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GENERALIZATION OF MAJORIZATION THEOREM VIA ABEL-GONTSCHAROFF POLYNOMIAL

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Abstract. In this paper we use Abel-Gontscharoff formula and Green function to give some identities for the difference of majorization inequality and present the generalization of majorization theorem for the class of n-convex. We use inequalities for the Čebyšev functional to obtain bounds for the identities related to generalizations of majorization inequalities. We present mean value theorems and n-exponential convexity for the functional obtained from the generalized majorization inequalities. At the end we discuss the results for particular families of functions and give means.

1. Introduction

For fixed $m \geq 2$ let

$$\mathbf{x} = (x_1, ..., x_m), \quad \mathbf{y} = (y_1, ..., y_m)$$

denote two real m-tuples. Let

$$x_{[1]} \geq x_{[2]} \geq ... \geq x_{[m]}, \quad y_{[1]} \geq y_{[2]} \geq ... \geq y_{[m]},$$

$$x_{(1)} \leq x_{(2)} \leq ... \leq x_{(m)}, \quad y_{(1)} \leq y_{(2)} \leq ... \leq y_{(m)}$$

be their ordered components.

Definition 1.1. *[23, p. 319] $\boldsymbol{x}$ is said to majorize $\boldsymbol{y}$ (or $\boldsymbol{y}$ is said to be majorized by $\boldsymbol{x}$), in symbol, $\boldsymbol{x} \succ \boldsymbol{y}$, if*

$$\sum_{i=1}^{l} y_{[i]} \leq \sum_{i=1}^{l} x_{[i]} \tag{1.1}$$

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holds for $l = 1, 2, ..., m-1$ *and*

$$\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} y_i.$$

Note that (1.1) is equivalent to

$$\sum_{i=m-l+1}^{m} y_{(i)} \leq \sum_{i=m-l+1}^{m} x_{(i)}$$

holds for $l = 1, 2, ..., m-1$.

The following theorem is well-known as the majorization theorem given by Marshall and Olkin [20, p. 14] (see also [23, p. 320]):

THEOREM 1.2. *Let* $\boldsymbol{x} = (x_1, ..., x_m)$, $\boldsymbol{y} = (y_1, ..., y_m)$ *be two* m*-tuples such that* $x_i,\ y_i \in [a, b]$ $(i = 1, ..., m)$. *Then*

$$\sum_{i=1}^{m} \phi(y_i) \leq \sum_{i=1}^{m} \phi(x_i) \tag{1.2}$$

holds for every continuous convex function $\phi : [a, b] \to \mathbb{R}$ *if and only if* $\mathbf{x} \succ \mathbf{y}$ *holds.*

The following theorem can be regarded as a weighted version of Theorem 1.2 and is proved by Fuchs in [14] ([20, p. 580], [23, p. 323]):

THEOREM 1.3. *Let* $\boldsymbol{x} = (x_1, ..., x_m)$, $\boldsymbol{y} = (y_1, ..., y_m)$ *be two decreasing real* m*-tuples with* $x_i,\ y_i \in [a, b]$ $(i = 1, ..., m)$ *and* $\mathbf{w} = (w_1, w_2, ..., w_m)$ *be a real* m*-tuple such that*

$$\sum_{i=1}^{l} w_i\, y_i \leq \sum_{i=1}^{l} w_i\, x_i \quad \textit{for } \ l = 1, ..., m-1, \tag{1.3}$$

and

$$\sum_{i=1}^{m} w_i\, y_i = \sum_{i=1}^{m} w_i\, x_i. \tag{1.4}$$

Then for every continuous convex function $\phi : [a, b] \to \mathbb{R}$, *we have*

$$\sum_{i=1}^{m} w_i\, \phi(y_i) \leq \sum_{i=1}^{m} w_i\, \phi(x_i). \tag{1.5}$$

The following integral version of Theorem 1.3 is a simple consequence of Theorem 12.14 in [22] (see also [23, p.328]):

THEOREM 1.4. *Let* $x, y : [a, b] \to [\alpha, \beta]$ *be decreasing and* $w : [a, b] \to \mathbb{R}$ *be continuous functions. If*

$$\int_a^{\nu} w(t)\, y(t)\, dt \leq \int_a^{\nu} w(t)\, x(t)\, dt \quad \textit{for every } \ \nu \in [a, b], \tag{1.6}$$

and

$$\int_a^b w(t)\, y(t)\, dt \;=\; \int_a^b w(t)\, x(t)\, dt \tag{1.7}$$

hold, then for every continuous convex function $\phi : [\alpha, \beta] \to \mathbb{R}$*, we have*

$$\int_a^b w(t)\, \phi\,(y(t))\; dt \;\leq\; \int_a^b w(t)\, \phi\,(x(t))\; dt. \tag{1.8}$$

For some other related results and generalization of majorization theorem see [20, p. 583], [1]-[6], [8, 13, 18, 19, 21].
Consider the Green function G defined on $[\alpha, \beta] \times [\alpha, \beta]$ by

$$G(t, s) = \begin{cases} \frac{(t-\beta)(s-\alpha)}{\beta-\alpha}\,, & \alpha \leq s \leq t; \\ \frac{(s-\beta)(t-\alpha)}{\beta-\alpha}, & t \leq s \leq \beta. \end{cases} \tag{1.9}$$

The function G is convex in s, it is symmetric, so it is also convex in t. The function G is continuous in s and continuous in t.

For any function $\phi : [\alpha, \beta] \to \mathbb{R}$, $\phi \in C^2([\alpha, \beta])$, we can easily show by integrating by parts that the following is valid

$$\phi(x) = \frac{\beta - x}{\beta - \alpha}\phi(\alpha) + \frac{x - \alpha}{\beta - \alpha}\phi(\beta) + \int_\alpha^\beta G(x, s)\phi''(s) ds, \tag{1.10}$$

where the function G is defined as above in (1.9) ([26]).

In this paper, n always denotes a positive integer number. Throughout, in what follows, we shall assume that the function ϕ that is n-times continuously differentiable on the interval $[\alpha, \beta]$ (i.e., $\phi \in C^n[\alpha, \beta]$), although this restriction is not necessary.

The Abel-Gontscharoff interpolation problem in the real case was introduced in 1935 by Whittaker [26] and subsequently by Gontscharoff [15] and Davis [12]. The following theorem is Abel-Gontscharoff theorem for two points with integral remainder.

