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The Arbelos with Overhang

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ABSTRACT

We consider a generalized arbelos consisting of three semicircles with collinear centers, in which only two of the three semicircles touch. Many Archimedean circles of the ordinary arbelos are generalized to our generalized arbelos.

Key words: arbelos, arbelos with overhang, Archimedean circles

MSC2010: 51M04, 51N20

Arbelos s privjeskom

SAŽETAK

Promatra se poopćeni arbelos koji se sastoji od tri polukružnice s kolinearnim središtima, pri čemu se dvije on njih dodiruju. Mnoge Arhimedove kružnice obićnog arbelosa su poopćene za poopćeni arbelos.

Ključne riječi: arbelos, arbelos s privjeskom, Arhimedove kružnice

1 Introduction

The *arbelos* is a plane figure consisting of three mutually touching semicircles with collinear centers. It has three points of tangency. In [5], [7] and [9], we have considered a generalized arbelos called a *collinear arbelos* consisting of three circles with collinear centers, in which one of the circles touches the remaining two circles, but the two circles do not touch in general. Thereby the collinear arbelos has two points of tangency.

In this paper, we consider the remaining case. We consider a configuration consisting of three semicircles with collinear centers, in which only two semicircles touch, i.e., it has only one point of tangency. Many Archimedean circles of the ordinary arbelos are generalized to our generalized arbelos, but also several new Archimedean circles of the ordinary arbelos are induced by this.

2 An arbelos with overhang

Let O be a point on the segment AB with $|AO| = 2a$ and $|BO| = 2b$. We use a rectangular coordinate system with origin O such that the coordinates of the points A and B are $(2a,0)$ and $(-2b,0)$, respectively. For two points P and Q, (PQ) and $P(Q)$ denote the circle with diameter PQ and the circle with center P passing through Q, respectively. However if their centers lie on the line AB, we consider them as semicircles lying in the region $y \geq 0$ unless otherwise stated. Let A' (resp. B') be a point on the half line with endpoint O passing through A (resp. B), and let $|A'O| = 2a'$ (resp. $|B'O| = 2b'$) (see Figure 1). Let $\gamma = (AB)$, and let δ'_α be the circle touching the semicircle $(A'O)$ externally γ internally and the perpendicular to AB passing through O from the side opposite to the point B. The circle δ'_β is defined similarly.

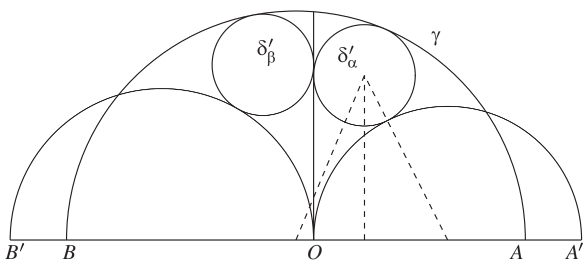

Figure 1

Proposition 1 *The two circles δ'_α and δ'_β are congruent if and only if $a' - a = b' - b$.*

Proof. Let r be the radius of δ'_α. The center of the circle with a diameter $A'O$ or AB, the center of $\delta_{\alpha'}$ and the foot of perpendicular from this point to AB form a right triangle. Hence by the Pythagorean theorem, we get

$$(r+a')^2 - (r-a')^2 = ((a+b)-r)^2 - (r-(a-b))^2.$$

Notice that if the foot of perpendicular coincides with the center of the circle with a diameter $A'O$ or AB, then one of the triangles degenerates to a segment. But the equation still holds. Solving the equation we get $r = ab/(a'+b)$. Similarly, δ'_β has radius $ab/(a+b')$. Therefore the two circles are congruent if and only if $a'+b = a+b'$. □

Let $\alpha = (AO)$, $\beta = (BO)$, and let $a' = a+h$, $b' = b+h$ with $-\min(a,b) < h$. We relabel a', b', A', B', δ'_α and δ'_β as a_h, b_h, A_h, B_h, δ^α_h and δ^β_h, respectively and let $\alpha_h = (A_hO)$ and $\beta_h = (B_hO)$. The configuration consisting of the three semicircles α_h, β_h and γ is denoted by $(\alpha_h, \beta_h, \gamma)$. We call $(\alpha_h, \beta_h, \gamma)$ an *arbelos with overhang* h, and $(\alpha_h, \beta_h, \gamma)$ is said to have overhang h. The ordinary arbelos (α, β, γ) has overhang 0. The perpendicular to AB passing through O is called the axis, which overlaps with the y-axis.

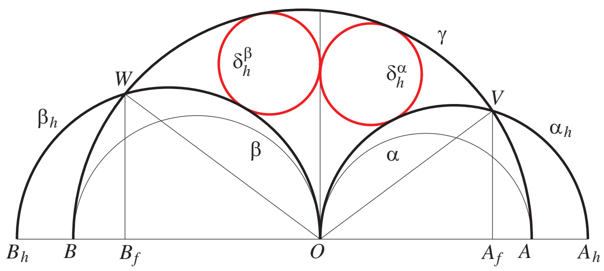

Figure 2

Now the circles δ^α_h and δ^β_h have the same radius $r^h_A = ab/(a+b+h)$ by Proposition 1. The two circles are a generalization of the twin circles of Archimedes of the ordinary arbelos (α, β, γ). Circles of radius r^h_A are said to be Archimedean circles of $(\alpha_h, \beta_h, \gamma)$ or Archimedean with respect to $(\alpha_h, \beta_h, \gamma)$. Also we say that $(\alpha_h, \beta_h, \gamma)$ has Archimedean circles of radius r^h_A. The common radius of Archimedean circles of (α, β, γ) is denoted by r_A, i.e., $r_A = ab/(a+b)$.

We define A_f and B_f as the points with coordinates $(2ab/b_h, 0)$ and $(-2ab/a_h, 0)$, respectively. Let γ have points V and W in common with the semicircles α_h and β_h respectively in the case $h \geq 0$ (see Figure 2). The points V and W have coordinates

$$(2ab/b_h, f(a,b)/b_h) \text{ and } (-2ab/a_h, f(a,b)/a_h),$$

respectively, where $f(a,b) = 2\sqrt{abh(a+b+h)}$. Therefore the points A_f and B_f are the feet of perpendiculars from V and W to the line AB, respectively. By the coordinates of V and W, we get $\tan\angle WOB = \tan\angle VOA$. Therefore $\angle WOB = \angle VOA$ holds.

