Abstract. The local connectedness of inverse limit spaces was studied in many papers ([2], [5], [7], [8], [11]).

The main purpose of the present paper is to prove the following theorem.

THEOREM 1.8 Let $X = \{X_\alpha, f_{\alpha\beta} \}$ be an inverse system such that the projections $f_{\alpha\beta}$ are irreducible fully closed mappings. In order that $\lim X$ be locally connected it is necessary that each $X_\alpha$ be locally connected and it is sufficient that each $X_\alpha$ be a locally connected space without local cut points.

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0. Introduction

Introduction contains some basic definitions and notations.

0.1. Let $Y$ be a subset of a space $X$. By $\text{Cl}_X Y$ or $\text{Cl} Y$ is denoted the closure of $Y$ in $X$. The boundary of the subset $A \subseteq X$ is denoted by $\text{Fr}(A)$.

0.2. The symbols $\mathbb{N}$ and $\mathbb{R}$ denote the sets of positive integers and real numbers.

0.3. If $A$ is a well-ordered set, then $\text{cf}(A)$ denotes the smallest ordinal number which is cofinal in $A$.

0.4. If $A$ is a well-ordered set, then $|A|$ denotes the cardinality of $A$.

0.5. If $\omega$ and $\kappa$ are cardinal numbers, then $\omega_\kappa$ denotes the cardinality of the set of all functions from $\kappa$ to $\omega$.

0.6. By $X = \{X_\alpha, f_{\alpha\beta}, A\}$ is denoted an inverse system and by $\lim X$ its limit.

0.7. An inverse system $X = \{X_\alpha, f_{\alpha\beta}, A\}$ is $\sigma$-directed if $A$ is $\sigma$-directed, i.e., for each sequence $a_1, a_2, \ldots, a_n, \ldots$ of the members of $A$ there is an $a \in A$ such that $a \geq a_i$, $i = 1, 2, \ldots$.

0.8. A topological space $X$ is called pseudocompact [4:263] if $X$ is completely regular and every real-valued function defined on $X$ is bounded.

0.9. A space $X$ is $\kappa$-compact if each open cover of $X$ of cardinality $\leq \kappa$ has a finite subcover.

1. Local connectedness of the inverse limit space

In the sequel we use the following characterization of local connectedness.

LEMMA 1.1 ([10:II, 242, Teorema 1.]). A space $X$ is locally connected if and only if each family $\{A_t : t \in T\}$ of subsets $A_t$ of $X$ has the property

\[
\text{Fr}(\bigcup \{A_t : t \in T\}) \subseteq \text{Cl}(\bigcup \{\text{Fr}A_t : t \in T\})
\]

(1)
REMARK 1.2 a) In the paper [9] it was proved that a non-locally connected space $X$ contains a family $\{A_t: t \in T\}$ of open disjoint subsets $A_t$ of $X$ for which (1) is not satisfied.

b) If $X$ is regular, then the family $\{A_t\}$ may be chosen so that $X - \text{Cl}(\bigcup\{A_t: t \in T\})$ is non-empty. Namely, if $X$ is not locally connected at some point $p$, then there are neighbourhoods $U$ and $V$ such that $x \in \text{Cl}(U \cup \bigcup V)$ and $V$ (as a subspace) is not locally connected at $p$. Now apply a) on the subspace $V$.

We start with the following theorem.

THEOREM 1.3 Let $X = \{X_\alpha, f_{\alpha \beta}, A\}$ be an inverse system such that the following condition (FR) is satisfied:

\[(FR) \text{ For each open subset } U \subseteq \text{lim} X \text{ and each } \alpha \in A \text{ it follows } f_\alpha(\text{Fr}(U)) = \text{Fr}(f_\alpha(U))\]

If the spaces $X_\alpha, \alpha \in A$, are locally connected, then $\text{lim} X$ is locally connected.