Theorem 1.5 ([7]). *Let* $n, k \in \mathbb{N}$*,* $n \geq 2$*,* $0 \leq k \leq n - 1$*, and* $\phi \in C^n([\alpha, \beta])$*. Then we have*

$$\phi(s) = \rho_{n-1}(\alpha, \beta, \phi, s) + R(\phi, s) \tag{1.11}$$

where $\rho_{n-1}(\alpha, \beta, \phi, s)$ *is the Abel-Gontscharoff interpolating polynomial for two points of degree* $n - 1$*, i.e.*

$$\rho_{n-1}(\alpha, \beta, \phi, s) = \sum_{i=0}^{k} \frac{(s-\alpha)^i}{i!}\phi^{(i)}(\alpha) + \sum_{j=0}^{n-k-2}\left[\sum_{i=0}^{j} \frac{(s-\alpha)^{k+1+i}\,(\alpha-\beta)^{j-i}}{(k+1+i)!\,(j-i)!}\right]\phi^{(k+1+j)}(\beta)$$

and the remainder is given by

$$R(\phi, s) = \int_{\alpha}^{\beta} G_n(s,t)\phi^{(n)}(t)dt$$

and $G_n(s,t)$ is defined by

(1.12)

$$G_n(s,t) = \frac{1}{(n-1)!}\begin{cases} \sum_{i=0}^{k} \binom{n-1}{i} (s-\alpha)^i (\alpha - t)^{n-i-1}, & \alpha \le t \le s; \\ -\sum_{i=k+1}^{n-1} \binom{n-1}{i} (s-\alpha)^i (\alpha - t)^{n-i-1}, & s \le t \le \beta. \end{cases}$$

Further, for $\alpha \le s, t \le \beta$ the following inequalities hold

$$(-1)^{n-k-1}\frac{\partial^i G_n(s,t)}{\partial s^i} \ge 0, \quad 0 \le i \le k, \tag{1.13}$$

$$(-1)^{n-i}\frac{\partial^i G_n(s,t)}{\partial s^i} \ge 0, \quad k+1 \le i \le n-1. \tag{1.14}$$

In order to recall the definition of $n-$convex function, first we write the definition of divided difference.

Definition 1.6. *[23, p. 15] Let ϕ be a real-valued function defined on $[\alpha, \beta]$. The divided difference of order n of the function ϕ at distinct points $[\alpha, \beta]$ is defined recursively by*

$$\phi[x_i] = \phi(x_i), \quad (i = 0, ..., n)$$

and

$$\phi[x_0, ..., x_n] = \frac{\phi[x_1, ..., x_n] - \phi[x_0, ..., x_{n-1}]}{x_n - x_0}.$$

The value $\phi[x_0, ..., x_n]$ is independent of the order of the points $x_0, ..., x_n$.

The definition may be extended to include the case that some (or all) the points coincide. Assuming that $\phi^{(j-1)}(x)$ exists, we define

$$\phi\underbrace{[x, ..., x]}_{j-times} = \frac{\phi^{(j-1)}(x)}{(j-1)!}. \tag{1.15}$$

Definition 1.7. *[23, p. 15] A function $\phi : [\alpha, \beta] \to \mathbb{R}$ is said to be n-convex, $n \ge 0$, on $[\alpha, \beta]$ if and only if for all choices of $(n+1)$ distinct points $x_0, ..., x_n \in [\alpha, \beta]$, the nth order divided difference is non negative that is*

$$\phi[x_0, x_1, ..., x_n] \ge 0.$$

In this paper we utilize Abel-Gontscharoff's theorem with the integral remainder and Green function to establish generalization of majorization theorem for the class of n-convex functions. We use inequalities for the Čebyšev functional to obtain bounds for the identities related to generalizations of majorization inequalities. We present mean value theorems and n-exponential convexity for the functional obtained from the generalized majorization inequalities which leads to exponential convexity and log-convexity for these functionals. Finally, we discuss the results for particular families of function and give classes of Cauchy type means and prove their monotonicity.

2. Main results

We begin this section with the proof of some identities related to generalizations of majorization inequality.

Theorem 2.1. *Let $n, k \in \mathbb{N}$, $n \geq 4$, $0 \leq k \leq n-1$, $\phi \in C^n([\alpha, \beta])$ and $\boldsymbol{w} = (w_1, ..., w_m)$, $\boldsymbol{x} = (x_1, ..., x_m)$ and $\boldsymbol{y} = (y_1, ..., y_m)$ be m-tuples such that $x_l,\ y_l \in [\alpha, \beta], w_l \in \mathbb{R}$ $(l = 1, ..., m)$. Also let G and G_n be defined by (1.9) and (1.12) respectively. Then*

$$
\begin{aligned}
\sum_{l=1}^{m} w_l\, \phi\left(x_l\right) - \sum_{l=1}^{m} w_l\, \phi\left(y_l\right) &= \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha} \sum_{l=1}^{m} w_l\left(x_l - y_l\right) \\
&+ \sum_{i=0}^{k} \frac{\phi^{(i+2)}(\alpha)}{i!} \int_{\alpha}^{\beta} \left[\sum_{l=1}^{m} w_l\left(G(x_l, s) - G(y_l, s)\right)\right] (s-\alpha)^{i}\, ds \\
&+ \sum_{j=0}^{n-k-4} \sum_{i=0}^{j} \frac{(-1)^{j-i}\,(\beta-\alpha)^{j-i}\,\phi^{(k+3+j)}(\beta)}{(k+1+i)!\,(j-i)!} \\
&\times \int_{\alpha}^{\beta} \left[\sum_{l=1}^{m} w_l\left(G(x_l, s) - G(y_l, s)\right)\right] (s-\alpha)^{k+1+i}\, ds \\
&+ \int_{\alpha}^{\beta}\int_{\alpha}^{\beta} \left[\sum_{l=1}^{m} w_l\left(G(x_l, s) - G(y_l, s)\right)\right] G_{n-2}(s,t)\phi^{(n)}(t) dt ds.
\end{aligned} \tag{2.1}
$$

Proof. Using (1.10) in $\sum_{l=1}^{m} w_l\, \phi\left(x_l\right) - \sum_{l=1}^{m} w_l\, \phi\left(y_l\right)$ we have

$$
\begin{aligned}
\sum_{l=1}^{m} w_l\, \phi\left(x_l\right) - \sum_{l=1}^{m} w_l\, \phi\left(y_l\right) &= \\
&\frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha} \sum_{l=1}^{m} w_l\left(x_l - y_l\right) \\
&+ \int_{\alpha}^{\beta} \left[\sum_{l=1}^{m} w_l G(x_l, s) - \sum_{l=1}^{m} w_l G(y_l, s)\right] \phi''(s) ds.
\end{aligned} \tag{2.2}
$$

By Theorem 1.5, $\phi''(s)$ can be expressed as

$$\phi''(s) = \sum_{i=0}^{k} \frac{(s-\alpha)^i}{i!}\phi^{(i+2)}(\alpha) + \sum_{j=0}^{n-k-4}\left[\sum_{i=0}^{j} \frac{(s-\alpha)^{k+1+i}(\alpha-\beta)^{j-i}}{(k+1+i)!\,(j-i)!}\right]\phi^{(k+3+j)}(\beta) + \int_\alpha^\beta G_{n-2}(s,t)\phi^{(n)}(t)dt. \tag{2.3}$$

Using (2.3) in (2.2) we get (2.1). ∎

Integral version of the above theorem can be stated as:

Theorem 2.2. *Let $n,k \in \mathbb{N}$, $n \geq 4$, $0 \leq k \leq n-1$, $\phi \in C^n([\alpha,\beta])$, and let $x, y : [a,b] \to [\alpha,\beta]$, $w : [a,b] \to \mathbb{R}$ be continuous functions and G, G_n be defined by (1.9) and (1.12) respectively. Then*

$$\begin{aligned}\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau &= \\ \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int_a^b w(\tau)(x(\tau)-y(\tau))d\tau& \\ + \sum_{i=0}^{k}\frac{\phi^{(i+2)}(\alpha)}{i!}\int_\alpha^\beta\left(\int_a^b w(\tau)(G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-\alpha)^i ds& \\ + \sum_{j=0}^{n-k-4}\sum_{i=0}^{j}\frac{(-1)^{j-i}(\beta-\alpha)^{j-i}\phi^{(k+3+j)}(\beta)}{(k+1+i)!\,(j-i)!}& \\ \times\int_\alpha^\beta\left(\int_a^b w(\tau)(G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-\alpha)^{k+1+i}ds& \\ + \int_\alpha^\beta\int_\alpha^\beta\left(\int_a^b w(\tau)(G(x(\tau),s)-G(y(\tau),s))d\tau\right)G_{n-2}(s,t)\phi^{(n)}(t)dtds.&\end{aligned} \tag{2.4}$$

In the following theorem we obtain generalizations of majorization inequality for n- convex functions.

