - c_g)(a_g - b_g)v_b \]
\[ + (a_g - b_g)(2a_gb_w - a_gb_w - b_gb_v)w_b + a_gb_g(b_g - c_g)d_b \]
\[ \lambda_2 = b_g(b_g - c_g)(a_g - b_g)u_b \]
\[ + b_g(b_g + c_gb_d - b_gb_d - c_gb_g)v_b \]
\[ - b_g(a_g - b_g)(b_g - c_g)v_b - a_gb_g(b_g - c_g)(a_g - b_g) \]
\[ \lambda_3 = (b_g + b_g)(b_g + c_gb_d - a_gb_g - 2b_gb_d - b_gb_d - b_gb_d - c_gb_d)u_b \]
\[ + b_g(b_g - c_g)(a_g - b_g) + c_gb_d(a_g - b_g) - c_gd_gb_g(b_g - c_g)v_b \]
\[ + (d_g + b_g)(b_g - a_g)(b_g - b_g)(b_g - c_g)v_b \]
\[ + b_g(b_g + c_gb_d - c_gb_d + b_g(c_ga_c_g(b_g + a_g) - a_c_g(c_gb_d - c_g)) \]
\[ - a_c_g(g_c_g(c_gb_d - c_g)) \]

We may then check directly that for example \( Q_{x0}^b \) is incident with \( A_3G_{12}^b \) by computing

\[
\begin{align*}
(b_g - a_g)u_b + b_gv_b + (a_g - 2b_g)w_b - b_gd_b + c_b(a_g - b_g) & \\
\left(\left( b_g(b_g - c_g)(d_gb_u - (b_g - c_b)v_b) \right) \right. & \\
+ b_g(v_b - w_b + d_b) & \\
\left. + b_g(v_b - w_b + d_b) \right) & \\
\left(b_g(a_g - b_g)((c_g - b_g)v_b + c_gb_d) \right. & \\
\left. \lambda_0 & \\
+ b_g(a_g - b_g)(b_g - c_g) & \\
\left(-b_g(c_g - b_g)v_b + (b_g - c_b)(a_g - b_g)u_b & \\
+ b_g(b_g - c_g)(b_g + c_gb_d - 2b_gb_d)v_b & \\
+ b_g(a_g - b_g)(b_g - c_g)v_b - c_gb_g(a_g - b_g)d_b & \\
\right) & \\
\left) & \\
\right) & \\
\right) & \\
\right) & \\
= 0 & \\
\end{align*}
\]

since \( d_gb_u^2 + (b_g - a_g)u_bw_b - b_gd_bd_b - c_gb_d(v_b) + c_gb_d(v_b) = 0 \) by using (2), and similarly for the other indices. In a parallel fashion, we find that the four blue/red quad points \( Q_{xj}^b \) have exactly the same formulas as the \( Q_{xj}^b \), except for the replacements \( a_g \rightarrow a_r, b_g \rightarrow b_r \) and \( c_g \rightarrow c_r \), and similarly for the other colours red and green.

\[ \square \]

Figure 6: Quad points and star lines

Now introduce the blue star line \( s_i^b \) to be the join of the corresponding blue/red quad point \( Q_{xj}^b \) and the blue/green quad point \( Q_{xj}^b \), and similarly for the other colours. There are then four blue star lines \( s_{i1}^b, s_{i2}^b, s_{i1}^g, s_{i2}^g \).

The blue star point \( B_{ij} \) is the meet of the two blue star lines \( s_{i1}^b \) and \( s_{i2}^b \), that is \( B_{ij} \equiv s_{i1}^b \cap s_{i2}^b \), and similarly for the other colours.

Note that following the introductory section on Quadrangles and quadrilaterals, we use the correspondence between the \( Q_{xj}^b \); and the tangents lines \( I_{xj}^a \); and the Incenters \( I_{xj}^a \) to match up the indices.

