GLASNIK MATEMATIČKI
Vol. 50(70)(2015), 1 – 15

PRIMITIVE BLOCK DESIGNS WITH AUTOMORPHISM GROUP PSL$(2, q)$

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Abstract. We present the results of a research which aims to determine, up to isomorphism and complementation, all primitive block designs with the projective line $F_q \cup \{\infty\}$ as the set of points and PSL $(2, q)$ as an automorphism group. The obtained designs are classified by the type of a block stabilizer. The results are complete, except for the designs with block stabilizers in the fifth Aschbacher's class. In particular, the problem is solved if q is a prime. We include formulas for the number of such designs with $q = p^{2^\alpha 3^\beta}$, α, β nonnegative integers.

1. Introduction and preliminaries

Our aim is to determine, up to isomorphism and complementation, all nontrivial primitive block designs on the projective line with PSL $(2, q)$ as an automorphism group. For each q we denote by npd(q) the number of such designs. We also determine which of the occurring 2-designs is even a 3-design.

Several authors have considered the action of group PSL $(2, q)$ on the projective line. For this research the most significant contribution is the work of Cameron et al. [5, 6], which we use in the part involving 3-designs with q odd. Focusing on primitive designs only, we extend the results taken from [5, 6] by solving the problem of isomorphism of the designs and by finding their full automorphism groups. Additionally, our method yields the series of 3-designs (Proposition 5.2) undetected in [5, 6]. The rest of the research comprises 2-designs and 3-designs with q even.

The obtained designs are presented following the type of a block stabilizer. We completely solved the problem in case when a block stabilizer is not in the fifth Aschbacher's class and, in particular, for q a prime number. In

2010 *Mathematics Subject Classification.* 05B05.
Key words and phrases. Block design, automorphism group, primitive action.

Section 5 a part of the designs is described by an explicit base block and an automorphism group. In Section 6 we give the data determining the full automorphism groups for the rest of the designs. Section 7 is a contribution to the calculation of the numbers $\mathrm{npd}(q)$. Formulas for $\mathrm{npd}(q)$ are determined in case $q = p^{2^\alpha 3^\beta}$, α, β nonnegative integers. For instance, Proposition 7.1 relates to the case $\alpha = \beta = 0$. The validity of the obtained formulas is illustrated through computer construction of designs up to $q = 103$, for which we used software packages GAP [9, 15] and MAGMA [3] and the libraries of primitive groups that they contain.

We start with a few basic notions and facts that are relevant for our study. More details on design theory the reader can find, for instance, in [2, 7], while for group theory we refer the reader to [1, 4, 8]. Our notation and terminology is in accordance with the cited literature.

A $t-(v, k, \lambda)$ *design* is a pair $D = (\Omega, \mathcal{B})$, where Ω is a set of v points, $\mathcal{B}$ a set of k-sets of Ω called blocks, such that any t different points are contained in exactly λ blocks, $t \leq k$ and $\lambda > 0$. Any $2 - (v, k, \lambda)$ design we simply call (v, k, λ) *block design*.

An *isomorphism* of t-designs $D = (\Omega, \mathcal{B})$ and $D' = (\Omega, \mathcal{B}')$ is a permutation of Ω which sends blocks of D to blocks of D'. An isomorphism from D to itself is called *automorphism*. The group of all automorphisms of D is denoted by $\mathrm{Aut}D$. For any $\omega \in \Omega$ by $G_\omega \leq \mathrm{Aut}D$ we denote a point stabilizer; $G_B \leq \mathrm{Aut}D$ denotes a block stabilizer, $B \in \mathcal{B}$.

A permutation group G acting transitively on a set X, $|X| \geq 2$, is primitive if each point stabilizer $G_x, x \in X$, is a maximal subgroup of G [8, p. 14]. We call a $t-$ design D *primitive* if there exists an automorphism group $G \leq \mathrm{Aut}D$ which acts primitively on the point and block sets.

It is known that 2-homogeneous permutation groups are primitive [8, p. 35], thus all 2-transitive permutation groups are primitive.

Proposition 1.1 ([8, p. 9]). *Let G be a permutation group acting transitively on a set X. Then a subgroup $L \leq G$ is transitive if and only if $G = LG_x$, $x \in X$.*

An overgroup of a primitive group is primitive. All primitive groups that have the same socle with a specified (transitive) permutation action form a *cohort* [8, p.138]. The primitive groups G of this research are almost simple, i.e. $T \lhd G \leq \mathrm{Aut}T$, T nonabelian simple, [13]. The definition of *Aschbacher's classes* can be found in [11].

We denote by F_q a finite field with q elements; we also set $q = p^f$, p a prime, $F_q^* = F_q \backslash \{0\}$ and $F_q^{(2)} = \{ x^2 \mid x \in F_q^* \}$. We consider primitive designs with respect to G, such that

$$\mathrm{PSL}(2, q) = T \trianglelefteq G \leq \mathrm{Aut}T = P\Gamma L(2, q). \tag{1.1}$$

The socle T of the cohort (1.1) is the group of all fractional linear transformations

$$t_{a,b,c,d} : z \mapsto \frac{az+b}{cz+d}$$

on the projective line $F_q \cup \{\infty\}$, $a, b, c, d \in F_q$, where $ad - bc$ is a square; $|T| = \frac{q(q^2-1)}{\gcd(2,q-1)}$.

Let ξ be a primitive element of F_q and let $\delta = t_{\xi,0,0,1}$. By ϕ we denote the automorphism $z \mapsto z^p$ of F_q which fixes ∞. Then we have

$$\mathrm{PGL}(2,q) = \langle \mathrm{PSL}(2,q), \delta \rangle = \langle T, \delta \rangle, \quad |\mathrm{PGL}(2,q)| = q(q^2-1);$$
$$P\Gamma L(2,q) = \langle \mathrm{PSL}(2,q), \delta, \phi \rangle = \langle T, \delta, \phi \rangle, \quad |P\Gamma L(2,q)| = fq(q^2-1);$$
$$P\Sigma L(2,q) = \langle \mathrm{PSL}(2,q), \phi \rangle = \langle T, \phi \rangle; \quad |P\Sigma L(2,q)| = \frac{fq(q^2-1)}{\gcd(2,q-1)}.$$

Our theoretical considerations are restricted to $q \geq 13$, $q \neq 23$.