Theorem 2.3. *Let $n,k \in \mathbb{N}$, $n \geq 4$, $0 \leq k \leq n-1$, $\boldsymbol{w} = (w_1,...,w_m)$, $\boldsymbol{x} = (x_1,...,x_m)$ and $\boldsymbol{y} = (y_1,...,y_m)$ be m-tuples such that x_l, $y_l \in [\alpha,\beta], w_l \in \mathbb{R}$ $(l = 1,...,m)$. Also let G and G_n be defined by (1.9) and (1.12) respectively. If $\phi : [\alpha,\beta] \to \mathbb{R}$ is $n-$convex, and*

$$\int_\alpha^\beta\left(\sum_{l=1}^{m} w_l\left(G(x_l,s)-G(y_l,s)\right)\right)G_{n-2}(s,t)ds \geq 0, \quad t \in [\alpha,\beta]. \tag{2.5}$$

Then

$$
\begin{aligned}
&\sum_{l=1}^{m} w_l\, \phi\left(x_l\right) - \sum_{l=1}^{m} w_l\, \phi\left(y_l\right) \geq \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha} \sum_{l=1}^{m} w_l\left(x_l - y_l\right) \\
&+ \sum_{i=0}^{k} \frac{\phi^{(i+2)}(\alpha)}{i!} \int_{\alpha}^{\beta} \left[\sum_{l=1}^{m} w_l\left(G(x_l,s)-G(y_l,s)\right)\right](s-\alpha)^{i}\, ds \\
&+ \sum_{j=0}^{n-k-4} \sum_{i=0}^{j} \frac{(-1)^{j-i}\,(\beta-\alpha)^{j-i}\,\phi^{(k+3+j)}(\beta)}{(k+1+i)!\,(j-i)!} \\
&\quad \times \int_{\alpha}^{\beta}\left[\sum_{l=1}^{m} w_l\left(G(x_l,s)-G(y_l,s)\right)\right](s-\alpha)^{k+1+i}\, ds.
\end{aligned}
\tag{2.6}
$$

If the reverse inequality in (2.5) holds, then also the reverse inequality in (2.6) holds.

Proof. Since the function ϕ is $n-$convex, therefore without loss of generality we can assume that ϕ is $n-$times differentiable and $\phi^{(n)}(x) \geq 0$ for all $x \in [\alpha, \beta]$ (see [23, p. 16 and p. 293]). Hence, we can apply Theorem 2.1 to obtain (2.6). □

Remark 2.1. As from (1.13) we have $(-1)^{n-k-3} G_{n-2}(s,t) \geq 0$, therefore for the case when n is even and k is odd or n is odd and k is even, it is enough to assume that $\sum_{l=1}^{m} w_l G(x_l, s) - \sum_{l=1}^{m} w_l G(y_l, s) \geq 0, s \in [\alpha, \beta]$, instead of the assumption (2.5) in Theorem 2.3. Similarly we can discuss for the reverse inequality in (2.6).

Integral version of the above theorem can be stated as:

Theorem 2.4. *Let $n, k \in \mathbb{N}$, $n \geq 4$, $0 \leq k \leq n-1$, $x, y : [a,b] \to [\alpha, \beta]$, $w : [a,b] \to \mathbb{R}$ be continuous functions and G, G_n be defined by (1.9) and (1.12) respectively. If $\phi : [\alpha, \beta] \to \mathbb{R}$ is $n-$convex, and*

$$
\int_{\alpha}^{\beta}\left(\int_{a}^{b} w(\tau)(G(x(\tau),s)-G(y(\tau),s))d\tau\right) G_{n-2}(s,t)ds \geq 0. \tag{2.7}
$$

Then

$$\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau \geq \tag{2.8}$$
$$\frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int_a^b w(\tau)(x(\tau)-y(\tau))d\tau$$
$$+\sum_{i=0}^{k}\frac{\phi^{(i+2)}(\alpha)}{i!}\int_\alpha^\beta\left(\int_a^b w(\tau)(G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-\alpha)^i ds$$
$$+\sum_{j=0}^{n-k-4}\sum_{i=0}^{j}\frac{(-1)^{j-i}(\beta-\alpha)^{j-i}\phi^{(k+3+j)}(\beta)}{(k+1+i)!\,(j-i)!}$$
$$\times\int_\alpha^\beta\left(\int_a^b w(\tau)(G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-\alpha)^{k+1+i}ds.$$

If the reverse inequality in (2.7) holds, then also the reverse inequality in (2.8) holds.

REMARK 2.2. As from (1.13) we have $(-1)^{n-k-3}G_{n-2}(s,t)\geq 0$, therefore for the case when n is even and k is odd or n is odd and k is even, it is enough to assume that $\int_a^b w(\tau)(G(x(\tau),s)-G(y(\tau),s))d\tau\geq 0, s\in[\alpha,\beta]$, instead of the assumption (2.7) in Theorem 2.4. Similarly we can discuss for the reverse inequality in (2.8).

We give generalization of majorization theorem for majorized m-tuples:

THEOREM 2.5. *Let $n,k\in\mathbb{N}$, $n\geq 4$, $0\leq k\leq n-1$ and $\boldsymbol{x}=(x_1,...,x_m)$, $\boldsymbol{y}=(y_1,...,y_m)$ be two m-tuples such that $\mathbf{y}\prec\mathbf{x}$ with x_l, $y_l\in[\alpha,\beta]$, $(l=1,...,m)$. Also let G be defined by (1.9). Consider $\phi:[\alpha,\beta]\to\mathbb{R}$ is $n-$convex.*

(i) *If n is even and k is odd or n is odd and k is even. Then*

$$\sum_{l=1}^{m}\phi(x_l)-\sum_{l=1}^{m}\phi(y_l)\geq \tag{2.9}$$
$$\sum_{i=0}^{k}\frac{\phi^{(i+2)}(\alpha)}{i!}\int_\alpha^\beta\left[\sum_{l=1}^{m}w_l\left(G(x_l,s)-G(y_l,s)\right)\right](s-\alpha)^i ds$$
$$+\sum_{j=0}^{n-k-4}\sum_{i=0}^{j}\frac{(-1)^{j-i}(\beta-\alpha)^{j-i}\phi^{(k+3+j)}(\beta)}{(k+1+i)!\,(j-i)!}$$
$$\times\int_\alpha^\beta\left[\sum_{l=1}^{m}w_l\left(G(x_l,s)-G(y_l,s)\right)\right](s-\alpha)^{k+1+i}ds.$$

Moreover if $\phi^{(i+2)}(\alpha) \geq 0$ for $i = 0, \ldots, k$ and $\phi^{(k+3+j)}(\beta) \geq 0$ if $j - i$ is even and $\phi^{(k+3+j)}(\beta) \leq 0$ if $j - i$ is odd for $i = 0, \ldots, j$ and $j = 0, \ldots, n - k - 4$, then the right hand side of (2.9) will be non negative, that is (1.2) holds.