**Theorem 8 (Star quadrilateral) The four blue star lines form a standard quadrilateral** \( s_{i1}^b, s_{i2}^b, s_{i1}^g, s_{i2}^g \). This holds also for the other colours.

**Proof.** The proof we have is surprisingly complicated. The star lines \( s_i^b \) have quite involved formulas; for example we find that

\[ s_0^b = Q_{x0}^b s_0^{a0} = \]

\[ E_{d_gb_u} + F_{c_gb_d} : \]
\[ (b_g - a_g)u_b + b_gv_b + (a_g - b_g)v_b + (b_g - c_g)v_b + c_gb_d : \]
\[ 2c_gb_d(b_g - a_g)u_b + (b_g - a_g)v_b + (a_g - b_g)v_b - c_gb_d : \]
\[ (b_g - a_g)u_b + b_gv_b + (a_g - b_g)v_b + (b_g - c_g)v_b + c_gb_d : \]
\[ 2c_gb_d(b_g - a_g)u_b + (b_g - a_g)v_b + (a_g - b_g)v_b - c_gb_d : \]
\[ E_{d_gb_u} + F_{c_gb_d} : \]
\[ \left( b_g - a_g \right)u_b + b_gv_b + (a_g - b_g)v_b + (b_g - c_g)v_b + c_gb_d : \]
\[ 2c_gb_d(b_g - a_g)u_b + (b_g - a_g)v_b + (a_g - b_g)v_b - c_gb_d : \]
\[ (b_g - a_g)u_b + b_gv_b + (a_g - b_g)v_b + (b_g - c_g)v_b + c_gb_d : \]
\[ 2c_gb_d(b_g - a_g)u_b + (b_g - a_g)v_b + (a_g - b_g)v_b - c_gb_d : \]
where \( E_0 \) and \( F_0 \) are both homogeneous polynomials of degree 6 in the variables \( a_i, b_i \) and \( c_i \), with the former having 32 terms and the latter 46 terms. After some trial and error we can present these in the somewhat pleasant, but still mysterious, forms:

\[
E_0 = -b_4 b_7 (a_4 c_7 - c_4 a_7 - b_7 c_4 + c_7 b_4) \cdot (b_4^2 + 4c_4^2 + a_4 c_6 - 6b_4 c_6) + b_4 b_7 (a_4 b_7 - b_7 a_4) (b_6^2 + 2c_6^2 + a_6 c_6 - 4b_6 c_6) - 2c_6 (a_6 c_6 b_7^2 - c_6 a_4 b_7^2 + a_6 a_4 c_6 b_7 - a_6 c_6 a_7 b_7) (b_6 - c_6)
\]

and

\[
F_0 = (a_6 c_6 b_7^2 - a_4 c_7^2 + a_4 a_7 c_6 b_7 - a_4 c_6 a_7 b_7) \cdot (b_6^2 - 4b_6 c_6 + 2c_6^2 + a_6 c_6) + b_6 b_7 (a_6 c_6 - c_6 a_6 - b_6 c_6 + c_6 b_6) \cdot (-5b_6^2 - 4c_6^2 + 2a_6 b_6 - 3a_6 c_6 + 10b_6 c_6) - 2b_7 b_6 (a_6 b_7 - c_6 b_7) (b_6 - c_6) (a_6 - 2b_6 + c_6).
\]

We can then calculate the blue star points, for example

\[
B_{03} = \begin{bmatrix}
\left\{ \begin{array}{c}
b_2 b_3 b_7 \left( a_2 b_3 - b_2 a_3 - c_2 a_7 + c_2 b_7 \right) \\
\left( \left( b_6 - c_6 \right)^2 + c_6 b_6 \right) u_6 - 2c_6 (b_6 - c_6) v_6
\end{array} \right\}, \\
0
\end{bmatrix}
\]

from which clearly \( B_{03} \) lies on \( l_3 \). The computations are similar for the other indices, and the other colours. \( \square \)