The circle touching γ internally and the segment AB at the point O has radius $2r_A$ [11]. The fact is generalized as follows.

Proposition 2 *If $h > 0$, the radius of the circle touching γ internally and the segments OV and OW is $2r^h_A$.*

Proof. Let r and $(0, c)$ be the radius and the coordinates of the center of the touching circle. Then we get

$$(a-b)^2 + c^2 = (a+b-r)^2. \tag{1}$$

Also by similar triangles, we get

$$\frac{r}{c} = \cos\angle VOA = \sqrt{\frac{ab}{(a+h)(b+h)}}. \tag{2}$$

Eliminating c from (1) and (2), and solving the resulting equation for r with $h > 0$, we get $r = 2r^h_A$. □

Let $\alpha_f = (A_fO)$ and $\beta_f = (B_fO)$. Archimedean circles of the ordinary arbelos $(\alpha_f, \beta, (A_fB))$ have radius

$$\frac{(ab/b_h)b}{ab/b_h + b} = \frac{ab}{a+b_h} = r^h_A.$$

Similarly Archimedean circles of the ordinary arbelos $(\alpha, \beta_f, (AB_f))$ have the same radius. Hence we get:

Proposition 3 *The ordinary arbeloi $(\alpha_f, \beta, (A_fB))$ and $(\alpha, \beta_f, (AB_f))$ share Archimedean circles with $(\alpha_h, \beta_h, \gamma)$.*

The circle touching the axis at the point O from the side opposite to the point B and also touching the tangent of β from the point A is an Archimedean circle of the ordinary arbelos (α, β, γ), which is denoted by W_6 in [4]. Hence by Proposition 3, we get the following proposition. By this proposition we can construct the point A_f (also B_f) even in the case $h < 0$ (see Figures 7 and 14).

Proposition 4 *The point A_f coincides with the point of intersection of the line AB and the external common tangent of β and the Archimedean circle of $(\alpha_h, \beta_h, \gamma)$ touching the axis at the point O from the side opposite to the point B.*

Since $|AB_f| : |AB_h| = a : a_h$ holds, we get the following proposition, which also enable us to construct the points A_f and B_f in the case $h < 0$.

Proposition 5 *The point B_f divides the segment AB_h in the ratio $a : |h|$ internally or externally, according as $h > 0$ or $h < 0$.*

3 Several twin circles

In this section we show that several twin circles exist for $(\alpha_h, \beta_h, \gamma)$, if $h > 0$. Let us assume $h > 0$, and let ε^α_1 be the circle touching the semicircles α externally α_h internally and the segment A_fV from the side opposite to the point A (see Figure 3). Let ε^α_2 be the circle touching the semicircles α externally α_h and γ internally. Also let ε^α_3 be the circle touching α_h and γ externally and the axis from the side opposite to the point B. The circles ε^β_1, ε^β_2 and ε^β_3 are defined similarly. The following proposition has a straightforward proof that is omitted.

Proposition 6 *If $h > 0$, the following statements hold.*
(i) *The circles ε_1^α and ε_1^β have the same radius*

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{h}{ab}+\frac{1}{h}\right)^{-1}=\left(\frac{1}{r_{\mathrm{A}}^{\mathrm{h}}}+\frac{1}{h}\right)^{-1}.$$

(ii) *The circles ε_2^α and ε_2^β have the same radius*

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{h}\right)^{-1}=\left(\frac{1}{r_{\mathrm{A}}}+\frac{1}{h}\right)^{-1}.$$

(iii) *The circles ε_3^α and ε_3^β have the same radius ab/h.*

The proposition also shows that the sum of the curvatures of the circles ε_2^α and ε_3^α equals the curvature of the circle ε_1^α.

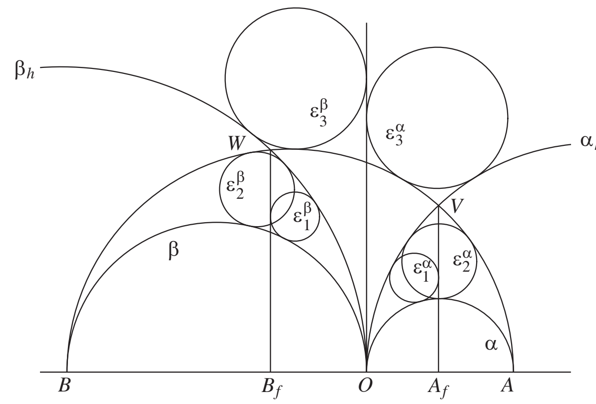

Figure 3

4 Bankoff circles

The circle orthogonal to the semicircles α, β and to the circle touching α and β externally and γ internally is an Archimedean circle of (α,β,γ) called the Bankoff triplet circle, which is denoted by W_3 in [4]. The maximal circle touching the external common tangent of α and β and the arc of γ cut by the tangent internally is an Archimedean circle of (α,β,γ) called the Bankoff quadruplet circle, which is denoted by W_4 in [4]. In this section we generalize the two circles (see Figures 4 and 6). Let $\gamma_f = (A_f B_f)$.

