

Roman domination number on cardinal product of paths and cycles

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Abstract. In this paper, the authors have determined certain upper and lower bounds for Roman domination numbers on cardinal products for any two graphs and some exact values for the cardinal product of paths and cycles. Roman domination was named after the historical fact that Roman legions were distributed across regions during the reign of the Roman Emperor Constantine in the 4th century A.D. Some areas had 1 or 2 legions, some had no legions, but every area had at least 1 neighboring area with 2 legions. Roman domination is used even today, not only in military situations, but also in protecting some locations against fire or crime.

Key words: Roman dominating function, Roman domination number γ_R , cardinal product of paths, cardinal product of paths and cycles

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1. Historical background and basic definitions

In the 4th century A.D., the Roman Empire was under the rule of Constantine the Great. During that time, the Empire suffered numerous barbaric attacks. Constantine saw that he had to arrange Roman legions in all the strategically important places for protection. Not only did this placement of Roman legions have to be successful in defending the Empire, it also had to be easy to maintain.

A location was considered secure if at least one Roman legion was stationed there. On the other hand, unsecure locations had no stationed legions, but had to be situated adjacent to at least one secure location. If an unsecure location was under attack, sending a legion from its secure neighbor would not be effective if it meant the neighboring location unsecure. Therefore, Constantine decreed that at least two legions had to be stationed at a location before one of them was sent to assist its neighbor under attack. In order to reduce costs of maintaining the army, Constantine had to use as few legions as possible, but still secure the whole Empire.

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This historical background motivated Ian Stuart (1999) to suggest a new variant of the domination known as Roman domination. By representing locations in the Empire as graph vertices and roads in the Empire as graph edges, the problem of defending the Roman Empire transforms into a problem of protecting (or dominating) a graph.

A *dominating function* on G is any function $f: V \rightarrow \{0, 1\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 1$. Such a function obviously induces the ordered partition (V_0, V_1) of V such that each vertex in V_0 is adjacent to at least one vertex in V_1 . Therefore, the set V_1 is called a *dominating set*.

There is a bijection between the set of all functions $f: V \rightarrow \{0, 1\}$ and the set of all ordered partitions (V_0, V_1) . Thus, the formulation $f = (V_0, V_1)$ is obtained. The *weight of f* equals $w(f) = \sum_{v \in V} f(v) = 0 \cdot |V_0| + 1 \cdot |V_1| = |V_1|$. Obviously, the most interesting dominating functions are those of minimum weight.

For a graph $G = (V, E)$, a *Roman dominating function* (RDF) is a function $f: V \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. Since this function also induces the ordered partition of V , $V_i = \{v \in V : f(v) = i\}$, $i \in \{0, 1, 2\}$, the formulation $f = (V_0, V_1, V_2)$ becomes valid. The weight of an RDF equals $w(f) = \sum_{v \in V} f(v) = 0 \cdot |V_0| + 1 \cdot |V_1| + 2 \cdot |V_2| = |V_1| + 2|V_2|$.

This definition implies that the same graph can be protected under several different RDFs, but the most important RDFs are those for which $w(f)$ achieves its minimum. This minimum weight is called *the Roman domination number* of G and we denote it by $\gamma_R(G)$. An RDF which satisfies $w(f) = \gamma_R(G)$ is called a *γ_R -function*. It is obvious that $\gamma_R(G) \leq |V_1| + 2|V_2|$ for any RDF $f = (V_0, V_1, V_2)$.

The main result describing a connection between domination and the Roman domination number of an arbitrary graph G is

$$\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G).$$

Graphs for which $\gamma_R(G) = 2\gamma(G)$ are called *Roman graphs*. It is obvious that $|V_1| = 0$ for every minimum weight RDF of any Roman graph.

Despite the fact that Roman domination has been studied for just a little over a decade, we already know the exact values of Roman domination numbers for many classes of graphs ([1],[2],[3]). However, the complexity of some graphs allows the author(s) to establish only an upper bound.

For more on the domination parameters and the terminology, see [5].

For arbitrary graphs G and H , the cardinal product of G and H is the graph $G \times H$ which satisfies the following

- Its vertex set is $V(G \times H) = V(G) \times V(H)$;
- Two vertices $(g, h), (g', h') \in V(G \times H)$ are adjacent if and only if g is adjacent to g' in G and h is adjacent to h' in H .