(ii) *If n and k both are even or both are odd, then reverse inequality holds in (2.9). Moreover if $\phi^{(i+2)}(\alpha) \leq 0$ for $i = 0, \ldots, k$ and $\phi^{(k+3+j)}(\beta) \leq 0$ if $j - i$ is even and $\phi^{(k+3+j)}(\beta) \geq 0$ if $j - i$ is odd for $i = 0, \ldots, j$ and $j = 0, \ldots, n-k-4$, then the right hand side of the reverse inequality in (2.9) will be non positive, that is the reverse inequality in (1.2) holds.*

Proof. By using (1.13) we have $(-1)^{n-k-3} G_{n-2}(s, t) \geq 0,\ \alpha \leq s, t \leq \beta$, therefore if n is even and k is odd or n is odd and k is even then $G_{n-2}(s, t) \geq 0$. Also as G is convex so by Theorem 1.2 and non negativity of G_{n-2}, the inequality (2.5) holds for $w_l = 1,\ l = 1, 2, .., m$. Hence by Theorem 2.3 for $w_l = 1, l = 1, 2, .., m$, the inequality (2.9) holds. By using the other conditions the non negativity of the right hand side of (2.9) is obvious.

Similarly we can prove (ii). ∎

In the following theorem we present generalization of Fuchs' majorization theorem.

Theorem 2.6. *Let $n, k \in \mathbb{N}$, $n \geq 4$, $0 \leq k \leq n - 1$, $\boldsymbol{x} = (x_1, ..., x_m)$, $\boldsymbol{y} = (y_1, ..., y_m)$ be decreasing and $\boldsymbol{w} = (w_1, ..., w_m)$ be any m-tuples such that $x_l,\ y_l \in [\alpha, \beta], w_l \in \mathbb{R}$ $(l = 1, ..., m)$ which satisfies (1.3) and (1.4). Also let G be defined by (1.9). Consider $\phi : [\alpha, \beta] \to \mathbb{R}$ is $n-$convex.*

(i) *If n is even and k is odd or n is odd and k is even. Then*

$$\sum_{l=1}^{m} w_l\, \phi(x_l) - \sum_{l=1}^{m} w_l\, \phi(y_l) \geq \tag{2.10}$$
$$\sum_{i=0}^{k} \frac{\phi^{(i+2)}(\alpha)}{i!} \int_{\alpha}^{\beta} \left[\sum_{l=1}^{m} w_l \left(G(x_l, s) - G(y_l, s)\right)\right] (s-\alpha)^i \, ds$$
$$+ \sum_{j=0}^{n-k-4} \sum_{i=0}^{j} \frac{(-1)^{j-i} (\beta-\alpha)^{j-i}\, \phi^{(k+3+j)}(\beta)}{(k+1+i)!\,(j-i)!}$$
$$\times \int_{\alpha}^{\beta} \left[\sum_{l=1}^{m} w_l \left(G(x_l, s) - G(y_l, s)\right)\right] (s-\alpha)^{k+1+i} \, ds.$$

Moreover if $\phi^{(i+2)}(\alpha) \geq 0$ for $i = 0, \ldots, k$ and $\phi^{(k+3+j)}(\beta) \geq 0$ if $j - i$ is even and $\phi^{(k+3+j)}(\beta) \leq 0$ if $j - i$ is odd for $i = 0, \ldots, j$ and $j = 0, \ldots, n - k - 4$, then the right hand side of (2.10) will be non negative, that is (1.5) holds.

(ii) *If n and k both are even or both are odd, then reverse inequality holds in (2.10). Moreover if $\phi^{(i+2)}(\alpha) \leq 0$ for $i = 0, \dots, k$ and $\phi^{(k+3+j)}(\beta) \leq 0$ if $j-i$ is even and $\phi^{(k+3+j)}(\beta) \geq 0$ if $j-i$ is odd for $i = 0, \dots, j$ and $j = 0, \dots, n-k-4$, then the right hand side of the reverse inequality in (2.10) will be non positive, that is the reverse inequality in (1.5) holds.*

PROOF. The proof is similar to the proof of Theorem 2.5 but use Theorem 1.3 instead of Theorem 1.2. ∎

The integral version of Theorem 2.6 can be stated as:

THEOREM 2.7. *Let $n, k \in \mathbb{N}$, $n \geq 4$, $0 \leq k \leq n-1$, $x, y : [a,b] \to [\alpha, \beta]$ be decreasing and $w : [a,b] \to \mathbb{R}$ be any continuous function. Also let G be defined by (1.9). Consider $\phi : [\alpha, \beta] \to \mathbb{R}$ is $n-$convex and*

$$\int_a^{\nu} w(\tau)y(\tau)\,d\tau \leq \int_a^{\nu} w(\tau)x(\tau)\,d\tau \quad \text{for } \nu \in [a,b], \tag{2.11}$$

$$\int_a^{b} w(\tau)x(\tau)\,d\tau = \int_a^{b} w(\tau)y(\tau)\,d\tau \tag{2.12}$$

(i) *If n is even and k is odd or n is odd and k is even. Then*

$$\begin{aligned}
&\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau \\
&\geq \sum_{i=0}^{k} \frac{\phi^{(i+2)}(\alpha)}{i!} \int_\alpha^\beta \left(\int_a^b w(\tau)(G(x(\tau),s) - G(y(\tau),s))d\tau \right) (s-\alpha)^i ds \\
&+ \sum_{j=0}^{n-k-4} \sum_{i=0}^{j} \frac{(-1)^{j-i}\,(\beta-\alpha)^{j-i}\,\phi^{(k+3+j)}(\beta)}{(k+1+i)!\,(j-i)!} \\
&\times \int_\alpha^\beta \left(\int_a^b w(\tau)(G(x(\tau),s) - G(y(\tau),s))d\tau \right) (s-\alpha)^{k+1+i}\,ds.
\end{aligned} \tag{2.13}$$

Moreover if $\phi^{(i+2)}(\alpha) \geq 0$ for $i = 0, \dots, k$ and $\phi^{(k+3+j)}(\beta) \geq 0$ if $j-i$ is even and $\phi^{(k+3+j)}(\beta) \leq 0$ if $j-i$ is odd for $i = 0, \dots, j$ and $j = 0, \dots, n-k-4$, then the right hand side of (2.13) will be non negative, that is integral version of (1.5) holds.