Theorem 1 *The following two circles are Archimedean with respect to $(\alpha_h,\beta_h,\gamma)$, and coincide.*
(i) *The circle orthogonal to the semicircles α, β and to the circle touching α and β externally and γ_f internally.*
(ii) *The circle orthogonal to the semicircles α_f, β_f and to the circle touching α_f and β_f externally and γ internally.*

Proof. Let δ be the circle touching α and β externally and γ_f internally, and let ε be the circle denoted by (i). We invert the figure in the circle with center O and radius $2\sqrt{ab}$, and label the images with an overline (see Figure 5). The x-coordinates of the points $\overline{A}$, $\overline{B}$, $\overline{A_f}$ and $\overline{B_f}$ are $2b$, $-2a$, $2b_h$ and $-2a_h$, respectively, and the circle $\overline{\gamma_f} = (\overline{A_f}\,\overline{B_f})$ has center $(b-a,0)$ and radius a_h+b_h. Let $(x_{\overline{\delta}}, y_{\overline{\delta}})$ and $r_{\overline{\delta}}$ be the coordinates of the center of the circle $\overline{\delta}$ and its radius. The circle $\overline{\delta}$ touches $\overline{\alpha}$ and $\overline{\beta}$, which are the perpendiculars to AB passing through the points $\overline{A}$ and $\overline{B}$, respectively. Therefore we get $x_{\overline{\delta}} = b-a$ and $r_{\overline{\delta}} = a+b$. Since $\overline{\delta}$ touches $\overline{\gamma_f}$ externally and the x-coordinates of their centers are the same, we get

$$y_{\overline{\delta}} = a_h+b_h+r_{\overline{\delta}} = 2(a+b+h).$$

Since $\overline{\varepsilon}$ is the line perpendicular to the line $\overline{\alpha}$ and passes through the point of tangency of $\overline{\alpha}$ and $\overline{\delta}$, it is parallel to AB and passes through the center of $\overline{\delta}$. Hence the distance between AB and the farthest point on ε equals $4ab/y_{\overline{\delta}} = 2r_{\mathrm{A}}^{\mathrm{h}}$. Therefore ε is Archimedean with respect to $(\alpha_h,\beta_h,\gamma)$. The part (ii) is proved similarly. □

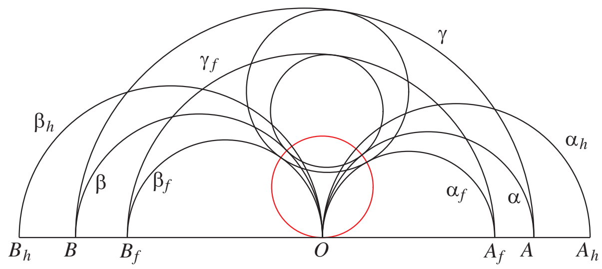

Figure 4

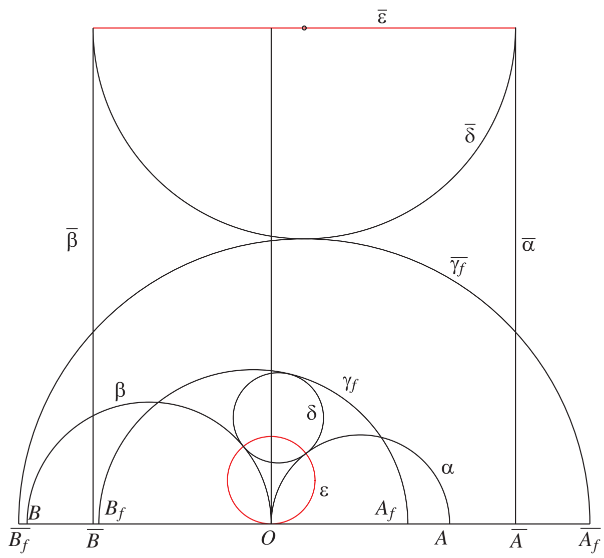

Figure 5

We call the circle in Theorem 1 the Bankoff triplet circle of $(\alpha_h,\beta_h,\gamma)$.

Theorem 2 *If $\mathcal{E}$ is the external common tangent of the semicircles α and β_h, or α_h and β, then the maximal circle touching $\mathcal{E}$ and the arc of γ cut by $\mathcal{E}$ (the part of γ between the two points of intersection of γ and $\mathcal{E}$) internally is Archimedean with respect to $(\alpha_h,\beta_h,\gamma)$.*

Proof. We prove the case $\mathcal{E}$ being the common tangent of α and β_h. The other case is proved similarly. Let d be the distance between $\mathcal{E}$ and the centers of γ, and let T be the point of intersection of the lines $\mathcal{E}$ and AB. If T lies in the region $x>0$, let $|AT|=t$. By similar triangles, we get

$$a/(t+a)=d/(t+a+b)=b_h/(t+2a+b_h).$$

Eliminating t and solving the resulting equations for d, we get $d=a+b-2r_{\rm A}^{\rm h}$. Therefore δ is an Archimedean circle of $(\alpha_h,\beta_h,\gamma)$. The case T lying in the region $x<0$ is proved similarly. If $\mathcal{E}$ and AB are parallel, then $a=b+h=d$ and $r_{\rm A}^{\rm h}=ab/(2a)=b/2$. Therefore we also get $d=a+b-2r_{\rm A}^{\rm h}$. □

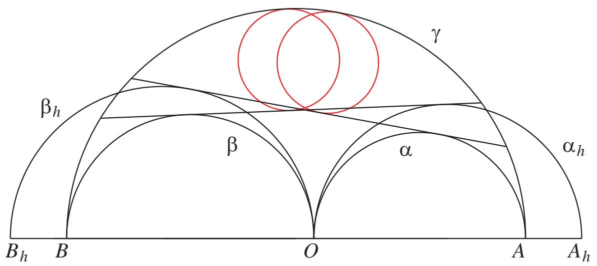

Figure 6

5 Miscellaneous Archimedean circles

In this section we consider miscellaneous Archimedean circles of $(\alpha_h,\beta_h,\gamma)$ obtained from points dividing given segments in the ratio $a:b_h$ or $a_h:b$ internally, some of which seem to be new even for the ordinary arbelos. Let I be the point of intersection of the axis and the semicircle γ. The minimal circle touching the axis and passing through the point of intersection of the semicircle α and the segment AI is an Archimedean circle of (α,β,γ), which is denoted by W_9 in [4]. Also the minimal circle touching the axis and passing through the point of intersection of the semicircles γ and $A(O)$ is an Archimedean circle of (α,β,γ), which is denoted by W_{13} in [4]. The two facts are generalized. Let A_m and B_m be the midpoints of the segments AA_h and BB_h, respectively (see Figure 7).