The cardinal product of two paths $P_m \times P_n$ has two connected components. If the vertices of P_m and P_n are denoted by $\{1, 2, 3, \dots, m\}$ and $\{1, 2, 3, \dots, n\}$, respectively, then the component of $P_m \times P_n$ containing the vertex $(1, 1)$ will be denoted

by K_1 , and the other component by K_2 . If at least one of the parameters m or n is even, components K_1 and K_2 are isomorphic. Otherwise, the component K_1 has one vertex more than the component K_2 .

In [1] we proved bounds for Roman domination numbers of the cardinal product of two paths and two cycles. Now, some bounds for cardinal product of any two graphs will be provided, and will take the exact values of Roman domination numbers for products of some graphs..

2. Bounds for Roman domination numbers on the cardinal product

The degree of a vertex x on graph G is the number of vertices on G which are adjacent to x . Using $\Delta(G)$ ($\delta(G)$) we denote the maximum (minimum) degree of vertices on G .

Lemma 1. *In [2], it has been proven that for any graph G of order n*

$$\gamma_R(G) \geq \frac{2n}{\Delta + 1}$$

$$\gamma_R(G) \leq n \frac{2 + \ln((1 + \delta(G))/2)}{\delta(G) + 1}$$

Theorem 1. *For any two graphs G and H of order m and n*

$$\gamma_R(G \times H) \geq \frac{2mn}{\Delta(G)\Delta(H) + 1}$$

$$\gamma_R(G \times H) \leq mn \frac{2 + \ln((1 + \delta(G)\delta(H))/2)}{\delta(G)\delta(H) + 1}$$

Proof. According to the definition of the cardinal product, vertex $(i, j) \in V(G \times H)$, $i \in V(G)$, $j \in V(H)$, is adjacent to zw vertices in graph $G \times H$, where z is the number of the vertices which are adjacent to i in graph G , and w is the number of the vertices which are adjacent to j in graph H , which implies that $\Delta(G \times H) = \Delta(G) \times \Delta(H)$. (The same is true for minimal degree.) But, since graph $G \times H$ is of order mn , the statement is true based on the previous Lemma. ■

Here, the exact values of Roman domination numbers for the products of certain graphs will be taken.

3. Specific values of Roman domination numbers

Observation 1. *Let C_n and P_n denote the cycle and path with n vertices, respectively. Then*

$$\gamma_R(C_n) = \gamma_R(P_n) = \lceil \frac{2n}{3} \rceil.$$

Following the investigations into the Cartesian product, cardinal products where one of the factors is a path are considered.

Theorem 2. For any tree T and any graph G without cycles of odd length we have

$$\gamma_R(P_2 \times T) = 2\gamma_R(T) = \gamma_R(P_2)\gamma_R(T)$$

and

$$\gamma_R(P_2 \times G) = 2\gamma_R(G) = \gamma_R(P_2)\gamma_R(G).$$

Proof. Obvious, since $P_2 \times T$ and $P_2 \times G$ consist of two disjoint copies of T and G , respectively. ■

Theorem 3. For the path P_2 and any odd cycle C_{2n+1} , $n \geq 1$,

$$\gamma_R(P_2 \times C_{2n+1}) = \lceil \frac{2(4n+2)}{3} \rceil = 2\lceil \frac{4n+2}{3} \rceil = \gamma_R(P_2)\gamma_R(C_{2n+1})$$

Proof. Note that the cardinal product of P_2 and C_{2n+1} is isomorphic to C_{4n+2} . Then Observation 1 implies that

$$\gamma_R(C_{4n+2}) = \lceil \frac{2(4n+2)}{3} \rceil = 2\lceil \frac{2n+1}{3} \rceil = \gamma_R(P_2)\gamma_R(C_{2n+1}). \blacksquare$$

Definition 1. For a fixed m , $1 \leq m \leq n$, the set $(P_k)_m = \{(i, m) | i = 1, \dots, k\}$ is called a column of $P_k \times P_n$. For a fixed l , $1 \leq l \leq k$, The set $(P_n)_l = \{(l, j) | j = 1, \dots, n\}$ is called a row of $P_k \times P_n$.