### 3 Explicit examples and some conjectures

#### 3.1 An example over \( Q(\sqrt{2}0, \sqrt{2}1, \sqrt{2}4, \sqrt{2}7, \sqrt{2}9) \)

We will now explore in detail a particular triangle which has both blue, red and green Incenters; for us this is not only an important tool for checking the consistency of our formulas, but also a way to get a sense of the level of complexity of various constructions. In fact this kind of explicit calculation of examples is much to be encouraged in this subject: especially as working over concrete fields, including finite fields and explicit extension fields of the rationals, allows us to appreciate the number theoretic aspects of our geometrical investigations. For example, finding a triangle with blue, red and green Incenters approximately is easy with a geometry package: finding a concrete example and working out all the points precisely is more challenging.

In particular we were unable, despite a reasonable computer search, to find any triangles with purely rational points that have blue, red and green Incenters! We would like to thank Michael Reynolds for his contributions to this search. So we tentatively conjecture that there are no such triangles.

In any case, to get an explicit example we use an algebraic extension field of the rationals; so by \( \sqrt{30} \) we mean an appropriate symbol in the extension field \( Q(\sqrt{30}) \) etc.. Note that although our use of square roots is entirely algebraic, our representation of these square roots as approximate rational numbers (we prefer to avoid discussion of “real numbers”), necessarily brings an approximate aspect into our diagrams.

![Figure 7: An example triangle \( X_1X_2X_3 \)](image)

**Example 1** One may check that the basic Triangle with points

\[
X_1 \equiv [-21/59, -58/59], \quad X_2 \equiv [-13/3, 2] \quad \text{and} \quad X_3 \equiv [35/3, -8/5]
\]

in \( Q(\sqrt{30}, \sqrt{2}1, \sqrt{2}4, \sqrt{2}7, \sqrt{2}9) \) has both blue, red and green Incenters. After translation by \( (21/59, 58/59) \) we obtain \( \tilde{X}_1 = [0,0], \tilde{X}_2 = [-704/177, 176/59] \) and \( \tilde{X}_3 = [2128/177, -182/295] \).

The matrix \( N \) and its inverse \( M \), where

\[
N = \begin{pmatrix}
\frac{704}{177} & \frac{176}{295} \\
\frac{2128}{177} & \frac{182}{295}
\end{pmatrix} = \begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}
\]

and

\[
M = N^{-1} = \begin{pmatrix}
\frac{11}{295} & \frac{5}{295} \\
\frac{5}{295} & \frac{11}{295}
\end{pmatrix}
\]

respectively send \([1,0]\) and \([0,1]\) to \( \tilde{X}_2 \) and \( \tilde{X}_3 \), and \( \tilde{X}_2 \) and \( \tilde{X}_3 \) to \([1,0]\) and \([0,1]\). From now on we discuss only the standard triangle \( A_1A_2A_3 \) associated to \( X_1X_2X_3 \); to convert back into the original coordinates, we would multiply by \( N \) and translate by \( (-21/59, -58/59) \). The bilinear forms in these new standard coordinates, for the blue, red and green
geometries respectively, are given by matrices

\[
D_b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} N^T = \begin{pmatrix} 274400 & -778848 \\ -778848 & 156645 \end{pmatrix} = \begin{pmatrix} a_b & b_b \\ b_b & c_b \end{pmatrix},
\]

\[
D_r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} N^T = \begin{pmatrix} 216332 & -720227 \\ -720227 & 11591144 \end{pmatrix} = \begin{pmatrix} a_r & b_r \\ b_r & c_r \end{pmatrix},
\]

\[
D_g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} N^T = \begin{pmatrix} 247808 & 2000768 \\ 2000768 & 523215 \end{pmatrix} = \begin{pmatrix} a_g & b_g \\ b_g & c_g \end{pmatrix}.
\]