Theorem 3 (i) *If I_α is the point of intersection of the axis and the semicircle (A_hB), then the point dividing the segment A_fI_α in the ratio $a:b_h$ internally lies on the semicircle α and its distance from the axis is $2r_{\rm A}^{\rm h}$.*
(ii) *The distance between the axis and the point of intersection of the semicircles $A_m(O)$ and γ is $2r_{\rm A}^{\rm h}$.*

Proof. Since I_α has coordinates $(0,2\sqrt{a_hb})$, the point dividing the segment A_fI_α in (i) has coordinates

$$\left(\frac{b_h\cdot 2ab/b_h}{a+b_h},\frac{a\cdot 2\sqrt{a_hb}}{a+b_h}\right)=\left(2r_{\rm A}^{\rm h},2r_{\rm A}^{\rm h}\sqrt{\frac{a_h}{b}}\right).$$

This proves (i). The point of intersection of γ and $A_m(O)$ has coordinates $\left(2r_{\rm A}^{\rm h},2\sqrt{(a-r_{\rm A}^{\rm h})(b+r_{\rm A}^{\rm h})}\right)$. This proves (ii). □

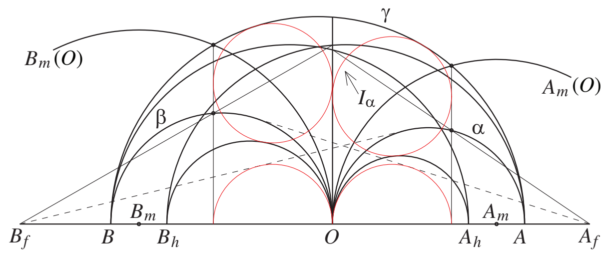

Figure 7

For a circle or a semicircle δ, its center is denoted by O_δ. The farthest point on δ from AB lying in the region $y\geq 0$ is denoted by T_δ. If the segments $T_\alpha T_\beta$ and $T_\gamma O_\gamma$ intersect at a point P, the circle (PT_γ) is an Archimedean circle of (α,β,γ), which is denoted by W_{20} in [4]. The fact is generalized (see Figure 8).

Theorem 4 *The segments $T_{\alpha_h}T_\beta$, $T_\alpha T_{\beta_h}$ and $T_\gamma O_\gamma$ intersect at a point P, which divides $T_{\alpha_h}T_\beta$ and $T_\alpha T_{\beta_h}$ in the ratios $b_h:a$ and $b:a_h$ internally, respectively. The circle (PT_γ) is Archimedean with respect to $(\alpha_h,\beta_h,\gamma)$.*

Proof. The points dividing $T_{\alpha_h}T_\beta$ in the ratio $b_h:a$ internally and $T_\alpha T_{\beta_h}$ in the ratio $b:a_h$ internally have the same coordinates $(a-b,a+b-2r_{\rm A}^{\rm h})$. □

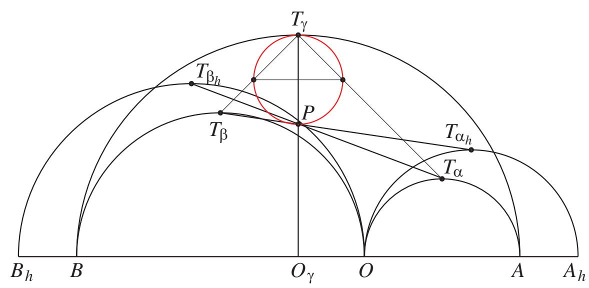

Figure 8

In the theorem, the endpoints of the diameter of (PT_γ) parallel to AB divide the segments $T_\gamma T_\alpha$ and $T_\gamma T_\beta$ in the ratios $a:b_h$ and $b:a_h$ internally, respectively.

Theorem 5 *Let T'_{α_h} and T'_β be the reflected images of the points T_{α_h} and T_β in the line AB, respectively. The following statements hold.*

(i) *Let C be the internal center of similitude of the semicircles γ and α_h. If D is the point of intersection of the lines CT'_β and AT_α, then D divides AT_α in the ratio $b : a_h$ internally and the circle touching α or AB at the point A and passing through D is Archimedean with respect to $(\alpha_h, \beta_h, \gamma)$.*
(ii) *The segments $A_f T_\beta$, AT_{β_f} and $T_\alpha O$ intersect at a point H, which divides $A_f T_\beta$ in the ratio $a : b_h$ internally AT_{β_f} and $T_\alpha O$ in the ratio $a_h : b$ internally, respectively. The circle touching α or AB at the point O and passing through H is Archimedean with respect to $(\alpha_h, \beta_h, \gamma)$.*
(iii) *If E is the point of intersection of the segments $T'_{\alpha_h} T_\gamma$ and OT_α, then E divides $T'_{\alpha_h} T_\gamma$ and OT_α in the ratio $a_h : b$ internally and the circle touching α or the line $O_\alpha T_\alpha$ at the point T_α and passing through E is Archimedean with respect to $(\alpha_h, \beta_h, \gamma)$.*

Proof. The point C has coordinates $(2aa_h/(2a+b_h), 0)$ (see Figure 9). If we regard β as a circle, then C coincides with the internal center of similitude of β and the Archimedean circle of $(\alpha_h, \beta_h, \gamma)$ touching α at the point A internally. Hence D has coordinates $(2a - r_A^h, r_A^h)$. This proves (i). The points dividing $A_f T_\beta$ in the ratio $a : b_h$ internally, AT_{β_f} and $T_\alpha O$ in the ratio $a_h : b$ internally have the same coordinates (r_A^h, r_A^h). This proves (ii). The points dividing OT_α and $T'_{\alpha_h} T_\gamma$ in the ratio $a_h : b$ internally have the same coordinates $(a - r_A^h, a - r_A^h)$. This proves (iii). $\square$

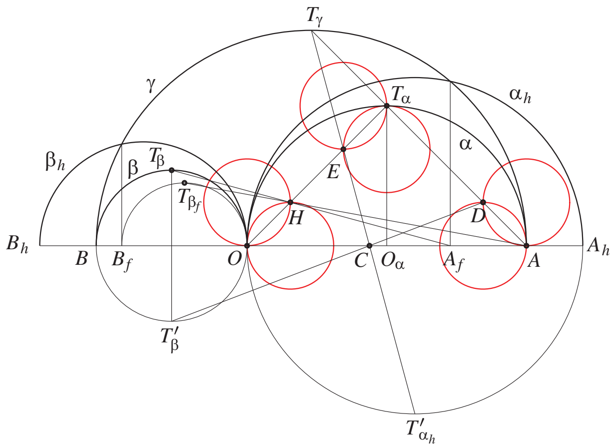

Figure 9

The circle touching AB at O and passing through H in (ii) is the Bankoff triplet circle of $(\alpha_h, \beta_h, \gamma)$.