Theorem 4. Let $n \geq 2$. Then

$$\gamma_R(P_3 \times P_n) = \begin{cases} 6\binom{n}{4} & n \equiv 0(\text{mod}4), \\ 6\lfloor \frac{n}{4} \rfloor + 2 & n \equiv 1(\text{mod}4), \\ 6\lfloor \frac{n}{4} \rfloor + 4 & n \equiv 2(\text{mod}4), \\ 6\lfloor \frac{n}{4} \rfloor + 5 & n \equiv 3(\text{mod}4) \end{cases}$$

Proof. First, it is necessary to find the upper bound for $\gamma_R(P_3 \times P_n)$ by constructing some Roman dominating sets. Later, the minimality will be proven.

Case 1. $n \equiv 0(\text{mod}4)$

Since, for this case, $P_3 \times P_n$ consists of two isomorphic components, we consider only K_1 . Set $S_1 = \{((2, 4j - 2) \in V_2; j = 1, \dots, \frac{n}{4}) \cup ((2, 4j) \in V_1; j = 1, \dots, \frac{n}{4})\}$ is a Roman dominating set and hence for $n \equiv 0(\text{mod}4)$,

$$\gamma_R(P_3 \times P_n) \leq 2|S_1| = 2(3\frac{n}{4}) = 6\frac{n}{4}.$$

Case 2. $n \equiv 1(\text{mod}4)$

On K_1 , the set $S_1 = \{((2, 4j - 2) \cup (2, n - 1) \in V_2; j = 1, \dots, \lfloor \frac{n}{4} \rfloor) \cup ((2, 4j) \in V_1; j = 1, \dots, \lfloor \frac{n}{4} \rfloor - 1)\}$ is deemed to be a Roman dominating set, and on K_2 the set $S_2 = \{((2, 4j - 1) \in V_2; j = 1, \dots, \lfloor \frac{n}{4} \rfloor) \cup ((2, 4j - 3) \cup (2, n) \in V_1; j = 1, \dots, \lfloor \frac{n}{4} \rfloor)\}$ is a Roman dominating set.

$$\text{Hence for } n \equiv 1(\text{mod}4), \gamma_R(P_3 \times P_n) \leq |S_1| \cup |S_2| = 2(3\lfloor \frac{n}{4} \rfloor + 1) = 6\lfloor \frac{n}{4} \rfloor + 2.$$

Case 3. $n \equiv 2(\text{mod}4)$

Since, for this case, $P_3 \times P_n$ consists of two isomorphic components, only K_1 is considered. Set $S_1 = \{((2, 4j-2) \cup (2, n) \in V_2; j = 1, \dots, \lfloor \frac{n}{4} \rfloor) \cup ((2, 4j) \in V_1; j = 1, \dots, \lfloor \frac{n}{4} \rfloor)\}$ is a Roman dominating set and hence for $n \equiv 2(\text{mod}4)$,

$$\gamma_R(P_3 \times P_n) \leq 2|S_1| = 2(3\lfloor \frac{n}{4} \rfloor + 2) = 6\lfloor \frac{n}{4} \rfloor + 4.$$

Case 4. $n \equiv 3(\text{mod}4)$

On K_1 , the set $S_1 = \{((2, 4j-2) \cup (2, n-1) \in V_2; j = 1, \dots, \lfloor \frac{n}{4} \rfloor) \cup ((2, 4j) \in V_1; j = 1, \dots, \lfloor \frac{n}{4} \rfloor)\}$ is considered a Roman dominating set, and on K_2 the set $S_2 = \{((2, 4j-1) \cup (2, n) \in V_2; j = 1, \dots, \lfloor \frac{n}{4} \rfloor) \cup ((2, 4j-3) \in V_1; j = 1, \dots, \lfloor \frac{n}{4} \rfloor + 1)\}$ is a Roman dominating set, hence for $n \equiv 3(\text{mod}4)$,

$$\gamma_R(P_3 \times P_n) \leq |S_1| \cup |S_2| = (3\lfloor \frac{n}{4} \rfloor + 2) + (3\lfloor \frac{n}{4} \rfloor + 3) = 6\lfloor \frac{n}{4} \rfloor + 5.$$

We now prove that

$$\gamma_R(P_3 \times P_n) \geq \begin{cases} 6\lfloor \frac{n}{4} \rfloor & n \equiv 0(\text{mod}4), \\ 6\lfloor \frac{n}{4} \rfloor + 2 & n \equiv 1(\text{mod}4), \\ 6\lfloor \frac{n}{4} \rfloor + 4 & n \equiv 2(\text{mod}4), \\ 6\lfloor \frac{n}{4} \rfloor + 5 & n \equiv 3(\text{mod}4) \end{cases}$$

Proof of the minimality:

Case 1. Let n is even.