Proof. Suppose that $\text{lim} X$ is not locally connected. By virtue of Remark 1.2, it follows that there exists an infinite family $\{A_t: t \in T\}$ of open subsets $A_t$ of $\text{lim} X$ such that there is a point $x \in \text{Fr}(\bigcup\{A_t: t \in T\}) - \text{Cl}(\bigcup\{\text{Fr}A_t: t \in T\})$. This means that there is an open set $U_x \subseteq X_\alpha$ such that $\text{Fr}(U_x) = \text{Fr}(x)$ and $f_{\alpha \beta}(U_x) \subseteq \text{Cl}(\bigcup\{\text{Fr}A_t: t \in T\})$. Clearly, $U_x = \text{Fr}(U_x) \subseteq \text{Fr}(\bigcup\{A_t: t \in T\})$. On the other hand from $x \in \text{Fr}(\bigcup\{A_t: t \in T\})$ and (FR) it follows that $f_\alpha(x) \in \text{Fr}(\bigcup\{f_\alpha(A_t): t \in T\})$. This means that the family $\{f_\alpha(A_t): t \in T\}$ has the property $\text{Fr}(\bigcup\{f_\alpha(A_t): t \in T\}) \subseteq \text{Cl}(\bigcup\{\text{Fr}f_\alpha(A_t): t \in T\})$. By Lemma 1.1, this is impossible since $X_\alpha$ is locally connected. The proof is completed.

If $M$ is connected set and $p$ is point of $M$ such that $M - p$ is not connected, then $p$ will be called a cut point of $M$ [20:41]. A point $x \in X$ is said to be a local cut point of $X$ if for each neighbourhood $U$ of $x$ there exists a neighbourhood $V$ of $x$, $x \in V \subset U$, such that $x$ is a cut point of $V$ [15]. Each $\mathbb{R}^n, n \geq 2$, is a space without local cut points. Each point of real line $\mathbb{R}$ is a local cut point. The Niemytzki plane is an example of a completely regular not normal space [4:62] without local cut points.

We say that a mapping $f: X \to Y$ onto $Y$ is irreducible if for each non-empty open subset $U \subseteq X$ the set $f^*(U) = \{y : y \in Y, f^{-1}(y) \subseteq U\}$ is non-empty.

The notion of fully closed mapping was introduced by V.V. Fedorčuk in [6].

A mapping $f: X \to Y$ is said to be fully closed if for each $y \in Y$ and each finite open cover $\{U_1, \ldots, U_n\}$ of $f^{-1}(y)$ the set $\{y\} \cup f^*(U_1) \cup \cdots \cup f^*(U_n)$ is a neighbourhood of $y$.

The space obtained by identifying to a point a closed subset $A$ of a space $X$ is denoted by $X/A$ [4:127]. The natural mapping $q: X \to X/A$ is a simple fully closed mapping. Each fully closed mapping is a limit of an inverse system of simple fully closed mappings [6:Teorema 2.1].

LEMMA 1.4 Each fully closed mapping $f: X \to Y$ is closed.

Proof. Let $y$ be any point of $Y$ and let $U$ be any open set such that $f^{-1}(y) \subseteq U$. By the definition of fully closed mapping we infer that $V = \{y\} \cup f^*(U)$ is a neighbourhood of $y$. Now we have $V = f^*(U)$ since $f^{-1}(y) \subseteq U$. This means that $V$ is an open set about $y$ such that $f^{-1}(V) \subseteq U$. By [4:Theorem 1.4.13] it follows that $f$ is closed.

LEMMA 1.5 [1:356]. If $f: X \to Y$ is closed and irreducible and if $U$ is an open subset of $X$, then $f(\text{Cl}(U)) = \text{Cl}(f(U))$.

LEMMA 1.6 [19:70]. Let $X$ be locally connected. If $f: X \to Y$ is closed and onto, then $Y$ is locally connected.
LEMMA 1.7 Let \( f : X \to Y \) be a fully closed irreducible mapping onto a space \( Y \) without local cut points. For each open \( U \subseteq X \) one has \( f(\text{Fr}U) = \text{Fr}(f^*(U)) \).