2. Construction method

The basis of our construction method is the following theorem.

Theorem 2.1. *[2, p.175] Let G be a permutation group on the finite set Ω, let $B \subset \Omega$ be a k-subset with at least two elements and let $G_B \leq G$ be a setwise stabilizer of B.*

If G is t-homogeneous and $k \geq t$, then $D = (\Omega, B^G = \{B^\gamma \mid \gamma \in G\})$ is a t-design with $b = |B^G| = |G| / |G_B|$ blocks and

$$\lambda = b\binom{k}{t} \Big/ \binom{v}{t} = |G| \binom{k}{t} \Big/ |G_B| \binom{v}{t}.$$

The set B is called a base block for D.

We take $\Omega = F_q \cup \{\infty\}$, $|\Omega| = q+1$. It is well-known that T-action on projective line is 2-homogeneous if $q \equiv 1 (\mathrm{mod}\, 4)$, whereas it is 3-homogeneous if $q \equiv 3 (\mathrm{mod}\, 4)$ or q is even. It is also known that $\mathrm{PGL}(2,q)$ acts 3-homogeneously on projective line for all q. If, for a given group G from (1.1), we select a subset $B \subset \Omega$ and construct the set $\mathcal{B} = B^G$, then the pair $D = (\Omega, \mathcal{B})$ is a 2 or 3-design with a base block B and $G \leq \mathrm{Aut}D$ (Theorem 2.1). G obviously acts primitively on the set of points of D. If the base block stabilizer $G_B \leq G$ is a maximal subgroup, then G acts primitively on blocks as well. Consequently, in order to construct a primitive design D in this way, it suffices to choose B to be a union of orbits of some maximal subgroup of G. Hence we denote by $D(G, B)$ such a design. We only consider possible non-trivial choices for B with the property $k = |B| \leq v/2$ because the complement of a primitive design is also primitive with the same full automorphism group. If the action of $\mathrm{Aut}D(G, B)$ is 3-homogeneous then the underlying design is a 3-design.

3. Minimal group for performing the construction

Suppose that for primitive design $D(G, B)$ we have $T \le G_1 \le G$. Then G_1 is normal subgroup of G, so G_1 acts transitively on blocks. Now, if $G_B \cap G_1$ is maximal in G_1, then the constructions by G_1 produce also all primitive designs admitting G as an automorphism group. The consequence of this simple observation is that, eventually, we only need to consider maximal subgroups of T and $\mathrm{PGL}(2, q)$ as setwise base block stabilizers to construct all desired primitive designs. Namely, if G is any group from (1.1) and $M \le G$ its maximal subgroup not contained in T, then $M \cap T$ is maximal in T with one single exception: normalizer $M = N_G(A_4) = S_4$ in $G = \mathrm{PGL}(2, q)$ and $q = p \equiv \pm 11, 19 (\mathrm{mod}\, 40)$ [10, Theorem 1.1 and Corollary 1.2].

Type	$H \le T$ (block stabilizer)	Asch.	G^{MIN}	ncc	G^{MAX}
1	$C_p^f \rtimes C_{\frac{q-1}{\gcd(2,q-1)}}$ (point stabilizer)	C_1	$\mathrm{PSL}(2,q)$	1	$P\Gamma L(2,q)$
2	$D_{\frac{2(q-1)}{\gcd(2,q-1)}}$ (two points setwise stabilizer)	C_2	$\mathrm{PSL}(2,q)$	1	$P\Gamma L(2,q)$
3	$D_{\frac{2(q+1)}{\gcd(2,q-1)}}$	C_3	$\mathrm{PSL}(2,q)$	1	$P\Gamma L(2,q)$
4	$\mathrm{PGL}(2,q_0), q = q_0^2$	C_5	$\mathrm{PSL}(2,q)$	2^1	$P\Sigma L(2,q)$
5	$\mathrm{PSL}(2,q_0), q = q_0^r$, $q_0 \neq 2, r$ odd prime	C_5	$\mathrm{PSL}(2,q)$	1	$P\Gamma L(2,q)$
6	$A_5, q = p^2 \equiv 49\,(\mathrm{mod}\, 60)$, $p \equiv 7, 13, 17, 23, 37, 43, 47, 53\,(\mathrm{mod}\, 60)$	C_9	$\mathrm{PSL}(2,q)$	2	$P\Sigma L(2,q)$
7	$A_5, q = p \equiv 1, 11, 19, 29, 31, 41, 49, 59\,(\mathrm{mod}\, 60)$	C_9	$\mathrm{PSL}(2,q)$	2	$\mathrm{PSL}(2,q)$
8	$A_4, q = p \equiv 13, 37, 43, 53, 67, 77, 83, 107\,(\mathrm{mod}\, 120)$	C_6	$\mathrm{PSL}(2,q)$	1	$\mathrm{PGL}(2,q)$
9	$S_4, q = p \equiv 1, 7, 17, 23, 31, 41, 47, 49, 71, 73, 79, 89, 97, 103, 113, 119\,(\mathrm{mod}\, 120)$	C_6	$\mathrm{PSL}(2,q)$	2	$\mathrm{PSL}(2,q)$
10	$S_4, q = p \equiv 11, 19, 29, 59$ $61, 91, 101, 109\,(\mathrm{mod}\, 120)$	C_6	$\mathrm{PGL}(2,q)$	1	$\mathrm{PGL}(2,q)$

Table 1

Maximal subgroups of T can, for instance, be read off from [10, Theorem 2.1 and Theorem 2.2]. Here we rewrite that division into nine isomorphism types following the orbit structure. The types are listed in the first nine rows of Table 1 and denoted by H. The corresponding Aschbacher's class ([11]) is indicated in the third column.

In Table 1 G^{MIN} denotes minimal groups from (1.1) with maximal subgroup H, i.e. minimal primitive automorphism groups of the prospective designs. The tenth row of Table 1 relates to the case $G^{MIN} = \mathrm{PGL}(2, q)$; here the socle T of cohort (1.1) does not act primitively on blocks.

Group G^{MAX} will be defined and explained in the next section. The fifth column reads the number ncc of conjugacy classes of H in G^{MIN}; $ncc \leq 2$ for any isomorphism type of maximal subgroup of T.