We will consider the component K_1 .

Lemma 2. *There is a minimum Roman dominating set D , such that D only contains vertices of the row $(2, i)$, $i \in \{2, 4, \dots, n\}$.*

Proof (of the Lemma). Let D be a minimal Roman dominating set that does not satisfy the assertion. Without loss of generality, D is assumed to contain a vertex of the row $(P_n)_1$. Let $(1, j)$ be this vertex for some fixed $j \in \{3, \dots, n-1\}$.

If $(1, j) \in V_1$ and $(3, j) \notin D$ then $(2, j-1)$ or $(2, j+1) \in V_2$. Then the set $D' = D \setminus (1, j)$ also Roman dominates K_1 and $|D'| \leq |D|$. So, D is not minimal.

If $(1, j) \in V_1$ and $(3, j) \in V_1$ then $(2, j-1)$ or $(2, j+1) \in D$.

a) Let $(2, j+1) \in V_2$ Then the set $D' = \{D \setminus ((1, j), (3, j))\}$ also Roman dominates K_1 . So, D is not minimal.

b) If $(2, j+1) \in V_1$, we can also take the same set D' as in a), and D is not minimal.

If $(1, j) \in V_1$ and $(3, j) \in V_2$, Then the set $D' = (D \setminus \{(1, j), (3, j)\}) \cup \{(2, j-1) \in V_1, (2, j+1) \in V_2\}$ also Roman dominates K_1 .

If $(1, j) \in V_2$ and $(3, j) \notin D$ then $(2, j-1)$ or $(2, j+1) \in V_2$. Let $(2, j+1) \in V_2$. Then the set $D' = D \setminus (1, j) \cup ((2, j-1) \in V_2)$ also Roman dominates K_1 .

If $(1, j) \in V_2$ and $(3, j) \in V_1$, then the set $D' = \{(D \setminus (1, j), (3, j)) \cup ((2, j+1) \in V_2, (2, j-1) \in V_1)\}$ also Roman dominates K_1 .

If $(1, j) \in V_2$ and $(3, j) \in V_2$, Then the set $D' = (D \setminus \{(1, j), (3, j)\}) \cup \{(2, j - 1), (2, j + 1) \in V_2\}$ also Roman dominates K_1 .

For $(1, 1) \in D$ is similarly. ■

Case 2. Let n be odd.

For both components, the assertion of Lemma 1 can be shown analogously.

From Lemma 1 and condition that each vertex from V_o must be adjacent to at least one vertex from V_2 , it follows that, to R-dominate vertices from the first and last row (on K_1), at least vertices $\{(2, 4j - 2); j = 1, \dots, \lfloor \frac{n}{4} \rfloor\}$ must be in V_2 , and then the remaining vertices on row (2,i) must be in V_1 . (For K_2 this is similar.) And this is exactly the structure from the set $S = S_1 \cup S_2$. ■

Theorem 5. Let $n \geq 2$. Then

$$\gamma_R(P_4 \times P_n) = \begin{cases} 2n + 2 & n = 5 \\ 2n & \text{otherwise} \end{cases}$$

Proof.

Since $P_4 \times P_n$ consists of two isomorphic components, only K_1 is considered.

Case 1. $n \equiv 0(\text{mod}6)$

Set $S_1 = \{(2, 6j + 2), (3, 6j + 6) : j = 0, 1, \dots, \lfloor \frac{n}{6} \rfloor - 1\} \in V_2; \cup \{(1, 6j + 6), (4, 6j + 2) : j = 0, 1, \dots, \lfloor \frac{n}{6} \rfloor - 1\} \in V_1\}$ is Roman dominating set and hence for $n \equiv 0(\text{mod}6)$, $\gamma_R(P_4 \times P_n) \leq 2|S_1| = 2(n)$.