(ii) *If n and k both are even or both are odd, then reverse inequality holds in (2.13). Moreover if $\phi^{(i+2)}(\alpha) \leq 0$ for $i = 0, \dots, k$ and $\phi^{(k+3+j)}(\beta) \leq 0$ if $j-i$ is even and $\phi^{(k+3+j)}(\beta) \geq 0$ if $j-i$ is odd for $i = 0, \dots, j$ and $j = 0, \dots, n-k-4$, then the right hand side of the reverse inequality in*

(2.13) will be non positive, that is the reverse inequality in the integral version of (1.5) holds.

3. Bounds for identites related to generalizations of majorization inequality

For two Lebesgue integrable functions $f, h : [\alpha, \beta] \to \mathbb{R}$ we consider the Čebyšev functional

$$\Lambda(f,h) = \frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)h(t)dt - \frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)dt \cdot \frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)dt.$$

In [11] the authors proved the following theorems:

Theorem 3.1. *Let $f : [\alpha, \beta] \to \mathbb{R}$ be a Lebesgue integrable function and $h : [\alpha, \beta] \to \mathbb{R}$ be an absolutely continuous function with $(\cdot - \alpha)(\beta - \cdot)[h']^2 \in L[\alpha, \beta]$. Then we have the inequality*

$$|\Lambda(f,h)| \leq \frac{1}{\sqrt{2}}[\Lambda(f,f)]^{\frac{1}{2}}\frac{1}{\sqrt{\beta-\alpha}}\left(\int_\alpha^\beta (x-\alpha)(\beta-x)[h'(x)]^2dx\right)^{\frac{1}{2}}. \tag{3.1}$$

The constant $\frac{1}{\sqrt{2}}$ in (3.1) is the best possible.

Theorem 3.2. *Assume that $h : [\alpha, \beta] \to \mathbb{R}$ is monotonic nondecreasing on $[\alpha, \beta]$ and $f : [\alpha, \beta] \to \mathbb{R}$ is absolutely continuous with $f' \in L_\infty[\alpha, \beta]$. Then we have the inequality*

$$|\Lambda(f,h)| \leq \frac{1}{2(\beta-\alpha)}\|f'\|_\infty \int_\alpha^\beta (x-\alpha)(\beta-x)dh(x). \tag{3.2}$$

The constant $\frac{1}{2}$ in (3.2) is the best possible.

In the sequel we use the above theorems to obtain generalizations of the results proved in the previous section.

For m-tuples $\mathbf{w} = (w_1, ..., w_m)$, $\mathbf{x} = (x_1, ..., x_m)$ and $\mathbf{y} = (y_1, ..., y_m)$ with $x_l, y_l \in [\alpha, \beta], w_l \in \mathbb{R}$ $(l = 1, ..., m)$ and the functions G, G_n as defined above, denote

$$\mathfrak{R}(t) = \sum_{l=1}^m w_l \int_\alpha^\beta \left(G(x_l, s) - G(y_l, s)\right) G_{n-2}(s,t)ds, \quad t \in [\alpha, \beta], \tag{3.3}$$

and for continuous functions $x, y : [a, b] \to [\alpha, \beta]$, $w : [a, b] \to \mathbb{R}$, denote

$$\tilde{\mathfrak{R}}(t) = \int_\alpha^\beta \left(\int_a^b w(\tau)(G(x(\tau), s) - G(y(\tau), s))d\tau\right) G_{n-2}(s,t)ds, \quad t \in [\alpha, \beta], \tag{3.4}$$

Consider the Čebyšev functionals $\Lambda(\mathfrak{R},\mathfrak{R})$ and $\Lambda(\tilde{\mathfrak{R}},\tilde{\mathfrak{R}})$ are given by:

$$\Lambda(\mathfrak{R},\mathfrak{R}) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{R}^2(t)dt - \left(\frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{R}(t)dt\right)^2, \tag{3.5}$$

$$\tilde{\Lambda}(\tilde{\mathfrak{R}},\tilde{\mathfrak{R}}) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \tilde{\mathfrak{R}}^2(t)dt - \left(\frac{1}{\beta-\alpha}\int_\alpha^\beta \tilde{\mathfrak{R}}(t)dt\right)^2. \tag{3.6}$$

THEOREM 3.3. *Let $n,k\in\mathbb{N}$, $n\ge 4$, $0\le k\le n-1$, $\phi\in C^n([\alpha,\beta])$ with $(\cdot-\alpha)(\beta-\cdot)[\phi^{(n+1)}]^2\in L[\alpha,\beta]$, $\boldsymbol{w}=(w_1,...,w_m)$, $\boldsymbol{x}=(x_1,...,x_m)$ and $\boldsymbol{y}=(y_1,...,y_m)$ be m-tuples such that $x_l,\ y_l\in[\alpha,\beta], w_l\in\mathbb{R}$ $(l=1,...,m)$ and let the functions G, $\mathfrak{R}$ and Λ be defined by (1.9), (3.3) and (3.5) respectively. Then*

$$\begin{aligned}\sum_{l=1}^m w_l\,\phi(x_l) - \sum_{l=1}^m w_l\,\phi(y_l) &= \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{l=1}^m w_l\,(x_l-y_l)\\ &+\sum_{i=0}^k \frac{\phi^{(i+2)}(\alpha)}{i!}\int_\alpha^\beta\left[\sum_{l=1}^m w_l\,(G(x_l,s)-G(y_l,s))\right](s-\alpha)^i\,ds\\ &+\sum_{j=0}^{n-k-4}\sum_{i=0}^{j}\frac{(\alpha-\beta)^{j-i}\,\phi^{(k+3+j)}(\beta)}{(k+1+i)!\,(j-i)!}\\ &\times\int_\alpha^\beta\left[\sum_{l=1}^m w_l\,(G(x_l,s)-G(y_l,s))\right](s-\alpha)^{k+1+i}\,ds\\ &+\frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{\beta-\alpha}\int_\alpha^\beta\mathfrak{R}(t)dt+\kappa_n(\phi;\alpha,\beta).\end{aligned} \tag{3.7}$$

where the remainder $\kappa_n(\phi;\alpha,\beta)$ satisfies the estimation

$$|\kappa_n(\phi;\alpha,\beta)| \le \frac{\sqrt{\beta-\alpha}}{\sqrt{2}}\left[\Lambda(\mathfrak{R},\mathfrak{R})\right]^{\frac12}\left|\int_\alpha^\beta (t-\alpha)(\beta-t)[\phi^{(n+1)}(t)]^2dt\right|^{\frac12}. \tag{3.8}$$

PROOF. If we apply Theorem 3.1 for $f\to\mathfrak{R}$ and $h\to\phi^{(n)}$ we obtain

$$\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{R}(t)\phi^{(n)}(t)dt - \frac{1}{\beta-\alpha}\int_\alpha^\beta\mathfrak{R}(t)dt\cdot\frac{1}{\beta-\alpha}\int_\alpha^\beta\phi^{(n)}(t)dt\right|$$

$$\le \frac{1}{\sqrt{2}}\left[\Lambda(\mathfrak{R},\mathfrak{R})\right]^{\frac12}\frac{1}{\sqrt{\beta-\alpha}}\left|\int_\alpha^\beta (t-\alpha)(\beta-t)[\phi^{(n+1)}(t)]^2dt\right|^{\frac12}.$$