To accomplish the construction of all our aimed designs it suffices to:

(i) *Compose base blocks as all possible unions of H-orbits, H of type* 2 *thru* 10. (Block stabilizer H of type 1 has two orbits whose lengths are 1 and q. The corresponding design is obviously trivial.)

(ii) *Generate the block set $\mathcal{B}$ from each base block B* by the action of

$$\begin{cases} T \text{ on } B, & \text{for } H \text{ of type 2 to 9;} \\ \mathrm{PGL}(2, q) \text{ on } B, & \text{for } H \text{ of type 10.} \end{cases}$$

4. Preliminary analysis of designs

In this section we give important facts about full automorphism groups and possible isomorphisms of the designs we consider. By H we denote a maximal subgroup of T.

The following assertion has been observed in [5, Introduction] and [12].

Proposition 4.1. *Let G belong to the cohort (1.1) and let $D = D(G, B)$ and $D' = D(G, B')$ be any two considered designs. If $\pi : D \to D'$ is an isomorphism, then $\pi \in P\Gamma L(2, q)$.*

Taking $D' = D$, from the proposition it follows that each considered design D has the property $\mathrm{Aut}D \leq P\Gamma L(2, q)$. Isomorphisms from $P\Gamma L(2, q)$ act on the set of blocks of a design in the sense that they preserve T as the generating group:

$$\left(B^T\right)^\pi = B^{T\pi} = B^{\pi\pi^{-1}T\pi} = \left(B^\pi\right)^T, \ \pi \in P\Gamma L(2, q). \tag{4.1}$$

For a given design $D = D(T, B)$, the set $stb(D) = \{T_{B^*} | \, B^* \in \mathcal{B} = B^T\}$ of stabilizers of all blocks of D we call D-stabilizer. It is the set of all subgroups conjugate to some block stabilizer $H = T_B$, i.e. $stb(D) = H^T$. For an isomorphism $\pi : D \to D'$ we have $(stb(D))^\pi = stb(D')$, cf. (4.1). This means that we obtain all designs up to isomorphism if the construction (i)–(ii) is performed only for a chosen representative of every conjugacy class of H in $P\Gamma L(2, q)$. Clearly, $P\Gamma L(2, q)$ acts on the classes of T. So if $H = T_B$ for the design $D(G,B)$ then by G^{MAX} we denote the set stabilizer of the conjugacy class of H in T. Obviously, $\mathrm{Aut}D \leq G^{MAX}$ and

$$G^{MAX} = \begin{cases} P\Gamma L(2,q), \text{ if there exists one conjugacy class of } H \text{ in } T; \\ \\ P\Sigma L(2,q), \text{ if } q \text{ is odd and there exist two conjugacy} \\ \text{classes of } H \text{ in } T. \end{cases}$$

From Proposition 1.1 we deduce

$$G^{MAX} = T \cdot N_{G^{MAX}}(H). \tag{4.2}$$

$N_{G^{MAX}}(H)$ acts on the set of H-orbits and, accordingly, on the block set of the design.

Proposition 4.2. *Let $D_1 = D(T,B_1)$ and $D_2 = D(T,B_2)$ be designs with $T_{B_1} = T_{B_2} = H$. Then D_1 and D_2 are isomorphic if and only if there exists $\pi \in N_{G^{MAX}}(H)$ so that $B_1^\pi = B_2$.*

Proof. Let $\varphi : D_1 \to D_2$ be an isomorphism. Then there exists $g \in T$ such that $B_1^\varphi = B_2^g$, and for the isomorphism $\pi = \varphi g^{-1} : D_1 \to D_2$ we have $B_1^\pi = B_2$. Now $H = T_{B_2} = T_{B_1^\pi} = T_{B_1}^\pi = H^\pi$, which proves $\pi \in N_{G^{MAX}}(H)$.

Conversely, let there exist $\pi \in N_{G^{MAX}}(H)$ such that $B_1^\pi = B_2$. Then

$$B_2^T = (B_1^\pi)^T \overset{(4.1)}{=} \left(B_1^T\right)^\pi,$$

i.e. π is an isomorphism. ∎

Corollary 4.3. $\mathrm{Aut}D = T \cdot \{\pi \in N_{G^{MAX}}(H) | \, B^\pi = B\}$.

Proof. Proposition 1.1 implies $\mathrm{Aut}D = T \cdot \mathrm{Aut}D_B$, while previous proposition with $D_1 = D_2 = D$ gives $\mathrm{Aut}D_B = \{\pi \in N_{G^{MAX}}(H) | \, B^\pi = B\}$. ∎

5. Designs with block stabilizers of types 2 thru 5

In this and the subsequent section we describe all primitive 2-designs D with $\mathrm{PSL}\,(2,q) \trianglelefteq \mathrm{Aut}D$ on $q+1$ points up to one undecided case. For the description and for solving the problem of possible isomorphism between two designs we use group $N_{G^{MAX}}(H)$. Henceforth that group we denote by K.

For H-types 2 thru 5 we explicitly give H-orbits and a base block of the design. The description is incomplete for H-type 5 in the sense that we found orbit lengths for H but not for K.

A) *H-type 2* (a dihedral group, two point stabilizer)

Proposition 5.1. *Let $q \geq 13$. A block design D with the socle* $\mathrm{PSL}\,(2,q)$ *of* $\mathrm{Aut}D$ *and the base block stabilizer H in the second Aschbacher's class exists if and only if $q \equiv 1 (\mathrm{mod}\, 4)$. Then D is 2-$(q+1, \frac{q-1}{2}, \frac{(q-1)(q-3)}{8})$ design which is unique up to isomorphism and complementation. Moreover,* $\mathrm{Aut}D = P\Sigma L\,(2,q)$.

PROOF. If D exists then

$$H = T_{\{0,\infty\}} = \left\{x \mapsto ax : a \in F_q^{(2)}\right\} \rtimes \left\langle x \mapsto \frac{-1}{x}\right\rangle,$$

$G^{MAX} = P\Gamma L\,(2,q)$ and $K = P\Gamma L\,(2,q)_{\{0,\infty\}}$.