Proof. By virtue of Lemmas 1.4 and 1.5 we have \( f(\text{Fr}U) \subseteq \text{Fr}(f^*(U)) \) since \( f \) is closed and irreducible mapping. In order to complete the proof it suffices to prove that \( f(\text{Fr}U) \supseteq \text{Fr}(f^*(U)) \). If \( y \in \text{Fr}(f^*(U)) = \text{Cl}(f^*(U)) - f^*(U) \), then by Lemma 1.5 it follows that there is a \( x \in \text{CIU} \) such that \( y = f(x) \). On the other hand from the relation \( y \notin f^*(U) \) it follows that \( f^{-1}(y) \not\subseteq U \). If we suppose that the set \( f^{-1}(y) \cap \text{Fr}U \) is empty, then we have that \( f^{-1}(y) = (f^{-1}(y) \cap U) \cup (f^{-1}(y) \cap (X \setminus \text{CIU})) \). The sets \( f^{-1}(y) \cap U \) and \( f^{-1}(y) \cap (X \setminus \text{CIU}) \) are open and disjoint. Now, we have the finite open cover \( \{U, V\} \), \( V = X \setminus \text{CIU} \), of \( f^{-1}(y) \). Thus, \( W = \{y\} \cup f^*(U) \cup f^*(V) \) is a neighborhood of \( y \). Moreover \( f^*(U) \) and \( f^*(V) \) are non-empty since \( f \) is irreducible. From the fact that \( U \) and \( V \) are disjoint it follows that \( f(U) \) and \( f(V) \) are disjoint open sets. Thus, the set \( W - \{y\} \) is disconnected. This means that \( f(\text{Fr}U) \cap \text{Fr}(f^*(V)) \) is non-empty. The proof is completed.

The following theorem is the main theorem of this Section.

THEOREM 1.8 Let \( X = \{X_\alpha, f_\alpha \beta, A\} \) be an inverse system such that the projections \( f_\alpha \beta \) are irreducible fully closed mappings. In order that \( \lim X \) be locally connected it is necessary that each \( X_\alpha \) be locally connected and it is sufficient that each \( X_\alpha \) be a locally connected spaces without local cut points.


Sufficiency. Suppose that \( \lim X \) is not locally connected. By Remark 1.2b) it follows that there exists an infinite family \( \{A_\alpha : \alpha \in T\} \) of open subsets \( A_\alpha \) of \( \text{lim} X \) such that there is a point \( x \in \text{Fr}(\bigcup\{A_\alpha : \alpha \in T\}) \cap \text{Cl}(\bigcup\{\text{Fr}A_\alpha : \alpha \in T\}) \). This means that there is an open \( U_\alpha \subseteq X_\alpha \) such that \( f_\alpha^{-1}(U_\alpha) \cap \bigcup\{\text{Fr}A_\alpha : \alpha \in T\} \) is the empty set. Clearly, \( (U_\alpha) \cap f_\alpha^{-1}(\bigcup\{\text{Fr}A_\alpha : \alpha \in T\}) \) is also empty set. By 1.7 we have \( U_\alpha \cap \bigcup\{\text{Fr}A_\alpha : \alpha \in T\} = \emptyset \). We infer that \( f_\alpha(x) \notin \text{Cl}(\bigcup\{\text{Fr}A_\alpha : \alpha \in T\}) \). On the other hand, for each neighbourhood \( V_\alpha \) of \( f_\alpha(x) \) and for \( U = f_\alpha^{-1}(V_\alpha) \), from \( x \in \text{Fr}(\bigcup\{A_\alpha : \alpha \in T\}) \) it follows that each neighbourhood \( U \) of \( x \) meets \( \text{Int}(\bigcup\{A_\alpha : \alpha \in T\}) = \emptyset \). We define \( W = U \cap V \). Clearly, \( W \) is a neighbourhood of \( f_\alpha(x) \) which contains \( f_\alpha(V) \). Furthermore, \( f_\alpha(V) \) meets some \( f_\alpha(A_\beta) \), i.e., \( f_\alpha(V) \) meets \( \bigcap\{f_\alpha(A_\beta) : \beta \in T\} \). Therefore, \( f_\alpha(V) \) is non-empty, we infer that \( V \) meets \( X_\alpha \setminus \bigcup\{f_\alpha(A_\beta) : \beta \in T\} \). Since \( f_\alpha(V) \) is non-empty, we infer that \( V \) meets \( X_\alpha \setminus \bigcup\{f_\alpha(A_\beta) : \beta \in T\} \). This means that \( f_\alpha(x) \in \text{Fr}(\bigcup\{f_\alpha(A_\beta) : \beta \in T\}) \). Finally, \( f_\alpha(x) \in \text{Fr}(\bigcup\{f_\alpha(A_\beta) : \beta \in T\}) \cap \text{Cl}(\bigcup\{\text{Fr}(f_\alpha(A_\beta)) : \beta \in T\}) \). This is impossible since \( Y \) has no local cut points. The proof is completed.