Therefore we have

$$\int_\alpha^\beta \mathfrak{R}(t)\phi^{(n)}(t)dt = \frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{\beta-\alpha}\int_\alpha^\beta\mathfrak{R}(t)dt+\kappa_n(\phi;\alpha,\beta)$$

where the remainder $\kappa_n(\phi;\alpha,\beta)$ satisfies the estimation (3.8). Now from the identity (2.1) we obtain (3.7). ∎

Integral case of the above theorem can be given:

THEOREM 3.4. *Let $n,k\in\mathbb{N}$, $n\geq 4$, $0\leq k\leq n-1$, $\phi\in C^n([\alpha,\beta])$ with $(\cdot-\alpha)(\beta-\cdot)[\phi^{(n+1)}]^2\in L[\alpha,\beta]$ and $x,y:[a,b]\to[\alpha,\beta]$, $w:[a,b]\to\mathbb{R}$ be continuous functions and let the functions G, $\tilde{\mathfrak{R}}$, $\tilde{\Lambda}$ be defined by (1.9), (3.4) and (3.6) respectively. Then*

$$\begin{aligned}
(3.9)\quad &\int_a^b w(\tau)\phi(x(\tau))d\tau-\int_a^b w(\tau)\phi(y(\tau))d\tau=\\
&\frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int_a^b w(\tau)(x(\tau)-y(\tau))d\tau\\
&+\sum_{i=0}^{k}\frac{\phi^{(i+2)}(\alpha)}{i!}\int_\alpha^\beta\left(\int_a^b w(\tau)(G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-\alpha)^i ds\\
&+\sum_{j=0}^{n-k-4}\sum_{i=0}^{j}\frac{(\alpha-\beta)^{j-i}\,\phi^{(k+3+j)}(\beta)}{(k+1+i)!\,(j-i)!}\\
&\times\int_\alpha^\beta\left(\int_a^b w(\tau)(G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-\alpha)^{k+1+i}\,ds\\
&+\frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{\beta-\alpha}\int_\alpha^\beta\tilde{\mathfrak{R}}(t)dt+\tilde{\kappa}_n(\phi;\alpha,\beta).
\end{aligned}$$

where the remainder $\tilde{\kappa}_n(\phi;\alpha,\beta)$ satisfies the estimation

(3.10)

$$|\tilde{\kappa}_n(\phi;\alpha,\beta)|\leq\frac{\sqrt{\beta-\alpha}}{\sqrt{2}}\left[\tilde{\Lambda}(\tilde{\mathfrak{R}},\tilde{\mathfrak{R}})\right]^{\frac{1}{2}}\left|\int_\alpha^\beta(t-\alpha)(\beta-t)[\phi^{(n+1)}(t)]^2dt\right|^{\frac{1}{2}}.$$

Using Theorem 3.2 we obtain the following Grüss type inequalities.

THEOREM 3.5. *Let $n,k\in\mathbb{N}$, $n\geq 4$, $0\leq k\leq n-1$, $\phi\in C^n([\alpha,\beta])$ such $\phi^{(n)}$ is increasing on $[\alpha,\beta]$ and let the functions G, $\mathfrak{R}$ and Λ be defined by (1.9), (3.3) and (3.5) respectively. Then the representation* (3.7) *holds and the remainder $\kappa_n(\phi;\alpha,\beta)$ satisfies the bound*

(3.11)

$$|\kappa_n(\phi;\alpha,\beta)|\leq\|\mathfrak{R}'\|_\infty\left\{\frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2}-\frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha}\right\}.$$

Proof. Applying Theorem 3.2 for $f \to \mathfrak{R}$ and $h \to \phi^{(n)}$ we obtain

$$\begin{aligned} &\left| \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{R}(t)\phi^{(n)}(t)dt - \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{R}(t)dt \cdot \frac{1}{\beta-\alpha}\int_\alpha^\beta \phi^{(n)}(t)dt \right| \\ &\leq \frac{1}{2(\beta-\alpha)}\|\mathfrak{R}'\|_\infty \int_\alpha^\beta (t-\alpha)(\beta-t)\phi^{(n+1)}(t)dt. \end{aligned} \tag{3.12}$$

Since

$$\begin{aligned} &\int_\alpha^\beta (t-\alpha)(\beta-t)\phi^{(n+1)}(t)dt = \int_\alpha^\beta [2t-(\alpha+\beta)]\phi^{(n)}(t)dt \\ &= (\beta-\alpha)\left[\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)\right] - 2\left(\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)\right), \end{aligned}$$

using the identity (2.1) and the inequality (3.12) we deduce (3.11). ◻

Integral case of the above theorem can be given:

Theorem 3.6. *Let $n, k \in \mathbb{N}$, $n \geq 4$, $0 \leq k \leq n-1$, $\phi \in C^n([\alpha,\beta])$ such that $\phi^{(n)}$ is increasing on $[\alpha,\beta]$ and let the functions G, $\tilde{\mathfrak{R}}$ $\tilde{\Lambda}$ be defined by (1.9), (3.4) and (3.6) respectively. Then we have the representation* (3.9) *and the remainder $\tilde{\kappa}_n(\phi;\alpha,\beta)$ satisfies the bound*

$$|\tilde{\kappa}_n(\phi;\alpha,\beta)| \leq \|\tilde{\mathfrak{R}}'\|_\infty \left\{ \frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha} \right\}. \tag{3.13}$$

Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be a function then the p-norm of ϕ is defined by

$$\|\phi\|_p = \begin{cases} \left(\int_\alpha^\beta |\phi(t)|^p dt\right)^{\frac{1}{p}}, & \text{for } 1 \leq p < \infty, \quad \text{if } |\phi|^p \text{ is } R\text{-integrable function,} \\ \text{essential supremum of } \phi, & \text{for } p = \infty, \quad \text{if } \phi \text{ is essentially bounded.} \end{cases}$$

We present the Ostrowski-type inequalities related to generalizations of majorization inequality.

Theorem 3.7. *Suppose that all assumptions of Theorem 2.1 hold. Assume (p,q) is a pair of conjugate exponents, that is $1 \leq p, q \leq \infty$, $1/p+1/q =$*

1. *Let* $\left|\phi^{(n)}\right|^p : [\alpha,\beta] \to \mathbb{R}$ *be an R-integrable function. Then we have:*

$$\begin{aligned}
(3.14)\quad & \Bigg|\sum_{l=1}^{m} w_l\,\phi(x_l) - \sum_{l=1}^{m} w_l\,\phi(y_l) - \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{l=1}^{m} w_l\,(x_l-y_l) \\
& - \sum_{i=0}^{k}\frac{\phi^{(i+2)}(\alpha)}{i!}\int_\alpha^\beta\left(\sum_{l=1}^{m} w_l\,(G(x_l,s)-G(y_l,s))\right)(s-\alpha)^i ds \\
& \qquad - \sum_{j=0}^{n-k-4}\sum_{i=0}^{j}\frac{(-1)^{j-i}(\beta-\alpha)^{j-i}\,\phi^{(k+3+j)}(\beta)}{(k+1+i)!\,(j-i)!} \\
& \qquad \times \int_\alpha^\beta\left[\sum_{l=1}^{m} w_l\,(G(x_l,s)-G(y_l,s))\right](s-\alpha)^{k+1+i}\,ds\Bigg| \\
& \qquad\qquad \le \left\|\phi^{(n)}\right\|_p \|\mathfrak{R}\|_q\,,
\end{aligned}$$

$$\textit{where}\ \ \mathfrak{R}(t) = \int_\alpha^\beta \sum_{l=1}^{m} w_l\,(G(x_l,s)-G(y_l,s))\,G_{n-2}(s,t)ds, \quad t\in[\alpha,\beta].$$

The constant on the right-hand side of (3.14) is sharp for $1 < p \le \infty$ *and the best possible for* $p = 1$.