The orbits of subgroup $\left\{x \mapsto ax \mid a \in F_q^{(2)}\right\}$ are $\{\infty\}$,$\{0\}$, $F_q^{(2)}$ and $F_q^{*}\diagdown F_q^{(2)}$. Consequently,

$$H-\text{orbits are} \begin{cases} \{0,\infty\} \text{ and } F_q^{*}, & \text{for } q \text{ even or } q \equiv 3 \,(\mathrm{mod}\,4)\,; \\ \\ \{0,\infty\},\ F_q^{(2)} \text{ and } F_q^{*}\diagdown F_q^{(2)}, & \text{for } q \equiv 1\,(\mathrm{mod}\,4). \end{cases}$$

Thus, nontrivial 2-designs exist for $q \equiv 1\,(\mathrm{mod}\,4)\,,\ q \geq 13$. Up to complementation it remains to consider base blocks consisting of one orbit each, that being $B_1 = F_q^{(2)}$ and $B_2 = F_q^{*}\diagdown F_q^{(2)}$.

Obviously the mapping $x \mapsto \xi x, \xi \in F_q^{*}\diagdown F_q^{(2)}$, lies in K and maps the orbit $F_q^{(2)}$ into $F_q^{*}\diagdown F_q^{(2)}$, so up to isomorphism and complementation there exists a unique $2-\left(q+1, \frac{q-1}{2}, \frac{(q-1)(q-3)}{8}\right)$ design $D = D\left(T, F_q^{(2)}\right)$. Using Corollary 4.3 we easily get Aut$D = P\Sigma L\,(2,q)$. ∎

B) *H-type 3* (a dihedral group of order $\dfrac{2\,(q+1)}{\gcd(2,q-1)}$)

PROPOSITION 5.2. *Let $q = p^f \geq 13$. A block design D with the socle* PSL$\,(2,q)$ *of* Aut$D$ *and the base block stabilizer H in the third Aschbacher's class exists if and only if $q \equiv 1(\mathrm{mod}\,4)$. D is unique up to isomorphism and complementation. If $p \equiv 1(\mathrm{mod}\,4)$, then D is a* 2-$(q+1, \frac{q+1}{2}, \frac{(q-1)^2}{8})$ *design with* Aut$D = P\Sigma L\,(2,q)$. *If $p \equiv 3(\mathrm{mod}\,4)$, then D is a* $3-\left(q+1, \frac{q+1}{2}, \frac{(q-3)(q-1)}{16}\right)$ *design with* Aut$D =$ PSL$\,(2,q)\cdot\Delta$, *where Δ is cyclic group of order $2f$ and* $|\mathrm{PSL}\,(2,q) \cap \Delta| = 2$.

PROOF. If $q \equiv 3\,(\mathrm{mod}\,4)$ or q is even, then H acts transitively on the projective line, which leaves the possibility $q \equiv 1\,(\mathrm{mod}\,4)$, so H is a dihedral group of order $q+1$ and $G^{MAX} = P\Gamma L\,(2,q)$. In [6, Lemma 14, (i)] we find that H acts in two orbits on Ω.

Let $F_q^{*} = \langle\xi\rangle$ and $A = \left\{x \mapsto \frac{ax+b\xi}{bx+a} : a,b \in F_q, a^2 - b^2\xi \in F_q^{(2)}\right\}$. Then A is a cyclic subgroup of H of order $\frac{q+1}{2}$ and $H = N_T\,(A)$. One H-orbit on Ω is $\{\infty\} \cup \left\{a \in F_q^{*} : a^2 \in \xi + F_q^{(2)}\right\}$. Up to isomorphism and complementation, design $D = D\left(T, \{\infty\} \cup \left\{a \in F_q^{*} : a^2 \in \xi + F_q^{(2)}\right\}\right)$ is a unique $2-\left(q+1, \frac{q+1}{2}, \frac{(q-1)^2}{8}\right)$ design with the base block stabilizer $A \rtimes \Delta$, where $\Delta = \{\,\delta_u \mid u \in \mathbb{Z}\}$ is a cyclic group of order $2f$ and δ_u is the action on Ω defined by $x \to \xi^{\frac{1-p^u}{2}} x^{p^u}$ $(\delta_{u_1} \circ \delta_{u_2} = \delta_{u_1+u_2})$. Corollary 4.3 implies $\mathrm{Aut}D = T \cdot \Delta$. If $p \equiv 3 \pmod 4$, then $\xi^{\frac{1-p}{2}}$ is not a square, so $\mathrm{Aut}D$ is 3-homogeneous. $\square$

REMARK 5.3. 3-designs from the above proposition are not given in [5,6]. The group $\mathrm{PSL}(2,q) \cdot \Delta$ has the same order as $P\Sigma L(2,q)$ but is different from this group.

C) *H-type 4* ($H \cong \mathrm{PGL}(2,q_0)$, $q = q_0^2$)

PROPOSITION 5.4. *Let* $q = q_0^2 \geq 13$. *Then, up to isomorphism and complementation, there exists a unique primitive block design D with automorphism group* $\mathrm{PSL}(2,q)$ *and a block stabilizer* $H = \mathrm{PGL}(2,q_0)$. $\mathrm{Aut}D = P\Sigma L(2,q)$ *and one of the following holds:*

(1) *q is even and D is a* $3-(q_0^2+1, q_0+1, 1)$ *design;*
(2) *q is odd and D is a* $2-(q_0^2+1, q_0+1, \frac{q_0+1}{2})$ *design.*

PROOF. If q is odd, in [6, Lemma 14, (i)] we find that H acts in two orbits on Ω. If q is even, then H consists of all fractional linear transformations in T with coefficients from F_{q_0}. Obviously, $\{\infty\} \cup F_{q_0}$ is a H-orbit. Let $\gamma \in F_q \setminus F_{q_0}$. γ generates F_q and every element in $F_q \setminus F_{q_0}$ can be presented in the form $a\gamma + b$, $a \in F_{q_0}^*$, $b \in F_{q_0}$. H contains the group $\left\{x \mapsto ax + b : a \in F_{q_0}^*, b \in F_{q_0}\right\}$, so $F_q \setminus F_{q_0}$ is also a H-orbit, i.e. H acts in two orbits on Ω also for q even. Thus, because $D(T, F_q \setminus F_{q_0})$ is complementary to $D = D(T, F_{q_0} \cup \{\infty\})$, D is a unique existing $2-\left(q_0^2+1, q_0+1, \frac{q_0+1}{\gcd(2,q_0-1)}\right)$ design up to isomorphism and complementation; $\mathrm{Aut}D = P\Sigma L(2,q)$. For q even, the group $P\Sigma L(2,q)$ is 3-homogeneous, so D is a $3-\left(q_0^2+1, q_0+1, 1\right)$ design. $\square$

Designs from the above proposition, (1), are called Möbius planes ([7, p. 82]).