THEOREM 1.9 Let \( X = \{X_\alpha, f_\alpha \beta, A\} \) be an inverse system with irreducible bonding mappings \( f_\alpha \beta \). Then the projections \( f_\alpha : \text{lim} X \to X_\alpha \), \( \alpha \in A \), are irreducible.

Proof. In order to prove that \( f_\alpha \) is irreducible it suffices to prove that for each open non-empty \( U \subseteq \text{lim} X \) the set \( f_\alpha(U) \) is non-empty. Let \( x \) be any point of \( U \). By virtue of the definition of a base of \( \text{lim} X \) there is a \( \beta \in A \) and open set \( U_\beta \subseteq X_\beta \) such that \( x \in f_\beta(U_\beta) \subseteq U \). Then \( \gamma \leq \beta \). Then for open set \( f_\beta^{-1}(U_\beta) = f_\beta^{-1}(U) \subseteq U \). Moreover, the set \( f_\beta(U_\beta) \) is non-empty since \( f_\gamma \) is irreducible. Clearly, \( f_\beta(f_\beta(U_\beta)) \subseteq U \). This means that \( f_\beta(U) \) is non-empty. The proof is completed.

An inverse system \( Y = \{Y_\alpha, g_\alpha \beta, B\} \) is said to be a subsystem of a system \( X = \{X_\alpha, f_\alpha \beta, A\} \) if \( B \subseteq A, Y_\alpha \subseteq X_\alpha \) and \( g_\alpha \beta = f_\alpha \beta |_{Y_\alpha} \). If \( B \) is cofinal in \( A \) and \( Y_\alpha = X_\alpha \) for each \( \alpha \in B \), then \( \text{lim} Y = \text{lim} X \). Similarly, if \( B \) is cofinal in \( A \) and \( Y_\alpha \subseteq X_\alpha \) for each \( \alpha \in B \), then \( \text{lim} Y \) is homeomorphic to a subset of \( \text{lim} X \).
We say that an inverse system \( X = \{ X_\alpha, f_\alpha \}, A \) is an \( N \)-system if each subsystem \( Y = \{ Y_\alpha, g_\alpha, B \}, Y_\alpha \neq \emptyset, Y_\alpha \) closed in \( X_\alpha \), has a non-empty limit \( \text{lim} Y \).

**THEOREM 1.10** Let \( X = \{ X_\alpha, f_\alpha \}, A \) be an inverse system with fully closed mappings \( f_\alpha \). If \( X \) is an \( N \)-system, then the projections \( f_\alpha: \text{lim} X \rightarrow X_\alpha, \alpha \in A \), are fully closed.