Proof. As we have

$$\mathfrak{R}(t) = \sum_{l=1}^{m} w_l \int_\alpha^\beta (G(x_l,s)-G(y_l,s))\,G_{n-2}(s,t)ds, \quad t\in[\alpha,\beta].$$

Using the identity (2.1) and applying Hölder's inequality we obtain

$$\begin{aligned}
& \Bigg|\sum_{l=1}^{m} w_l\,\phi(x_l) - \sum_{l=1}^{m} w_l\,\phi(y_l) - \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{l=1}^{m} w_l\,(x_l-y_l) \\
& - \sum_{i=0}^{k}\frac{\phi^{(i+2)}(\alpha)}{i!}\int_\alpha^\beta\left(\sum_{l=1}^{m} w_lG(x_l,s)-\sum_{l=1}^{m} w_lG(y_l,s)\right)(s-\alpha)^i ds \\
& - \sum_{j=0}^{n-k-4}\sum_{i=0}^{j}\frac{(-1)^{j-i}(\beta-\alpha)^{j-i}\,\phi^{(k+3+j)}(\beta)}{(k+1+i)!\,(j-i)!} \\
& \quad \times \int_\alpha^\beta\left[\sum_{l=1}^{m} w_lG(x_l,s)-\sum_{l=1}^{m} w_lG(y_l,s)\right](s-\alpha)^{k+1+i}\,ds\Bigg| \\
& = \left|\int_\alpha^\beta \mathfrak{R}(t)\phi^{(n)}(t)dt\right| \le \left\|\phi^{(n)}\right\|_p \|\mathfrak{R}\|_q\,.
\end{aligned}$$

For the proof of the sharpness of the constant $\|\mathfrak{R}\|_q$ let us find a function ϕ for which the equality in (3.14) is obtained.

For $1 < p < \infty$ take ϕ to be such that

$$\phi^{(n)}(t) = \operatorname{sgn} \mathfrak{R}(t) \, |\mathfrak{R}(t)|^{\frac{1}{p-1}} .$$

For $p = \infty$ take $\phi^{(n)}(t) = \operatorname{sgn} \mathfrak{R}(t)$.

For $p = 1$ we prove that

$$\left| \int_{\alpha}^{\beta} \mathfrak{R}(t)\phi^{(n)}(t)dt \right| \leq \max_{t\in[\alpha,\beta]} |\mathfrak{R}(t)| \left(\int_{\alpha}^{\beta} \left| \phi^{(n)}(t) \right| dt \right) \tag{3.15}$$

is the best possible inequality. As $\mathfrak{R}(t)$ is continuous on $[\alpha, \beta]$ so assume that $|\mathfrak{R}(t)|$ attains its maximum at $t_0 \in [\alpha, \beta]$. First we assume that $\mathfrak{R}(t_0) > 0$. For ε small enough we define $\phi_\varepsilon(t)$ by

$$\phi_\varepsilon(t) = \begin{cases} 0, & \alpha \leq t \leq t_0, \\ \frac{1}{\varepsilon\, n!}(t-t_0)^n, & t_0 \leq t \leq t_0+\varepsilon, \\ \frac{1}{n!}(t-t_0)^{n-1}, & t_0+\varepsilon \leq t \leq \beta. \end{cases}$$

Then for ε small enough

$$\left| \int_{\alpha}^{\beta} \mathfrak{R}(t)\phi^{(n)}(t)dt \right| = \left| \int_{t_0}^{t_0+\varepsilon} \mathfrak{R}(t)\frac{1}{\varepsilon}dt \right| = \frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} \mathfrak{R}(t)dt.$$

Now from the inequality (3.15) we have

$$\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} \mathfrak{R}(t)dt \leq \mathfrak{R}(t_0)\int_{t_0}^{t_0+\varepsilon} \frac{1}{\varepsilon}dt = \mathfrak{R}(t_0).$$

Since,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} \mathfrak{R}(t)dt = \mathfrak{R}(t_0)$$

the statement follows. In the case $\mathfrak{R}(t_0) < 0$, we define $\phi_\varepsilon(t)$ by

$$\phi_\varepsilon(t) = \begin{cases} \frac{1}{n!}(t-t_0-\varepsilon)^{n-1},, & \alpha \leq t \leq t_0, \\ -\frac{1}{\varepsilon n!}(t-t_0-\varepsilon)^n, & t_0 \leq t \leq t_0+\varepsilon, \\ 0, & t_0+\varepsilon \leq t \leq \beta, \end{cases}$$

and the rest of the proof is the same as above. ∎

Integral case can be given as:

Theorem 3.8. *Suppose that all assumptions of Theorem 2.2 hold. Assume (p, q) is a pair of conjugate exponents, that is $1 \leq p, q \leq \infty$, $1/p+1/q =$*

1. *Let* $\left|\phi^{(n)}\right|^p : [\alpha, \beta] \to \mathbb{R}$ *be an R-integrable function. Then we have:*

(3.16)
$$\begin{aligned}
&\left| \int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau \right.\\
&- \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int_a^b w(\tau)(x(\tau)-y(\tau))d\tau\\
&- \sum_{i=0}^{k}\frac{\phi^{(i+2)}(\alpha)}{i!}\int_\alpha^\beta \left(\int_a^b w(\tau)(G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-\alpha)^i ds\\
&- \sum_{j=0}^{n-k-4}\sum_{i=0}^{j}\frac{(-1)^{j-i}(\beta-\alpha)^{j-i}\phi^{(k+3+j)}(\beta)}{(k+1+i)!\,(j-i)!}\\
&\quad \left. \times \int_\alpha^\beta \left[\int_a^b w(\tau)(G(x(\tau),s)-G(y(\tau),s))d\tau\right](s-\alpha)^{k+1+i} ds \right|\\
&\le \left\|\phi^{(n)}\right\|_p \left\|\tilde{\mathfrak{R}}\right\|_q,
\end{aligned}$$

where $\tilde{\mathfrak{R}}(t) = \int_\alpha^\beta \left(\int_a^b w(\tau)(G(x(\tau),s)-G(y(\tau),s))d\tau\right) G_{n-2}(s,t)ds,$

$t \in [\alpha, \beta]$. *The constant on the right-hand side of (3.16) is sharp for* $1 < p \le \infty$ *and the best possible for* $p = 1$.