The determinants of \(D_b\), \(D_r\), and \(D_g\) are \(\Delta_b = \frac{97140736}{87025}\) and \(\Delta_r = \frac{-2140736}{87025}\), while the mixed traces are \(d_b = \frac{6070}{87025}\) and \(d_g = \frac{526}{87025}\). The orthocenters of \(\triangle A_1A_2A_3\) are

\[
H_b = \begin{pmatrix} 8825537 & 84337 \\ 1015920 & 25488 \end{pmatrix}, \quad H_r = \begin{pmatrix} 87833227 & 55537 \\ 11214720 & 25488 \end{pmatrix},
\]

\[
H_g = \begin{pmatrix} 7105 & 377 \\ 3894 & 177 \end{pmatrix}.
\]

Blue, red and green Incenters exist over \(\mathbb{F} = \mathbb{Q}(\sqrt{30}, \sqrt{2\bar{7}}, \sqrt{741}, \sqrt{2470}, \sqrt{82297})\) and we may choose

\[
u_b = 1875104 \begin{pmatrix} 31329 \\ 177 \end{pmatrix}, \quad v_b = 14432 \begin{pmatrix} 177 \\ 4425 \end{pmatrix}, \quad w_b = 873628 \begin{pmatrix} 885 \\ 217 \end{pmatrix},
\]

\[
u_r = 17248 \begin{pmatrix} \sqrt{2470} \\ \sqrt{82297} \end{pmatrix}, \quad v_r = 2464 \begin{pmatrix} \sqrt{2470} \\ \sqrt{82297} \end{pmatrix}, \quad w_r = 83628 \begin{pmatrix} 885 \\ 217 \end{pmatrix}, \quad w_r = 873628 \begin{pmatrix} 885 \\ 217 \end{pmatrix},
\]

\[
u_g = 19712 \begin{pmatrix} \sqrt{2470} \\ \sqrt{82297} \end{pmatrix}, \quad v_g = 2816 \begin{pmatrix} \sqrt{30} \\ 295 \end{pmatrix}, \quad w_g = 448 \begin{pmatrix} \sqrt{741} \\ 295 \end{pmatrix}.
\]

Then the four blue Incenters, the four red Incenters and the four green Incenters of \(\triangle A_1A_2A_3\) respectively are

\[
I_{b1} = \begin{pmatrix} 761 \end{pmatrix}, \quad I_{b2} = \begin{pmatrix} 761 \end{pmatrix}, \quad I_{b3} = \begin{pmatrix} 761 \end{pmatrix}, \quad I_{b4} = \begin{pmatrix} 761 \end{pmatrix},
\]

\[
I_{r1} = \begin{pmatrix} 5327 \end{pmatrix}, \quad I_{r2} = \begin{pmatrix} 5327 \end{pmatrix}, \quad I_{r3} = \begin{pmatrix} 5327 \end{pmatrix}, \quad I_{r4} = \begin{pmatrix} 5327 \end{pmatrix},
\]

\[
I_{g1} = \begin{pmatrix} 5327 \end{pmatrix}, \quad I_{g2} = \begin{pmatrix} 5327 \end{pmatrix}, \quad I_{g3} = \begin{pmatrix} 5327 \end{pmatrix}, \quad I_{g4} = \begin{pmatrix} 5327 \end{pmatrix}.
\]

The Incenter circle quadrances are

\[
K_b = \frac{18154129609}{28196100}, \quad K_r = \frac{-116818191}{28196100},
\]

\[
K_g = \frac{1182272}{28196100}.
\]

The blue, red and green Incenter Circles themselves have respective equations

\[
4840000x^2 - 19447120xy + 28376929y^2 = 0,
\]

\[
+19447120x + 19447120y - 9723560 = 0,
\]

\[
19360x^2 - 62524xy + 12103y^2 = 0,
\]

\[
+62524x + 62524y - 31262 = 0,
\]

\[
19360x^2 - 2572240xy + 4032553y^2 = 0,
\]

\[
+2572240x + 2572240y - 1286120 = 0.
\]