**Proof.** Let \( x_\alpha \) be any point of \( X_\alpha \) and let \( \{ U_1, ..., U_\nu \} \) be an open finite cover of a set \( f_\alpha^{-1}(x_\alpha) \). Let us note that from the fact that \( X \) is \( N \)-system it follows that \( f_\alpha^{-1}(x_\alpha) \) is non-empty. Namely, a subsystem \( \{ f_\alpha^{-1}(x_\alpha), f_\alpha^{-1}(x_\alpha), \alpha \leq \beta \leq \gamma \} \) has a non-empty limit homeomorphic to \( f_\alpha^{-1}(x_\alpha) \). By the definition of a base in \( \text{lim} X \) we infer that for \( U_i, 1 \leq i \leq \nu \), there is \( \alpha(i) \in A \) and an open non-empty set \( U_\alpha(i) \) in \( X_\alpha(i) \) such that \( f_\alpha^{-1}(U_\alpha(i)) \subseteq U_i \). We may assume that \( U_\alpha(i) \) is maximal open set such that \( f_\alpha^{-1}(U_\alpha(i)) \subseteq U_i \). Let \( \beta \geq \alpha(1), ..., \alpha(s) \). There is a maximal open \( U_\beta \supseteq f_\alpha^{-1}(U_\alpha(i)) \) such that \( f_\alpha^{-1}(U_\beta) \subseteq U_i \). Moreover, \( U_n = \bigcup \{ U_\gamma \gamma \leq \beta \} \) is non-empty for each \( \gamma \geq \beta \). Then, we have a subsystem \( Y = \{ Y_\gamma, f_\gamma \gamma \gamma, B \} \). By the assumption that \( X \) is \( N \)-system, it follows that \( \text{lim} Y \) is non-empty. Each point \( y \in \text{lim} Y \) we shall identify with a point of \( \text{lim} X \) such that \( \text{lim} Y = \bigcap \{ f_\gamma^{-1}(Y_\gamma) : \gamma \geq \beta \} \). For each \( y \in \text{lim} Y \) we have \( y \in f_\alpha^{-1}(x_\alpha) \) and \( y \in \text{lim} X \), namely \( f_\gamma^{-1}(x_\alpha) \). This is impossible. Thus, there is a \( \gamma \in A \) such that \( Y_\gamma \) is empty, i.e., \( f_\gamma^{-1}(x_\alpha) \); \( f_\gamma^{-1}(x_\alpha) \). Now, \( U_\alpha = \bigcup \{ U_\gamma : 1 \leq i \leq \nu \} \). In \( \text{lim} X \), \( U_\alpha = \bigcup \{ U_\gamma \gamma \leq \beta \} \) is a neighbourhood of \( x_\alpha \). From the maximality of \( U_\gamma \) it follows that \( f_\alpha^{-1}(U_\gamma) \subseteq f_\gamma^{-1}(Y_\gamma) \). Finally, we infer that \( U = \{ x_\alpha \} \subseteq \text{lim} X \), i.e., \( f_\alpha \) is fully closed. The proof is completed.

**REMARK 1.11** If in 1.10. the mappings \( f_\alpha \) are fully closed with compact fibers \( f_\alpha^{-1}(x_\alpha) \) (i.e. fully closed and perfect), then see [6].

2. Applications of the main theorem

In this Section we apply Theorem 1.8. on the inverse systems with fully closed irreducible bonding mappings.

**THEOREM 2.1** Let \( X = \{ X_\alpha, f_{m \alpha}, N \} \) be an inverse sequence such that the mappings \( f_{m \alpha} \) are fully closed irreducible mappings and the spaces \( X_\alpha \) are regular countably compact spaces. In order that \( \text{lim} X \) be locally connected it is necessary that each \( X_\alpha \) be locally connected and it is sufficient that each \( X_\alpha \) be a locally connected space without local cut points.

**Proof.** From Theorem 8. of [17] it follows that the projections \( f_n: \text{lim} X \rightarrow X_n, n \in N \), are closed. Theorem 1.9. establishes that \( f_n \) is irreducible. Moreover, \( X \) is a \( N \)-system [13]. This means that \( f_n \) is fully closed. Theorem 1.8. completes the proof.

We say that a mapping \( f: X \rightarrow Y \) is perfect if it is closed and each fiber \( f^{-1}(y), y \in Y \), is compact [4:236]. A mapping \( f \) is said to be fully perfect if \( f \) is perfect and fully closed.

**THEOREM 2.2** Let \( X = \{ X_\alpha, f_\alpha \}, A \) be an inverse system with fully perfect irreducible bonding mappings \( f_\alpha \). In order that \( \text{lim} X \) be locally connected it is necessary that each \( X_\alpha \) be locally connected and it is sufficient that each \( X_\alpha \) be a locally connected space without local cut points.

**Proof.** The projections \( f_\alpha \) are fully perfect [6] and irreducible, [Theorem 1.9.]. Theorem 1.8. completes the proof.