6 Archimedean circles touching γ

In [12], we gave necessary and sufficient conditions that a circle touching the semicircle γ internally is Archimedean with respect to (α, β, γ). In this section we generalize this.

Let $\alpha(z)$ and $\beta(z)$ be the semicircles constructed in the region $y \geq 0$ touching the axis at the point O and having the centers with coordinates $(za, 0)$ and $(-zb, 0)$, respectively for a real number z[1]. Let $\mathcal{C}(m,n)$ be the circle touching the semicircles γ internally and $\alpha(m)$ and $\beta(n)$ at points different from O such that the points of tangency on $\alpha(m)$, γ and $\beta(n)$ lie counterclockwise in this order for real numbers m and n. The radius of $\mathcal{C}(m,n)$ is expressed as follows [12, Theorem 1]:

$$\frac{ab(ma+nb)}{ma^2+nb^2+mnab}. \tag{3}$$

Let $\alpha_h(z)$ and $\beta_h(z)$ be the semicircles constructed in the region $y \geq 0$ touching the axis at the point O and having the centers with coordinates $(za_h, 0)$ and $(-zb_h, 0)$, respectively for a real number z. Let $\mathcal{C}_h(m,n)$ be the circle touching the semicircles γ internally and $\alpha_h(m)$ and $\beta_h(n)$ at points different from O such that the points of tangency on $\alpha_h(m)$, γ and $\beta_h(n)$ lie counterclockwise in this order for real numbers m and n.

Theorem 6 *The circle $\mathcal{C}_h(m,n)$ has radius*

$$\frac{ab(ma_h+nb_h)}{maa_h+nbb_h+mna_hb_h}. \tag{4}$$

Proof. Notice that $\alpha_h(m) = \alpha(ma_h/a)$ and $\beta_h(n) = \beta(nb_h/b)$. Replacing m and n by ma_h/a and nb_h/b respectively in (3), we get (4). $\square$

Theorem 7 *The circle $\mathcal{C}_h(m,n)$ is Archimedean with respect to $(\alpha_h, \beta_h, \gamma)$ if and only if*

$$\frac{1}{m}+\frac{1}{n}=1. \tag{5}$$

Proof. The theorem follows from Theorem 6, because

$$\frac{ab(ma_h+nb_h)}{maa_h+nbb_h+mna_hb_h} - r_A^h = \frac{(m+n-mn)a_hb_hr_A^h}{maa_h+nbb_h+mna_hb_h}. \quad \square$$

Corollary 1 *The following circles are Archimedean with respect to $(\alpha_h, \beta_h, \gamma)$.*
(i) *The circle touching the semicircles $A_h(O)$ and $B_h(O)$ externally and γ internally.*
(ii) *The circle touching γ internally and the two distinct circles of radius $a_h + b_h$ touching the axis at the point O externally.*

Proof. The part (i) follows from the fact $A_h(O) = \alpha_h(2)$ and $B_h(O) = \beta_h(2)$ (see Figure 10). The part (ii) follows from the fact that $m = (a_h+b_h)/a_h$ and $n = (a_h+b_h)/b_h$ satisfy (5). $\square$

[1] The notations are slightly changed from [12]

The circle described in (i) is a generalization of Schoch circle of the ordinary arbelos which is denoted by W_{15} in [4]. The circle described in (ii) is a generalization of the Archimedean circle of the ordinary arbelos in [10].

Let $\gamma_h = (A_hB_h)$. If $m > 0$ (resp. $m < 0$), let $P_{\alpha_h}(m)$ be the external (resp. internal) center of similitude of the semicircles $\alpha_h(m)$ and γ_h. Similarly the point $P_{\beta_h}(m)$ is defined.

Theorem 8 *The points $P_{\alpha_h}(m)$ and $P_{\beta_h}(n)$ coincide if and only if* (5) *holds.*

Proof. The semicircles $\alpha_h(m)$ and γ_h have radii ma_h and $a_h + b_h$ and centers with x-coordinates ma_h and $a_h - b_h$, respectively. Hence $P_{\alpha_h}(m)$ has x-coordinate

$$\frac{(a_h+b_h)ma_h - ma_h(a_h-b_h)}{-ma_h+(a_h+b_h)} = \frac{2ma_hb_h}{a_h+b_h-ma_h}.$$

Similarly $P_{\beta_h}(n)$ has x-coordinate

$$\frac{(a_h+b_h)(-nb_h) - nb_h(a_h-b_h)}{-nb_h+(a_h+b_h)} = \frac{-2na_hb_h}{a_h+b_h-nb_h}.$$

While

$$\begin{aligned} &\frac{2ma_hb_h}{a_h+b_h-ma_h} - \frac{-2na_hb_h}{a_h+b_h-nb_h} \\ &= \frac{2(m+n-mn)(a_h+b_h)a_hb_h}{(a_h+b_h-ma_h)(a_h+b_h-nb_h)}. \end{aligned}$$

Therefore the proof is complete. □

Corollary 2 *The circle $\mathcal{C}_h(m,n)$ is Archimedean with respect to $(\alpha_h, \beta_h, \gamma)$ if and only if the points $P_{\alpha_h}(m)$ and $P_{\beta_h}(n)$ coincide.*

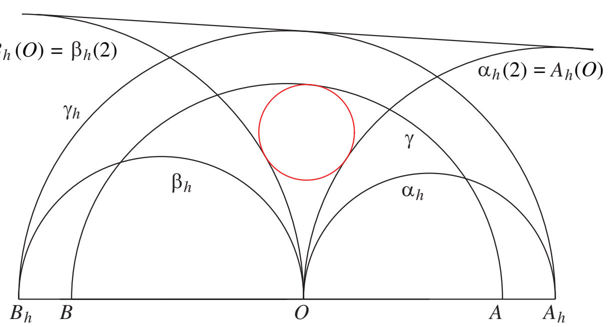

Figure 10: $m = n = 2$

If the external common tangents from γ_h to both $\alpha_h(m)$ and $\beta_h(n)$ exist, then the circle $\mathcal{C}_h(m,n)$ is Archimedean with respect to $(\alpha_h, \beta_h, \gamma)$ if and only if the two tangents coincide (see Figure 10).