The four tangent lines \(I_{b}^{(i)}\) are

\[
I_{b1}^{(1)} = (1570 : -11823 : 8323),
\]

\[
I_{b1}^{(2)} = (-127512 : -33761 : 58261),
\]

\[
I_{b1}^{(3)} = (-18216 : -11823 : 8323),
\]

\[
I_{b1}^{(4)} = (-1570 : 4823 : 8323).
\]
The meets of these four tangent lines agree with the following meets with the side lines of $\Delta A_1A_2A_3$:

\[
G_{01} = t_g^0 = t_g^1 = \begin{pmatrix} 1034074074039 \\ 868804574039 \\ 1034074074039 \\ 868804574039 \end{pmatrix},
\]

\[
G_{23} = t_g^{23} = t_g^{23} = \begin{pmatrix} 1034074074039 \\ 868804574039 \\ 1034074074039 \\ 868804574039 \end{pmatrix},
\]

\[
G_{03} = t_g^{03} = t_g^{03} = \begin{pmatrix} 1034074074039 \\ 868804574039 \\ 1034074074039 \\ 868804574039 \end{pmatrix},
\]

\[
G_{13} = t_g^{13} = t_g^{13} = \begin{pmatrix} 1034074074039 \\ 868804574039 \\ 1034074074039 \\ 868804574039 \end{pmatrix}.
\]

The blue/red quad points $Q_{ij}^b$ associated to $t_g^0, t_g^1, t_g^2, t_g^3$ respectively are

\[
Q_{00} = [18005811535; 12129669559],
\]

\[
Q_{01} = [18005811535; 12129669559],
\]

\[
Q_{03} = [18005811535; 12129669559],
\]

\[
Q_{13} = [18005811535; 12129669559].
\]

The respective blue/green quad points $Q_{ij}^g$ are

\[
Q_{00}^g = [4161500; 11762777],
\]

\[
Q_{01}^g = [4161500; 11762777],
\]

\[
Q_{03}^g = [4161500; 11762777],
\]

\[
Q_{13}^g = [4161500; 11762777].
\]

The blue star lines are then

\[
Q_0 = Q_{00}^g Q_{00}^b = (1796063533088; 868804574039; -1034074074039)
\]

\[
Q_1^b = (272084614990; 1199343574039; -1034074074039)
\]

\[
Q_2^b = (272084614990; 868804574039; -1034074074039)
\]

\[
Q_3^b = (1796063533088; 1199343574039; -1034074074039)
\]

and they meet at the blue star points

\[
B_{01} = \begin{pmatrix} 1652695000000 \\ 761989459049 \end{pmatrix},
\]

\[
B_{23} = \begin{pmatrix} 1652695000000 \\ 761989459049 \end{pmatrix},
\]

\[
B_{02} = \begin{pmatrix} 1034074074039 \\ 868804574039 \end{pmatrix},
\]

\[
B_{03} = \begin{pmatrix} 1034074074039 \\ 868804574039 \end{pmatrix}.
\]

Note the pleasant rationality of the previous objects.

### 3.2 An example over $\mathbb{F}_{13}$

Now we look at an example over a finite field.

**Theorem 9 (Null quadrances incircles)** Suppose that the field $\mathbb{F}$ contains an element $i$, where $i^2 = -1$, and the characteristic of $\mathbb{F}$ is not 2. If

\[
K_b = b_b(a_b - b_b)(a_b - c_b) \quad K_r = b_r(a_r - b_r)(a_r - c_r)
\]

\[
K_r = b_r(a_r - b_r)(a_r - c_r)
\]

\[
K_g = b_g(a_g - b_g)(a_g - c_g)
\]

then the standard triangle $\Delta A_1A_2A_3$ has four distinct blue, red and green Incircles.