D) *H-type 5* ($H \cong \mathrm{PSL}(2,q_0)$, $q = q_0^r$, $q_0 \neq 2$, $r > 2$ prime)

For this H-type we only partly solved the problem by finding orbit structure for H. Orbit structure for $K = N_{P\Gamma L(2,q)}(H)$ remained beyond our reach because of the great number of combinatorial possibilities for the action of the automorphisms of F_q (contained in K) on H-orbits.

If q is odd, from [6, Lemma 14, (ii)] it follows that $\{\infty\} \cup F_{q_0}$ is the only H-orbit which is not regular. In case q is even, H consists of all elements in T with coefficients $a, b, c, d \in F_{q_0}$, $q_0 \neq 2$ is a prime power. Obviously, $\{\infty\} \cup F_{q_0}$ is a H-orbit. Let $\gamma \in F_q \setminus F_{q_0}$. Because r is prime, γ generates F_q. Let $x \mapsto \frac{ax+b}{cx+d}$ be an element of H which stabilizes γ. Then $\frac{a\gamma+b}{c\gamma+d} = \gamma$, i.e. $c\gamma^2 + (d-a)\gamma - b = 0$. $c \neq 0$ would imply that γ is a root of a polynomial of degree 2 with coefficients in F_{q_0}, which is a contradiction. Thus $c = 0$, $a = d$ and $b = 0$, which means that points in $F_q \setminus F_{q_0}$ have trivial stabilizer and that $\{\infty\} \cup F_{q_0}$ is the only non regular H-orbit also in case q is even.

Consequently, for H-orbit lengths we find $(q_0+1)^1\left|\mathrm{PSL}(2,q_0)\right|^{s_r}$, where $s_r = \frac{q_0^{r-1}-1}{q_0^2-1}\cdot\gcd(2,q_0-1)$. Substituting $q_0 = p^{f/r}$ we can write $s_r = \frac{p^{f(1-1/r)}-1}{p^{2f/r}-1}\cdot\gcd(2,p-1)$ as well.

Design $D = D(T,\{\infty\}\cup F_{q_0})$ is $3-(q_0^r+1, q_0+1, 1)$ design called spherical geometry, [7, p. 82]; $\mathrm{Aut}D = P\Gamma L(2,q)$.

In case $r=3$ we can easily describe all existing designs because $s_3\in\{1,2\}$. If q is even then $s_3 = 1$, so there exist only spherical geometry and its complement. If q is odd then $s_3 = 2$. Let $\gamma\in F_q\setminus F_{q_0}$ generate F_q. There exists $\pi\in F_{q_0}^*$ such that $\pi\notin F_q^{(2)}$. Now γ and $\pi\gamma$ lie in different orbits as an equation $\frac{a\gamma+b}{c\gamma+d}=\pi\gamma$ with $a,b,c,d\in F_{q_0}$ is impossible. In this case there exists exactly one more design (up to isomorphism and complementation), that being $D^+ = D\left(T,\gamma^{\mathrm{PSL}(2,q_0)}\right)$. D^+ is $2-\left(q_0^3+1, \frac{q_0(q_0^2-1)}{2}, \frac{(q_0^3-1)(q_0(q_0^2-1)-2)}{4}\right)$ design. If $q\equiv 3 \pmod 4$, then D^+ is $3-\left(q_0^3+1, \frac{q_0(q_0^2-1)}{2}, \frac{(q_0(q_0^2-1)-2)(q_0(q_0^2-1)-4)}{8}\right)$ design.

6. On the designs obtained for H-types 6 thru 10

In this section we consider designs with block stabilizers H from the last five rows of Table 1, i.e. $H\cong A_4, S_4, A_5\le \mathrm{PSL}(2,q)$ and $H\cong S_4\le\mathrm{PGL}(2,q)$. We determine the number of designs and their full automorphism groups using orbit lengths of groups H and K; here either $K=H$ or $[K:H]=2$. Orbit lengths for groups H and K, in case of H-types 7-10, can be found in [5,6], as well as H-orbit lengths in case of H-type 6 ($q=p^2$). On the other hand, if H is of type 6 then K-orbit sizes can not be read off from the papers of Cameron et al. Therefore, subsequently in the section, we give in detail only the determining of K-orbit lengths for H-type 6.

Let us begin with the calculation of the numbers $\mathrm{npd}_H(q)$ of nontrivial primitive t-designs having a particular block stabilizer H, regarded up to isomorphism and complementation. Let θ be the number of H-orbits. If $K=H$, obviously $\mathrm{npd}_H(q) = (2^\theta-2)/2 = 2^{\theta-1}-1$. In case $[K:H]=2$ let $l\ge 0$ be the number of H-orbits that K fixes setwise. Then $\theta-l$ is even, say $\theta-l=2j$, $j\ge1$. Let $\mathcal{O}=\{o_{11},o_{21},o_{12},o_{22},\dots,o_{1j},o_{2j}\}$ be the set of H-orbits that K does not fix setwise, where $\{o_{1i},o_{2i}\}, i=1,\dots,j$ are K-orbits on $\mathcal{O}$. If we denote by Λ the number of nonisomorphic designs with base blocks $\widetilde{B}\subseteq\mathcal{O}$, then we have $\mathrm{npd}_H(q) = [2^l\cdot\Lambda-2]/2$. In order to calculate Λ one can observe $2\times j$ matrices $A=[A_{mi}]$ whose $0,1$ entries correspond to the specific base block $\widetilde{B}$ in the sense that

$$A_{mi} = \begin{cases} 1, & o_{mi}\subseteq\widetilde{B} \\ 0, & o_{mi}\not\subseteq\widetilde{B}\end{cases}.$$

The action of $K\backslash H$ on $\mathcal{O}$ and the consequent development of $\widetilde{B}$ reflect in the entries of A as swapping the position of the rows of A. Eventually, we use the following lemma to determine Λ.