7 Woo's Archimedean circles

We assume $a \neq b$ in this section. Peter Woo found that the circle touching $\alpha(z)$ and $\beta(z)$ externally with center on the line $x = (b-a)r_A/(b+a)$ is Archimedean with respect to (α, β, γ) for a non-negative real number z [4]. The line is called the Schoch line of (α, β, γ). The fact was generalized for a real number $z \geq -r_A/(a+b)$ in [12]. We generalize this.

A circle is said to touch $\alpha_h(z)$ appropriately if it touches $\alpha_h(z)$ externally in the case $z > 0$ and it touches the reflected image of $\alpha_h(z)$ in the line AB internally in the case $z < 0$. The same notion of appropriate tangency applies to $\beta_h(z)$. Let $s_h = (b_h - a_h)r^h_A/(b_h + a_h)$. We call the line $x = s_h$ the Schoch line of $(\alpha_h, \beta_h, \gamma)$.

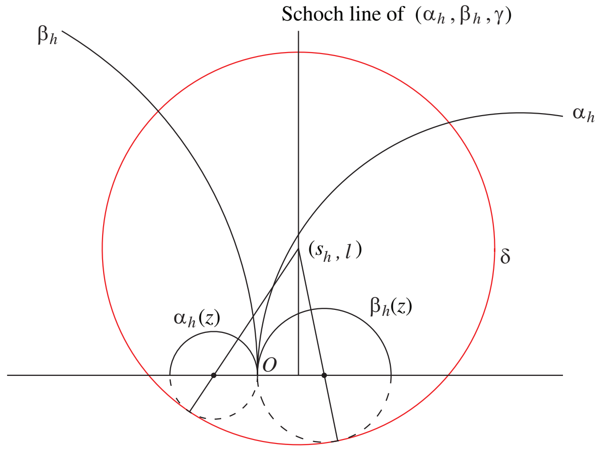

Figure 11: $z < 0$

Theorem 9 *Let δ be the circle touching $\alpha_h(z)$ and $\beta_h(z)$ appropriately and having its center on the Schoch line of $(\alpha_h, \beta_h, \gamma)$ for a real number $z \neq 0$. The following statements hold.*

(i) *The circle δ is Archimedean with respect to $(\alpha_h, \beta_h, \gamma)$.*

(ii) *The circle δ exists if and only if $-a_hb_h/(a_h+b_h)^2 \leq z < 0$ or $0 < z$.*

Proof. If r is the radius of δ and l is the y-coordinate of its center (see Figure 11), then we get

$$(za_h+r)^2 - (s_h - za_h)^2 = (zb_h+r)^2 - (s_h+zb_h)^2 = l^2. \quad (6)$$

Solving the equation for r, we get $r = s_h(b_h+a_h)/(b_h - a_h) = r^h_A$. This proves (i). From (6) we also get

$$\begin{aligned} l^2 &= \frac{4a_hb_hs_h(s_h+(b_h-a_h)z)}{(a_h-b_h)^2} \\ &= \frac{4a_h^2b_h^2(a_hb_h+(a_h+b_h)^2z)}{(a_h+b_h)^4}. \end{aligned}$$

Therefore l satisfying (6) is real if and only if $-a_hb_h/(a_h+b_h)^2 \leq z < 0$ or $0 < z$. This proves (ii). □

Notice that the circle δ in Theorem 9 is not uniquely determined if $a = b$. We get an infinite set of Archimedean circles of $(\alpha_h, \beta_h, \gamma)$, whose centers lie on the line $x = s_h$ by the theorem. However the Archimedean circle of $(\alpha_h, \beta_h, \gamma)$ with center $(s_h, 2a_h b_h \sqrt{a_h b_h}/(a_h + b_h)^2)$ is not a member of this set. In fact, there are infinitely many circles passing through O with center on this line. But it seems to be natural to consider this circle as a member of this set.

8 Dilation

In [8], we have shown that if σ is a dilation with center O, the circle touching the semicircles $(A^\sigma O)$ externally (AB^σ) internally and the axis from the side opposite to the point B is an Archimedean circle of (α, β, γ), where A^σ and B^σ are the images of A and B by σ, respectively. In this section we generalize this fact.

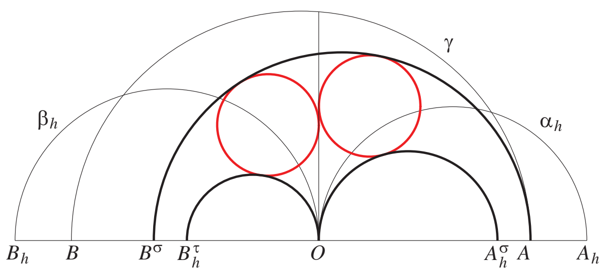

Figure 12: $t = 2/3$

Theorem 10 *Let σ and τ be the dilations with center O and A with the same ratio of magnification t, respectively. Then the following statements are true.*
(i) *The circle touching the semicircles $(A_h^\sigma O)$ externally (AB^σ) internally and the axis from the side opposite to the point B is Archimedean with respect to $(\alpha_h, \beta_h, \gamma)$.*
(ii) *If $t > a/(a + b_h)$, then $((A_h^\sigma O), (B_h^\tau O), (AB^\sigma))$ is an arbelos with overhang $ta_h - a$, and has Archimedean circles of radius $r_{\rm A}^{\rm h}$.*
(iii) *If $t = a/a_h$, then $((A_h^\sigma O), (B_h^\tau O), (AB^\sigma))$ coincides with $(\alpha, \beta_f, (AB_f))$, and the points B_h^τ and B^σ also coincide with the point B_f.*