**COROLLARY 2.3** Let \( X = \{ X_\alpha, f_\alpha \}, A \) be an inverse system of compact spaces \( X_\alpha \) and fully closed irreducible mappings \( f_\alpha \). In order that \( \text{lim} X \) be locally connected it is necessary that each \( X_\alpha \) be locally connected and it is sufficient that each \( X_\alpha \) be a locally connected space without local cut points.
A topological space is a \( q \)-space [16] if for each \( x \in X \) there is a sequence \( U_1, U_2, \ldots \) of open sets such that \( x \in U_n \) \( i \in \mathbb{N} \), with the property: if \( x_n \in U_n, x_n \to x_m \) for \( m \neq n \), then there is an accumulation point of \( \{x_n, n \in \mathbb{N}\} \).

**Lemma 2.4** (15). Let \( f: X \to Y \) be a closed mapping of a normal space \( X \) onto \( T_1 \) q-space \( Y \), then \( Fr f^{-1}(y) \) is countably compact for each \( y \in Y \).

**Corollary 2.5** If \( f \) in Lemma 2.5. is closed and irreducible and if \( X \) is \( T_1 \), then \( f^{-1}(y) \) is countably compact for each \( y \in Y \).

**Proof.** By [1, Exercise 112], we have that \( |f^{-1}(y)| = 1 \) or \( Fr f^{-1}(y) = f^{-1}(y) \). The proof is completed.

We say that a space \( X \) is iso-compact if each countably compact closed subspace \( Y \subseteq X \) is compact.

**Corollary 2.6** Let \( f: X \to Y \) be a closed irreducible mapping of a normal \( T_1 \) iso-compact space \( X \) onto a \( T_1 \) q-space \( Y \), then \( f^{-1}(y), y \in Y \), is compact i.e., \( f \) is perfect and irreducible.

**Theorem 2.7** Let \( X = \{X_\alpha, f_\alpha \}, A \} \) be an inverse system of a \( T_1 \) normal iso-compact q-spaces with fully closed irreducible mappings \( f_\alpha \). In order that \( limX \) be locally connected, it is necessary that each \( X_\alpha \) be locally connected and it is sufficient that each \( X_\alpha \) be a locally connected space without local cut points.

**Proof.** By 2.6. and 1.11., it follows that the projections \( f_\alpha \) are fully closed. Apply Theorem 1.8.

**Corollary 2.8** Let \( X = \{X_\alpha, f_\alpha \}, A \} \) be an inverse system of metric spaces \( X_\alpha \) and fully closed irreducible mappings \( f_\alpha \). In order that \( limX \) be locally connected, it is necessary that each \( X_\alpha \) be locally connected and it is sufficient that each \( X_\alpha \) be a locally connected space without local cut points.

**Proof.** A metric space \( X \) is a q-space since \( X \) is first-countable. A metric space \( X \) is iso-compact since a metric countably compact space \( X \) is compact [4;320]. Apply Theorem 2.7.

**Remark 2.9** Corollary 2.8. holds if we replace" metric" by" paracompact q-space" or by" first-countable paracompact".

**Lemma 2.10** (15). Let \( f: X \to Y \) be a closed mapping of a normal space \( X \) onto a \( T_1 \)-q-space \( Y \). If \( |f^{-1}(y)| \leq \aleph_0 \), \( y \in Y \), then \( Fr f^{-1}(y) \) is compact, for each \( y \in Y \).

**Proof.** By 2.4. \( Fr f^{-1}(y) \) is countably compact. Since each countably compact space is compact, we infer that \( Fr f^{-1}(y) \) is compact, for each \( y \in Y \).

**Theorem 2.11** Let \( X = \{X_\alpha, f_\alpha \}, A \} \) be an inverse system of \( T_1 \)-normal q-spaces \( X_\alpha \) and of fully closed irreducible mappings \( f_\alpha \) with countable fibers \( f_\alpha \)(\( x_\alpha \)), for each \( x_\alpha \in X_\alpha \). In order that \( limX \) be locally connected, it is necessary that each \( X_\alpha \) be locally connected and it is sufficient that each \( X_\alpha \) be a locally connected space without local cut points.