Lemma 6.1. *Let $\mathcal{A}$ be the set of all $2 \times j$ matrices with $0, 1$ entries. For $A_1, A_2 \in \mathcal{A}$ we define $A_1 \sim A_2$ if and only if $A_1 = A_2$ or A_2 is obtained from A_1 by swapping the rows. Then $\sim$ is an equivalence relation and $|\mathcal{A}/{\sim}| = 2^{2j-1} + 2^{j-1}$.*

Proof. Obviously $|\mathcal{A}| = 2^{2j}$. It is easily checked that $\sim$ is an equivalence relation with 1 or 2 elements in each equivalence class. A class consists of only one matrix if the columns of that matrix are of the form $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so the number of singleton classes is 2^j. Let's denote the number of classes with two elements by μ. Then we have $2^j + 2\mu = 2^{2j}$ or $\mu = 2^{2j-1} - 2^{j-1}$. From $|\mathcal{A}/{\sim}| = 2^j + \mu$ we finally obtain $|\mathcal{A}/{\sim}| = 2^{2j-1} + 2^{j-1}$. ∎

From the lemma we conclude that for a given block stabilizer H with $[K : H] = 2$ we have $\Lambda = 2^{2j-1} + 2^{j-1}$ and consequently $\mathrm{npd}_H(q) = [2^l \cdot (2^{2j-1} + 2^{j-1}) - 2]/2$.

e) *H-type 6 ($H \cong A_5$)*

Here $q = p^2 \equiv 49 \pmod{60}$, $p \equiv 7, 13, 17, 23, 37, 43, 47, 53 \pmod{60}$, [10, Theorem 2.2]. There are two conjugacy classes of H in T, so $G^{MAX} = P\Sigma L(2, q)$; $K = S_5$. According to [6, Lemma 11, (i)] the only possible combination of H-orbit lengths is: $20^1 30^1 60^{\frac{q-49}{60}}$. H-orbits of length 20 and 30 are obviously fixed by K- action. On K-orbits of length 20, 30 and 60 H acts transitively, so for any point ω from these orbits $K_\omega \not\leq H$ holds. Let $\tau \in K \setminus H$ be an involution. All such involutions are conjugate in K, thus they fix the same number of points, say φ. The list of possibilities for K-action on the orbits of length m is obtained by computer calculations ([9], [3]):

m	20	20	30	30	60
K_ω	S_3	C_6	C_4	C_2^2	C_2
φ	6	2	0	6	6

If K acts on an orbit of length 20, then τ has at least two fixed points in that orbit. Without loss of generality we may take that τ fixes points 0 and ∞; namely, such a choice of K can be obtained by the action of some element from G^{MAX}. Then $x^\tau = ax^p$, where $a \in F_q^{(2)}$ and $a^{p+1} = 1$. The equation $x^{p-1} = a^{-1}$ has $p - 1$ solutions in F_q^*. These solutions are fixed points of τ, so that τ has exactly $p + 1$ fixed points. Let φ_1, φ_2 be the numbers of fixed points in orbits of length 20 and 30, respectively; $\varphi_1 \in \{2, 6\}$, $\varphi_2 \in \{0, 6\}$. Let d be the number of orbits of length 60 fixed by involution τ. Then the equation

$$\varphi_1 + \varphi_2 + 6d = p + 1 \tag{6.1}$$

holds. By substituting all admissible φ_1 and φ_2 into equation (6.1) and then solving it for d we finally obtain orbit lengths for K:

1. $p \equiv 13, 37 \pmod{60} \to 20^1 30^1 60^{\frac{p-1}{6}} 120^{\frac{1}{2}\left(\frac{q-49}{60} - \frac{p-1}{6}\right)}$,
2. $p \equiv 17, 53 \pmod{60} \to 20^1 30^1 60^{\frac{p-5}{6}} 120^{\frac{1}{2}\left(\frac{q-49}{60} - \frac{p-5}{6}\right)}$,
3. $p \equiv 7, 43 \pmod{60} \to 20^1 30^1 60^{\frac{p-7}{6}} 120^{\frac{1}{2}\left(\frac{q-49}{60} - \frac{p-7}{6}\right)}$,
4. $p \equiv 23, 47 \pmod{60} \to 20^1 30^1 60^{\frac{p-11}{6}} 120^{\frac{1}{2}\left(\frac{q-49}{60} - \frac{p-11}{6}\right)}$.

Obviously the following assertion holds.

PROPOSITION 6.2. *Let $p \geq 5$ and let S_5 be a maximal subgroup of $P\Sigma L(2, p^2)$. If orbit lengths of S_5 are $20^1 30^1 60^d 120^{d_1}$, $d_1 = \frac{1}{2}\left(\frac{q-49}{60} - d\right)$, then for the number of primitive designs with $q = p^2$ we have the following:*

1) *If* $p \not\equiv 49 \pmod{60}$, *then* $\mathrm{npd}(q) = 3$.
2) *If* $p \equiv 49 \pmod{60}$, *then* $\mathrm{npd}(q) = 2 + 2^{d+1}(2^{2d_1 - 1} + 2^{d_1 - 1})$.

For $q = p^2 \equiv 49 \pmod{60}$ we obtain the series of $2-\left(q+1, k, \frac{(q-1)k(k-1)}{120}\right)$ designs D with $\mathrm{PSL}(2, q) \leq \mathrm{Aut}D \leq P\Sigma L(2, q)$.

For H-types 7 through 10 we only note possible orbit lengths and the corresponding number of primitive designs $\mathrm{npd}_H(q)$ which we need for counting the total number of designs. $\mathrm{Aut}D$ is easily read off from Table 1, except for H-type 8 where it is necessary to take into account Corollary 4.3 to obtain $\mathrm{Aut}D$.