Proof. Let r be the radius of the touching circle in (i) (see Figure 12). Then we get

$$(ta_h + r)^2 - (ta_h - r)^2 = (bt + a - r)^2 - ((-bt + a) - r)^2.$$

Solving the equation, we get $r = r_{\rm A}^{\rm h}$. This proves (i). If $t > a/(a + b_h)$, then $|AB_h^\tau| = 2t(a + b_h) > 2a = |AO|$. Hence the point B_h^τ lies on the half line with endpoint O passing through B. While $|A_h^\sigma O| - |AO| = 2(ta_h - a)$ and $|B_h^\tau O| - |B^\sigma O| = 2t(a + b_h) - 2(tb - a) = 2(ta_h - a)$. Hence the configuration $((A_h^\sigma O), (B_h^\tau O), (AB^\sigma))$ is an arbelos with overhang $ta_h - a$. The rest of (ii) follows from (i). If $t = a/a_h$, the points A and A_h^σ coincide, i.e., $(A_h^\sigma O) = \alpha$. While $a/a_h > a/(a_h + b) = a/(a + b_h)$ holds. Therefore we get an ordinary arbelos $(\alpha, (B_h^\tau O), (AB^\sigma))$, whose Archimedean circles have radius $r_{\rm A}^{\rm h}$ by (ii). While $(\alpha, \beta_f, (AB_f))$ is also an ordinary arbelos having Archimedean circles of the same radius by Proposition 3. Therefore the two ordinary arbeloi coincide. The rest of (iii) is obvious. □

9 New type of Archimedean circles

Quang Tuan Bui has found a pair of new type of Archimedean circles such that the endpoints of the diameter parallel to the line AB lie on a given circle [1], which has been rediscovered by us [6]. One of the circles is obtained as follows: If the line $T_\alpha O_\alpha$ intersects the semicircle γ at a point S and the lines SA and SO intersect the semicircle α at points T and U respectively, the circle (TU) is Archimedean with respect to (α, β, γ). The fact is generalized (see Figures 13 and 14). Notice that $h + r_{\rm A}^{\rm h} > 0$ and $a - r_{\rm A}^{\rm h} > 0$.

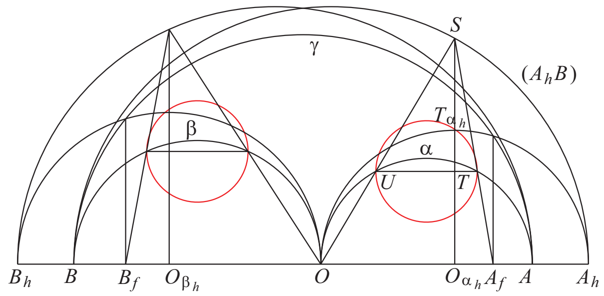

Figure 13

Theorem 11 (i) *Let S be the point of intersection of the semicircle $(A_h B)$ and the the line $T_{\alpha_h} O_{\alpha_h}$. If T is the point dividing the segment SA_f in the ratio $(h + r_{\rm A}^{\rm h}) : (a - r_{\rm A}^{\rm h})$ internally and U is the point of intersection of the line SO and the semicircle α, then T lies on α and the line TU is parallel to AB and the circle (TU) is Archimedean with respect to $(\alpha_h, \beta_h, \gamma)$.*
(ii) *Let S be the point of intersection of the semicircle (AB_h) and the line $T_\alpha O_\alpha$. If the line SO intersects the semicircle α_f at a point U and the line parallel to AB passing through U intersects α_f at a point T again, then the circle (TU) is Archimedean with respect to $(\alpha_h, \beta_h, \gamma)$.*

Proof. The point S in (i) has coordinates $(a_h, g(a,b))$, where $g(a,b) = \sqrt{a_h(a_h + 2b)}$. Hence the points T and U have coordinates $(a + r_{\rm A}^{\rm h}, g(a,b) r_{\rm A}^{\rm h}/b)$ and $(a - r_{\rm A}^{\rm h}, g(a,b) r_{\rm A}^{\rm h}/b)$, respectively. This proves (i). The point U in (ii) has x-coordinate $ab/b_h - r_{\rm A}^{\rm h}$. This proves (ii). □

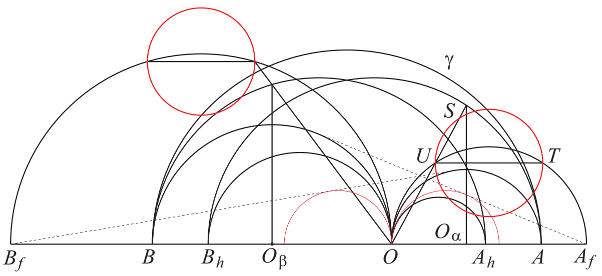

Figure 14

Notice that $(h+r_{\rm A}^{\rm h}):(a-r_{\rm A}^{\rm h})=b:a$ if $h=0$.

10 Power type Archimedean circles

From now on we consider all the semicircles with centers on the line AB as circles. If two congruent circles of radius r touching at a point D also touch a given circle δ at points different from D, we say that D generates circles of radius r with δ, and the two circles are said to be generated by D with δ. If the two generated circles are Archimedean with respect to $(\alpha_h,\beta_h,\gamma)$, we say that D generates Archimedean circles of $(\alpha_h,\beta_h,\gamma)$ with δ.
Frank Power has found that the point T_α generates Archimedean circles of (α,β,γ) with the circle γ [13]. Quang Tuan Bui has found that the circles (AO_β), (BO_α) and the axis belong to the same intersecting pencil of circles and the points of intersection generate Archimedean circles of (α,β,γ) with γ [3]. We generalize the two facts. The following lemma is needed [7].

Lemma 1 *For a circle δ of radius r, a point D generates circles of radius $||DO_\delta|^2-r^2|/(2r)$ with δ.*

The parts (i) and (ii) of the next theorem are generalizations of Power's result and Bui's result, respectively (see Figure 15).