**Proof.** Apply 2.10. and 2.2.

**Corollary 2.12** Let \( X = \{X_\alpha, f_\alpha \}, A \} \) be an inverse system of \( T_1 \) - normal first-countable spaces \( X_\alpha \) and of closed irreducible mappings with countable fibers \( f_\alpha \)(\( x_\alpha \)). In order that \( limX \) be locally connected it is necessary that each \( X_\alpha \) be locally connected and it is sufficient that each \( X_\alpha \) be a locally connected space without local cut points.
3. Concluding remarks

We close this paper with two lemmas.

**Lemma 3.1** Let $X$ be a normal space and let $Y$ be a locally connected space without local cut points. If $f: X \to Y$ is a fully closed irreducible mapping, then $f$ is monotone.

**Proof.** Suppose that for some point $y \in Y$ the set $f^{-1}(y)$ is not connected. This means that there is a pair of disjoint closed (in $f^{-1}(y)$) sets $F_1, F_2$ such that $f^{-1}(y) = F_1 \cup F_2$. Clearly, the sets $F_1$ and $F_2$ are closed in $X$. There exist a pair $U, V$ of disjoint open sets in $X$ such that $F_1 \subseteq U$ and $F_2 \subseteq V$. Now, we have a finite open cover $\{U, V\}$ of $f^{-1}(y)$. A set $W = \{y\} \cup f^*(U) \cup f^*(V)$ is a neighbourhood of $y$. Moreover, $f^*(U)$ and $f^*(V)$ are non-empty since $f$ is irreducible. From the fact that $U$ and $V$ are disjoint, it follows that $f^*(U)$ and $f^*(V)$ are disjoint open sets. Thus, the set $W \setminus \{y\}$ is disconnected. This means that for each neighborhood $W_1$ of $y$ there exist open sets $U_1 = f^{-1}(W_1) \cap U$ and $V_1 = f^{-1}(W_1) \cap V$ such that $\{U_1, V_1\}$ is a finite open cover of $f^{-1}(y)$. Thus, $W_2 = \{y\} \cup f^*(U_1) \cup f^*(V_1)$ is a neighborhood of $y$ contained in $W$. Moreover, we have $f^*(U_1) \subseteq f^*(U)$, $f^*(V_1) \subseteq f^*(V)$. This means that $W_2 - \{y\}$ is disconnected. This is impossible since $Y$ is locally connected and has no cut-points. Thus the set $f^{-1}(y)$ is connected. The proof is completed.

**Lemma 3.2** Let $f: X \to Y$ be a closed monotone irreducible mapping. If $U \subseteq X$ is open, then $f(Fr(U)) = Fr(f^*(U))$.

**Proof.** If $x \in Fr(U)$, then $f(x) \in Cl f^*(U)$. It is clear that $f(x) \not\in f^*(U)$ since $f(x) \in f^*(U)$ implies $f^{-1}(x) \subseteq U$, i.e., $x \in U$. This is impossible because $x \in Fr(U) = Cl U - U$. Thus, $f(Fr(U)) \subseteq Fr(f^*(U))$. In order to complete the proof it suffices to prove that $f(Fr(U)) = Fr(f^*(U))$. If $y \in Fr(f^*(U)) = Cl f^*(U) - f^*(U)$, then by Lemma 1.5., it follows that there is an $x \in Cl f(U)$ such that $y = f(x)$. On the other hand, from the relation $y \not\in f^*(U)$, it follows that $f^{-1}(y) \not\subseteq U$. If we suppose that the set $f^{-1}(y)\cap Fr(U)$ is empty, then we have $f^{-1}(y) = (f^{-1}(y) \cap U) \cup (f^{-1}(y) \cap (X-Cl U))$. The sets $f^{-1}(y) \cap U$ and $f^{-1}(y) \cap (X-Cl U)$ are open and disjoint in $f^{-1}(y)$. This is impossible since $f^{-1}(y)$ is connected. Thus, $f^{-1}(y) \cap Fr(U)$ is non-empty. The proof is completed.

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