F) *H-type 7 ($H \cong A_5$)*

$q = p \equiv$	$H = K \cong A_5$ orbit lengths	$\mathrm{npd}_H(q)$
$1 \pmod{60}$	$12^1 20^1 30^1 60^{\frac{q-61}{60}}$	$2^{\frac{q+59}{60}} - 1$
$11 \pmod{60}$	$12^1 60^{\frac{q-11}{60}}$	$2^{\frac{q-11}{60}} - 1$
$19 \pmod{60}$	$20^1 60^{\frac{q-19}{60}}$	$2^{\frac{q-19}{60}} - 1$
$29 \pmod{60}$	$30^1 60^{\frac{q-29}{60}}$	$2^{\frac{q-29}{60}} - 1$
$31 \pmod{60}$	$12^1 20^1 60^{\frac{q-31}{60}}$	$2^{\frac{q+29}{60}} - 1$
$41 \pmod{60}$	$12^1 30^1 60^{\frac{q-41}{60}}$	$2^{\frac{q+19}{60}} - 1$
$49 \pmod{60}$	$20^1 30^1 60^{\frac{q-49}{60}}$	$2^{\frac{q+11}{60}} - 1$
$59 \pmod{60}$	$60^{\frac{q+1}{60}}$	$2^{\frac{q-59}{60}} - 1$

G) *H-type 8 ($H \cong A_4$)*

$q = p \equiv$	$H \cong A_4$	$K \cong S_4$	$\mathrm{npd}_H(q)$
$53, 77 \pmod{120}$	$6^1 12^{\frac{q-5}{12}}$	$6^1 24^{\frac{q-5}{24}}$	$2^{\frac{q-17}{12}} + 2^{\frac{q-29}{24}} - 1$
$83, 107 \pmod{120}$	$12^{\frac{q+1}{12}}$	$12^1 24^{\frac{q-11}{24}}$	$2^{\frac{q-23}{12}} + 2^{\frac{q-35}{24}} - 1$
$13, 37 \pmod{120}$	$4^2 6^1 12^{\frac{q-13}{12}}$	$6^1 8^1 24^{\frac{q-13}{24}}$	$2^{\frac{q-1}{12}} + 2^{\frac{q-13}{24}} - 1$
$43, 67 \pmod{120}$	$4^2 12^{\frac{q-7}{12}}$	$8^1 12^1 24^{\frac{q-19}{24}}$	$2^{\frac{q-7}{12}} + 2^{\frac{q-19}{24}} - 1$

H) *H-type 9 ($H \cong S_4$)*

$q = p \equiv$	$H = K \cong S_4$ orbit lengths	$\mathrm{npd}_H(q)$
$1, 49, 73, 97 \pmod{120}$	$6^1 8^1 12^1 24^{\frac{q-25}{24}}$	$2^{\frac{q+23}{24}} - 1$
$7, 31, 79, 103 \pmod{120}$	$8^1 24^{\frac{q-7}{24}}$	$2^{\frac{q-7}{24}} - 1$
$17, 41, 89, 113 \pmod{120}$	$6^1 12^1 24^{\frac{q-17}{24}}$	$2^{\frac{q+7}{24}} - 1$
$23, 47, 71, 119 \pmod{120}$	$24^{\frac{q+1}{24}}$	$2^{\frac{q-23}{24}} - 1$

I) *H-type 10 ($H \cong S_4$)*

$q = p \equiv$	$H = K \cong S_4$ orbit lengths	$\mathrm{npd}_H(q)$
$29, 101 \pmod{120}$	$6^1 24^{\frac{q-5}{24}}$	$2^{\frac{q-5}{24}} - 1$
$11, 59 \pmod{120}$	$12^1 24^{\frac{q-11}{24}}$	$2^{\frac{q-11}{24}} - 1$
$61, 109 \pmod{120}$	$6^1 8^1 24^{\frac{q-13}{24}}$	$2^{\frac{q+11}{24}} - 1$
$19, 91 \pmod{120}$	$8^1 12^1 24^{\frac{q-19}{24}}$	$2^{\frac{q+5}{24}} - 1$

7. Survey of results

The q-range covered theoretically in this research is $q \geq 13, q \neq 23$. Cases with $q < 13$ and $q = 23$ are solved using programming and computation in GAP and MAGMA. In this way the nonexistence of primitive designs with $q = 4, 7, 8, 11$, and 23 is proved. For $q = 23$ it is interesting that $\mathrm{PSL}(2, 23) < M_{24}$ holds, [14], cf. Proposition 4.1. However, an exhausting computer search shows the nonexistence of primitive design with an automorphism group having $\mathrm{PSL}(2, 23)$ as the socle.

Below we give the number of primitive designs obtained through exhaustive computer search for all $q \leq 103$. The designs and the related documentation are available at: `http://www.pmfst.hr/~sbraic/t-designs/`.

q	4	5	7	8	9	11	13	16	17	19	23	25	27
npd (q)	0	1	0	0	2	0	4	1	3	1	0	3	2

q	29	31	32	37	41	43	47	49	53	59	61	64	67
npd (q)	3	2	0	11	6	9	1	4	11	3	12	2	35

q	71	73	79	81	83	89	97	101	103
npd (q)	4	17	8	3	35	18	33	20	15

The sole existing design for $q = 5$ has a block stabilizer H of type 3. It is a $2-(6,3,2)$ design to which extends the validity of Proposition 5.2. Out of two designs existing for $q = 9$, one is $2-(10,4,2)$ design with a block stabilizer H of type 4, described in Proposition 5.4, (2). The other is $3-(10,5,3)$ design with a block stabilizer $H = C_5 \rtimes C_4$. The subgroup $H \cap T$ is not maximal in $T = \mathrm{PSL}(2,9)$, whereas H is maximal in M_{10}, which is the full automorphism group of this design.

The total number of nontrivial primitive t-designs, up to isomorphism and complementation, for a given q is the sum of npd (q) over all H-types. Due to the incompleteness of results for block stabilizers H of type 5, in the following proposition we give that number only for $q = p$. The proof is pure combinatorics.