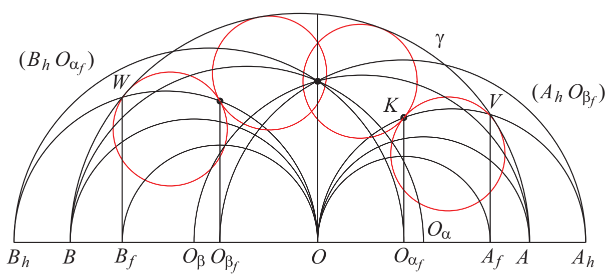

Figure 15

Theorem 12 (i) *If the circle α_h and the line $T_{\alpha_f}O_{\alpha_f}$ have a point in common, the point generates Archimedean circles of (α,β,γ) with γ.*
(ii) *The circles (AO_β), (BO_α), $(A_hO_{\beta_f})$, $(B_hO_{\alpha_f})$ and the axis belong to the same intersecting pencil of circles, and the points of intersection generate Archimedean circles of (α,β,γ) with γ.*

Proof. Let K be the point in (i) lying on α_h and $T_{\alpha_f}O_{\alpha_f}$. Then $|KO_{\alpha_f}|^2=a_h^2-(a_h-ab/b_h)^2$. Therefore $|KO_\gamma|^2-(a+b)^2=|O_\gamma O_{\alpha_f}|^2+|KO_{\alpha_f}|^2-(a+b)^2=(ab/b_h-(a-b))^2+a_h^2-(a_h-ab/b_h)^2-(a+b)^2=-2ab$. Therefore K generates Archimedean circles of (α,β,γ) with γ by Lemma 1. The part (ii) follows from the fact that the powers of the point O with respect to (AO_β), $(A_hO_{\beta_f})$ and $(B_hO_{\alpha_f})$ are the same. □

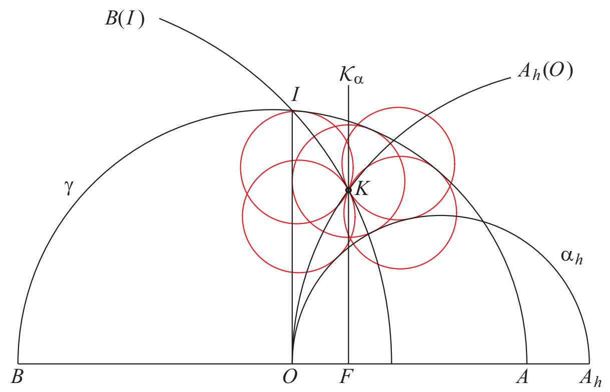

Figure 16

Recall that I is the point of intersection of the axis and the circle γ lying in the region $y>0$. Quang Tuan Bui has also found that the points of intersection of the circles (AO) and $B(I)$ generate Archimedean circles of (α,β,γ) with the circle γ [2]. Let J be the point of intersection of the circle $B(I)$ and the line AB lying in the region $x>0$. If $a<b$, we can choose h so that $4a_h<|OJ|$ holds. Then the circles $A_h(O)$ and $B(I)$ have no points in common. Let $\mathcal{K}_\alpha$ be the perpendicular to the line AB from the center of the circle δ_h^α. Quang Tuan Bui's result is generalized as follows.

Theorem 13 (i) *The circles $A_h(O)$, $B(I)$ and the line $\mathcal{K}_\alpha$ belong to the same pencil of circles. If the pencil is intersecting, the points of intersection generate Archimedean circles of $(\alpha_h,\beta_h,\gamma)$ with each of the circles γ and α_h.*
(ii) *The circles $A_m(O)$, $B_m(I)$ and the line $\mathcal{K}_\alpha$ belong to the same intersecting pencil of circles, and the points of intersection generate Archimedean circles of (α,β,γ) with the circle γ.*

Proof. The circles $A_h(O)$ and $B(I)$ are expressed by the equations

$$(x-2a_h)^2+y^2=4a_h^2 \tag{7}$$

and

$$(x+2b)^2+y^2=4b^2+4ab, \tag{8}$$

respectively (see Figure 16). Subtracting (8) from (7) and rearranging, we get $x=r_{\rm A}^{\rm h}$. Therefore $A_h(O)$, $B(I)$ and

$\mathcal{K}_\alpha$ belong to the same pencil of circles. Let us assume that the pencil is intersecting and K is one of the points of intersection. Let F be the foot of the perpendicular from K to AB. Then $|KF|^2 = 4a_h^2 - (r_A^h - 2a_h)^2$. Then $|KO_\gamma|^2 - (a+b)^2 = (r_A^h - (a-b))^2 + |KF|^2 - (a+b)^2 = -2(a+b)r_A^h$. Therefore K generates Archimedean circles of $(\alpha_h, \beta_h, \gamma)$ with γ by Lemma 1. The rest of (i) follows from $|KO_{\alpha_h}|^2 - a_h^2 = (r_A^h - a_h)^2 + |KF|^2 - a_h^2 = 2a_h r_A^h$. We prove (ii). The circles $A_m(O)$ and $B_m(I)$ are expressed by the equations

$$(x-(2a+h))^2 + y^2 = (2a+h)^2 \tag{9}$$

and

$$(x+2b+h)^2 + y^2 = (2b+h)^2 + 4ab, \tag{10}$$

respectively. Subtracting (10) from (9), we get $x = r_A^h$. Therefore $A_m(O)$, $B_m(I)$ and $\mathcal{K}_\alpha$ belong to the same pencil. Substituting $x = r_A^h$ in (9), and using $a+h > 0$, we get $y^2 = (2a+h)^2 - (r_A^h - (2a+h))^2 = r_A^h(2(2a+h) - r_A^h) > r_A^h(2a - r_A^h) > 0$. Therefore $A_m(O)$ and $\mathcal{K}_\alpha$ intersect. The rest of (ii) can be proved similarly as the proof of (i). Let K be one of the points of intersection in (ii), and let F be the foot of the perpendicular from K to AB. Then $|KF|^2 = r_A^h(2(2a+h) - r_A^h)$. Therefore $|KO_\gamma|^2 - (a+b)^2 = (r_A^h - (a-b))^2 + |KF|^2 - (a+b)^2 = -2ab$. This proves (ii). □

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