**Proof.** If $K_b = 0$ then from the definition of the blue incircle center $C_b$, which is $Q_b(H_b,X) = K_b$, $C_b$ is a null circle, so it is a product of lines. Similarly, if $K_r = 0$ then $C_r$ is a null circle, and if $K_g = 0$ then $C_g$ is a null circle. These null lines have distinct direction vectors $(1, \pm i), (1, 1)$ and $(0, 1), (0, 1)$ respectively, and they are never parallel since $\text{char}(\mathbb{F}) \neq 2$, so $i \neq \pm 1$. Therefore, any two null circles meet in exactly four points.

Here is an example found by Michael Reynolds [6] which illustrates explicitly the above theorem.

**Example 2** The triangle $X_1X_2X_3$ with points $X_1 \equiv [3,4], X_2 \equiv [1,9]$ and $X_3 \equiv [12,3]$ in $\mathbb{F}_{13}$ has four blue, red and green Incircles. In $\mathbb{F}_{13}$ the squares are 0, 1, 3, 4, 9, 10, 12, and in particular $-1 = 12 = 5^2$ is a square. After translation by $(3,4)$ we obtain $\tilde{X}_1 = [0,0], \tilde{X}_2 = [11,5]$ and $\tilde{X}_3 = [9,12]$. The matrix $N$ and its inverse $M$

\[
N = \begin{pmatrix} 11 & 5 \\ 9 & 12 \end{pmatrix}, 
\]

\[
M = N^{-1} = \begin{pmatrix} 10 & 11 \\ 12 & 7 \end{pmatrix}
\]

send $[1,0]$ and $[0,1]$ to $\tilde{X}_2$ and $\tilde{X}_3$, and $\tilde{X}_2$ and $\tilde{X}_3$ to $[1,0]$ and $[0,1]$ respectively. The bilinear form in these new standard coordinates for the blue, red and green geometries
respectively are
\[ D_b = N \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} N^T = \begin{pmatrix} 3 & 3 \\ 3 & 4 \end{pmatrix}, \]
\[ D_r = N \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} N^T = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}, \]
\[ D_g = N \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} N^T = \begin{pmatrix} 6 & 8 \\ 8 & 8 \end{pmatrix}. \]

We can see immediately that \( K_b = K_r = K_g = 0 \) from the definitions
\[ Q_b(H_b, I_{ir}) = \frac{b_b(a_b - b_b)(b_b - c_b)}{\Delta_b} \equiv K_b, \]
\[ Q_r(H_r, I_{rb}) = \frac{r_b(a_r - b_r)(b_r - c_r)}{\Delta_r} \equiv K_r, \]
\[ Q_g(H_g, I_{rg}) = \frac{g_b(a_g - b_g)(b_g - c_g)}{\Delta_g} \equiv K_g, \quad i = 0, 1, 2, 3. \]

The four blue, red and green Incenters respectively are
\[ I_0^b = [4, 8], \quad I_0^r = [3, 6], \quad I_0^g = [8, 10], \quad I_0^b = [11, 4], \]
\[ I_0^r = [10, 9], \quad I_1^r = [8, 2], \quad I_2^r = [6, 5], \quad I_3^r = [4, 12], \]
\[ I_0^g = [9, 8], \quad I_1^g = [5, 3], \quad I_2^g = [12, 11], \quad I_3^g = [2, 4], \]
and the blue, red and green Incenter Circles respectively have equations
\[ C_b : (y - x + 1)(x + 3y - 1) = 0, \]
\[ C_r : (x - 6y)(x + 6y) = 0, \]
\[ C_g : (x + 2y - 2)(x + 5y - 5) = 0. \]

From Michael Reynolds’ computer investigations, we tentatively conjecture that for finite fields \( \mathbb{F}_p \) where \( p \equiv 3 \mod 4 \), there are no triangles which have both blue, red and green Incenters, and for finite fields \( \mathbb{F}_p \) where \( p \equiv 1 \mod 4 \), blue, red and green Incenters exist precisely when \( K_b = K_r = K_g = 0 \), as in the above example.