Proposition 7.1. *If $q \geq 7$ is prime then the following formulas hold:*

1. $q \equiv 1 \pmod{120} \Rightarrow \mathrm{npd}\,(q) = 2^{\frac{q+59}{60}} + 2^{\frac{q+23}{24}}$,
2. $q \equiv 7, 103 \pmod{120} \Rightarrow \mathrm{npd}\,(q) = 2^{\frac{q-7}{24}} - 1$,
3. $q \equiv 11 \pmod{120} \Rightarrow \mathrm{npd}\,(q) = 2^{\frac{q-11}{60}} + 2^{\frac{q-11}{24}} - 2$,
4. $q \equiv 13, 37 \pmod{120} \Rightarrow \mathrm{npd}\,(q) = 2^{\frac{q-1}{12}} + 2^{\frac{q-13}{24}} + 1$,
5. $q \equiv 17, 113 \pmod{120} \Rightarrow \mathrm{npd}\,(q) = 2^{\frac{q+7}{24}} + 1$,
6. $q \equiv 19 \pmod{120} \Rightarrow \mathrm{npd}\,(q) = 2^{\frac{q-19}{60}} + 2^{\frac{q+5}{24}} - 2$,
7. $q \equiv 23, 47 \pmod{120} \Rightarrow \mathrm{npd}\,(q) = 2^{\frac{q-23}{24}} - 1$,
8. $q \equiv 29 \pmod{120} \Rightarrow \mathrm{npd}\,(q) = 2^{\frac{q-29}{60}} + 2^{\frac{q-5}{24}}$,
9. $q \equiv 31 \pmod{120} \Rightarrow \mathrm{npd}\,(q) = 2^{\frac{q+29}{60}} + 2^{\frac{q-7}{24}} - 2$,
10. $q \equiv 41 \pmod{120} \Rightarrow \mathrm{npd}\,(q) = 2^{\frac{q+19}{60}} + 2^{\frac{q+7}{24}}$,
11. $q \equiv 43, 67 \pmod{120} \Rightarrow \mathrm{npd}\,(q) = 2^{\frac{q-7}{12}} + 2^{\frac{q-19}{24}} - 1$,
12. $q \equiv 49 \pmod{120} \Rightarrow \mathrm{npd}\,(q) = 2^{\frac{q+11}{60}} + 2^{\frac{q+23}{24}}$,
13. $q \equiv 53, 77 \pmod{120} \Rightarrow \mathrm{npd}\,(q) = 2^{\frac{q-17}{12}} + 2^{\frac{q-29}{24}} + 1$,
14. $q \equiv 59 \pmod{120} \Rightarrow \mathrm{npd}\,(q) = 2^{\frac{q-59}{60}} + 2^{\frac{q-11}{24}} - 2$,
15. $q \equiv 61 \pmod{120} \Rightarrow \mathrm{npd}\,(q) = 2^{\frac{q+59}{60}} + 2^{\frac{q+11}{24}}$,
16. $q \equiv 71 \pmod{120} \Rightarrow \mathrm{npd}\,(q) = 2^{\frac{q-11}{60}} + 2^{\frac{q-23}{24}} - 2$,
17. $q \equiv 73, 97 \pmod{120} \Rightarrow \mathrm{npd}\,(q) = 2^{\frac{q+23}{24}} + 1$,
18. $q \equiv 79 \pmod{120} \Rightarrow \mathrm{npd}\,(q) = 2^{\frac{q-19}{60}} + 2^{\frac{q-7}{24}} - 2$,

19. $q \equiv 83, 107 \,(\mathrm{mod}\, 120) \Rightarrow \mathrm{npd}\,(q) = 2^{\frac{q-23}{12}} + 2^{\frac{q-35}{24}} - 1,$
20. $q \equiv 89 \,(\mathrm{mod}\, 120) \Rightarrow \mathrm{npd}\,(q) = 2^{\frac{q-29}{60}} + 2^{\frac{q+7}{24}},$
21. $q \equiv 91 \,(\mathrm{mod}\, 120) \Rightarrow \mathrm{npd}\,(q) = 2^{\frac{q+29}{60}} + 2^{\frac{q+5}{24}} - 2,$
22. $q \equiv 101 \,(\mathrm{mod}\, 120) \Rightarrow \mathrm{npd}\,(q) = 2^{\frac{q+19}{60}} + 2^{\frac{q-5}{24}},$
23. $q \equiv 109 \,(\mathrm{mod}\, 120) \Rightarrow \mathrm{npd}\,(q) = 2^{\frac{q+11}{60}} + 2^{\frac{q+11}{24}},$
24. $q \equiv 119 \,(\mathrm{mod}\, 120) \Rightarrow \mathrm{npd}\,(q) = 2^{\frac{q-59}{60}} + 2^{\frac{q-23}{24}} - 2.$

The number $\mathrm{npd}(p^2)$ is given in Proposition 6.2. Notice that $q = p^f, f > 2$ can appear only for H-types $2, 3, 4$ and 5.

Proposition 7.2. *Let α, β be nonnegative integers. Then for the number of primitive designs with $q = p^{2^\alpha 3^\beta}$ we have the following:*

1. $\mathrm{npd}(2^{2^\alpha}) = 1, \alpha \geq 2.$
2. $\mathrm{npd}(2^{3^\alpha}) = 1, \alpha \geq 2.$
3. $\mathrm{npd}(2^{2^\alpha 3^\beta}) = 2, \alpha, \beta \geq 1.$
4. $\mathrm{npd}(p^{2^\alpha}) = 3, p \neq 2$ *and* $\alpha \geq 2.$
5. $\mathrm{npd}(p^{3^\alpha}) = 2, p \equiv 3(\mathrm{mod}\, 4)$ *and* $\alpha \geq 1.$
6. $\mathrm{npd}(p^{3^\alpha}) = 4, p \equiv 1(\mathrm{mod}\, 4)$ *and* $\alpha \geq 1.$
7. $\mathrm{npd}(p^{2^\alpha 3^\beta}) = 5, p \neq 2$ *and* $\alpha, \beta \geq 1.$

Proposition 7.3. *Let $q \geq 4$. Then* $\mathrm{npd}(q) = 0$ *if and only if* $q = 7, 11, 23$ *or $q = 2^r, r$ a prime.*

Proof. If $q = 7, 11, 23$ or $q = 2^r, r$ a prime, we use Proposition 7.1, Proposition 5.1 and Proposition 5.2 to obtain $\mathrm{npd}(q) = 0$. Conversely, let $\mathrm{npd}(q) = 0$. If $q = p$, we simply solve the equalities $\mathrm{npd}\,(q) = 0$ in Proposition 7.1. If $q = p^f, f \geq 2$, then there exists a prime $r | \ f$ so that $q = q_0^r$ $(q_0 = p^{f/r}$, this relates to H-types 4 and 5) and it is known that 3-designs $D(T, \{\infty\} \cup F_{q_0})$ called spherical geometries exist, [7, p. 82]. The spherical geometry is not primitive design only in case $p = 2$ and f is a prime. □

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Received: 29.1.2014.