Case 2. $n \equiv 1(\text{mod}6)$

The set $S'_1 = (S_1 \setminus ((1, n - 2) \in V_1) \cup ((2, n - 1) \in V_2))$ is considered. This set is a Roman dominating set and hence for $n \equiv 1(\text{mod}6)$, $\gamma_R(P_4 \times P_n) \leq 2|S'_1| = 2(n)$.

Case 3. $n \equiv 2(\text{mod}6)$

The set $S'_1 = (S_1 \setminus (\{(4, n - 6), (1, n - 3)\} \in V_1, (3, n - 3) \in V_2) \cup (\{(3, n - 5), (2, n - 2), (3, n - 1) \in V_2\}))$ is considered. This set is a Roman dominating set and hence for $n \equiv 2(\text{mod}6)$, $\gamma_R(P_4 \times P_n) \leq 2|S'_1| = 2(n)$.

Case 4. $n \equiv 3(\text{mod}6)$

The set $S'_1 = (S_1 \cup ((2, n) \in V_2, (4, n) \in V_1))$ is considered. This set is a Roman dominating set and hence for $n \equiv 3(\text{mod}6)$, $\gamma_R(P_4 \times P_n) \leq 2|S'_1| = 2(n)$.

Case 5. $n \equiv 4(\text{mod}6)$

The set $S'_1 = (S_1 \cup ((2, n - 2), (3, n - 1) \in V_2))$ is considered. This set is a Roman dominating set and hence for $n \equiv 4(\text{mod}6)$, $\gamma_R(P_4 \times P_n) \leq 2|S'_1| = 2(n)$.

Case 6. $n \equiv 5(\text{mod}6)$

The set $S'_1 = (S_1 \setminus ((4, n-6) \in V_1) \cup ((3, n-7), (3, n-2), (2, n-1) \in V_2))$ is considered.

This set is a Roman dominating set and hence for $n \equiv 5(\text{mod}6)$,

$$\gamma_R(P_4 \times P_n) \leq 2|S'_1| = 2(n).$$

Proof of minimality:

Let $n \equiv 0(\text{mod}6)$ (or $n \equiv 3(\text{mod}6)$). If we partition the graph $P_4 \times P_n$ into 4×3 blocks, it follows that in S_1 , on each such block there exists 1 vertex from V_2 (weight 2) which Roman dominates 5 vertices, and 1 vertex from V_1 (weight 1) which Roman dominates only itself, and it lies in the first or the last row. In [1] it has been proved that the optimal Roman dominating set has exactly such structure. For other cases of n each new column increases the weight of the graph by 1. A smaller increase is not possible. ■

For $P_5 \times P_n$ and $P_6 \times P_n$ we have the following bounds:

$$\gamma_R(P_5 \times P_n) \leq \begin{cases} 8\left(\frac{n}{3}\right) & n \equiv 0(\text{mod}3), \\ 8\lfloor \frac{n}{3} \rfloor + 4 & n \equiv 1(\text{mod}3), \\ 8\lceil \frac{n}{3} \rceil & n \equiv 2(\text{mod}3) \end{cases}$$

$$\gamma_R(P_6 \times P_n) \leq \begin{cases} 10\left(\frac{n}{3}\right) & n \equiv 0(\text{mod}3), \\ 10\lfloor \frac{n}{3} \rfloor + 2 & n \equiv 1(\text{mod}3), \\ 10\lceil \frac{n}{3} \rceil + 4 & n \equiv 2(\text{mod}3) \end{cases}$$

We also have the conjecture that equality holds in both bounds, but still remains to be proven.

4. Conclusion

This paper provides bounds of Roman domination numbers for the product of any two graphs. Furthermore, exact values (not bounds) are given for the product of P_2 with any graph, and for $P_3 \times P_n$ and $P_4 \times P_n$. Upper bounds for the Roman domination number of $P_5 \times P_n$ and $P_6 \times P_n$ are also given, and conjecture is that they are exact values. This paper is a continuation of the paper in [1]. Bounds have been given only for Roman domination numbers of the cardinal product of two paths and two cycles.

Even today, Roman domination can be applied in the real life, and not only in a military sense. For instance, in unsecure parts of a town, where police are often called to intervene, there should be at least two teams of police. Meaning that when, one team goes out, the other team can react to another call. This service arrangement already exists in emergency medical stations. Such stations have several teams ready to intervene. The Roman domination can even be generalized by postulating that 3 or more teams dominate an area.

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