4 Spieker circles and Nagel circles

Now we recall from [5] that the central dilation \( \delta_{-1/2} \) about the centroid takes the Orthocenter to the Circumcenter, and the Incenters to the Spieker centers. In standard coordinates
\[ \delta_{-1/2}([x, y]) = (1/2)(-x, -y) \]
The inverse central dilation \( \delta_{-2} \) takes the Orthocenter to the De Longchamps point \( X_{20} \), and takes the Incenters to the Nagel points. In standard coordinates
\[ \delta_{-2}([x, y]) = [1 - 2x, 1 - 2y]. \]

**Theorem 10 (Spieker circles)** If a triangle has four blue Inceters \( I_0^b, I_1^b, I_2^b \) and \( I_3^b \), then the four blue Spieker centers all lie both on a red Spieker circle with center the red Circumcenter \( C_r \), and on a green Spieker circle with center the green Circumcenter \( C_g \). If both say blue and red Incenters exist, then all 8 blue and red Spieker points lie on the same green circle. The same holds for the other colours.

**Proof.** We see that if we use the central dilation formula to transform Incenter circles centred at the Orthocenters, we get the Spieker circles centred at Circumcentres, so this theorem is a direct consequence of the Incenter circles theorem and the fact that a dilation preserves circles of any colour.

Here are the formulas for the coloured Circumcenters in standard coordinates:
\[ C_b = \frac{1}{2\Delta_b}[c_b(a_b - b_b), a_b(c_b - b_b)] \]
\[ C_r = \frac{1}{2\Delta_r}[c_r(a_r - b_r), a_r(c_r - b_r)] \]
\[ C_g = \frac{1}{2\Delta_g}[c_g(a_g - b_g), a_g(c_g - b_g)]. \]

Theorem 11 (Nagel circles) If a triangle has four blue Inceters \( I_0^b, I_1^b, I_2^b \) and \( I_3^b \), then the four blue Nagel centers all lie both on a red Nagel circle with center the red De Longchamps point \( X_{20} \), and on a green Nagel circle with center the green De Longchamps point \( X_{20g} \). If both say blue and red Incenters exist, then all 8 blue and red Nagel points lie on the same green circle. The same holds for the other colours.

**Proof.** In the same fashion as in the previous theorem, if we use the inverse central dilation \( \delta_{-2} \) to transform Incenter circles centred at the Orthocenters, we get the Nagel circles centred at De Longchamps points. \( \square \)
Here are the formulas for the blue, red and green De Longchamps points:

\[ X_{20b} = \frac{1}{\Delta_b} \left[ b_b^2 - 2b_b c_b + a_b c_b, b_b^2 - 2a_b b_b + a_b c_b \right] \]

\[ X_{20r} = \frac{1}{\Delta_r} \left[ b_r^2 - 2b_r c_r + a_r c_r, b_r^2 - 2a_r b_r + a_r c_r \right] \]

\[ X_{20g} = \frac{1}{\Delta_g} \left[ b_g^2 - 2b_g c_g + a_g c_g, b_g^2 - 2a_g b_g + a_g c_g \right]. \]

In Figure 10 we see the relations between the three coloured Orthocenters, Circumcenters and De Longchamps points. The lines joining these are the three coloured Euler lines. Note that the centroids of the triangles of Orthocenters, Circumcenters and De Longchamps points all agree with the centroid \( G \) of the original triangle \( A_1 A_2 A_3 \). We conclude with a simple observation about De Longchamps points.

**Theorem 12 (Orthocenters as midpoints)** For any triangle, a coloured orthocenter \( H \) is the midpoint of the two De Longchamps points \( X_{20} \) of the other two colours.

**Proof.** This follows by considering the action of the central dilation \( \delta_{-2} \) which takes the circumcenter \( C_i \) to the orthocenter \( H_i \), and the orthocenter \( H_i \) to the De Longchamps point \( X_{20i} \). \( \square \)

**References**


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