4. $n-$Exponetial convexity and exponential convexity

We begin this section by giving some definitions and notions which are used frequently in the results. For more details see e.g. [9], [16] and [24].

Definition 4.1. *A function* $\phi : I \to \mathbb{R}$ *is n-exponentially convex in the Jensen sense on* I *if*

$$\sum_{i,j=1}^{n} \xi_i \xi_j \, \phi\left(\frac{x_i + x_j}{2}\right) \ge 0,$$

hold for all choices $\xi_1, \dots, \xi_n \in \mathbb{R}$ *and all choices* $x_1, \dots, x_n \in I$. *A function* $\phi : I \to \mathbb{R}$ *is n-exponentially convex if it is n-exponentially convex in the Jensen sense and continuous on* I.

Definition 4.2. *A function* $\phi : I \to \mathbb{R}$ *is exponentially convex in the Jensen sense on* I *if it is n-exponentially convex in the Jensen sense for all* $n \in \mathbb{N}$.

A function $\phi : I \to \mathbb{R}$ *is exponentially convex if it is exponentially convex in the Jensen sense and continuous.*

PROPOSITION 4.3. *If $\phi : I \to \mathbb{R}$ is an n-exponentially convex in the Jensen sense, then the matrix $\left[\phi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{m}$ is a positive semi-definite matrix for all $m \in \mathbb{N}, m \leq n$. Particularly,*

$$\det\left[\phi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{m} \geq 0$$

for all $m \in \mathbb{N}$, $m = 1, 2, ..., n$.

REMARK 4.1. It is known that $\phi : I \to \mathbb{R}$ is a log-convex in the Jensen sense if and only if

$$\alpha^2\phi(x) + 2\alpha\beta\phi\left(\frac{x+y}{2}\right) + \beta^2\phi(y) \geq 0,$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a positive function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense. A positive function is log-convex if and only if it is 2-exponentially convex.

We use an idea from [16] to give an elegant method of producing an n-exponentially convex functions and exponentially convex functions applying the above functionals on a given family with the same property (see [24]):

Motivated by inequalities (2.6) and (2.8), under the assumptions of Theorems 2.3 and 2.4 we define the following linear functionals:

$$\begin{aligned}
F_1(\phi) &= \sum_{l=1}^{m} w_l\,\phi(x_l) - \sum_{l=1}^{m} w_l\,\phi(y_l) - \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{l=1}^{m} w_l\,(x_l - y_l) \\
&- \sum_{i=0}^{k} \frac{\phi^{(i+2)}(\alpha)}{i!}\int_{\alpha}^{\beta}\left[\sum_{l=1}^{m} w_l\,(G(x_l,s) - G(y_l,s))\right](s-\alpha)^i\,ds \\
&- \sum_{j=0}^{n-k-4}\sum_{i=0}^{j} \frac{(-1)^{j-i}\,(\beta-\alpha)^{j-i}\,\phi^{(k+3+j)}(\beta)}{(k+1+i)!\,(j-i)!} \\
&\times \int_{\alpha}^{\beta}\left[\sum_{l=1}^{m} w_l\,(G(x_l,s) - G(y_l,s))\right](s-\alpha)^{k+1+i}\,ds
\end{aligned} \tag{4.1}$$

$$
\begin{aligned}
(4.2)\quad \digamma_2(\phi) = &\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau \\
&- \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int_a^b w(\tau)(x(\tau)-y(\tau))d\tau \\
&- \sum_{i=0}^{k}\frac{\phi^{(i+2)}(\alpha)}{i!}\int_\alpha^\beta\left(\int_a^b w(\tau)(G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-\alpha)^i ds \\
&- \sum_{j=0}^{n-k-4}\sum_{i=0}^{j}\frac{(-1)^{j-i}(\beta-\alpha)^{j-i}\phi^{(k+3+j)}(\beta)}{(k+1+i)!\,(j-i)!} \\
&\times \int_\alpha^\beta\left(\int_a^b w(\tau)(G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-\alpha)^{k+1+i}ds.
\end{aligned}
$$

Remark 4.2. Under the assumptions of Theorems 2.3 and 2.2, it holds $\digamma_i(\phi) \geq 0$, $i = 1, 2$ for all $n-$convex functions ϕ.

Lagrange and Cauchy type mean value theorems related to defined functionals are given in the following theorems.

Theorem 4.4. *Let $\phi : [\alpha, \beta] \to \mathbb{R}$ be such that $\phi \in C^n([\alpha, \beta])$. If the inequality in (2.6)(i=1), (2.8)(i=2) hold, then there exist $\xi_i \in [\alpha, \beta]$ such that*

$$
(4.3)\qquad \digamma_i(\phi) = \phi^{(n)}(\xi_i)\digamma_i(\varphi), \quad i = 1, 2,
$$

where $\varphi(x) = \frac{x^n}{n!}$ and $\digamma_1$, $\digamma_2$ are defined by (4.1) and (4.2) respectively.

Proof. Similar to the proof of Theorem 4.1 in [17]. □

Theorem 4.5. *Let $\phi, \psi : [\alpha, \beta] \to \mathbb{R}$ be such that $\phi, \psi \in C^n([\alpha, \beta])$. If the inequality in (2.6)(i=1) and (2.8)(i=2) hold, then there exist $\xi_i \in [\alpha, \beta]$ such that*

$$
(4.4)\qquad \frac{\digamma_i(\phi)}{\digamma_i(\psi)} = \frac{\phi^{(n)}(\xi_i)}{\psi^{(n)}(\xi_i)}, \quad i = 1, 2.
$$

provided that the denominators are non-zero and $\digamma_1$, $\digamma_2$ are defined by (4.1) and (4.2) respectively.

Proof. Similar to the proof of Corollary 4.2 in [17]. □

Now we will produce $n-$exponentially and exponentially convex functions applying defined functionals. We use an idea from [24]. In the sequel I and J will be intervals in $\mathbb{R}$.

Theorem 4.6. *Let $\Omega = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$ such that the function $t \mapsto [x_0, \dots, x_k; \phi_t]$ is $n-$exponentially convex in the Jensen sense on J for every*

$(k+1)$ mutually different points $x_0, \ldots, x_k \in I$. Then for the linear functionals $\digamma_i(\phi_t)$ $(i = 1, 2)$ as defined by (4.1) and (4.2), the following statements hold:

(i) The function $t \mapsto \digamma_i(\phi_t)$ is n-exponentially convex in the Jensen sense on J and the matrix $[\digamma_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m$ is a positive semi-definite for all $m \in \mathbb{N}, m \le n$, $t_1, .., t_m \in J$. Particularly,

$$\det[\digamma_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m \ge 0 \text{ for all } m \in \mathbb{N},\ m = 1, 2, ..., n.$$

(ii) If the function $t \mapsto \digamma_i(\phi_t)$ is continuous on J, then it is n-exponentially convex on J.

Proof. (i) For $\xi_j \in \mathbb{R}$ and $t_j \in J$, $j = 1, \ldots, n$, we define the function

$$h(x) = \sum_{j,l=1}^{n} \xi_j \xi_l \phi_{\frac{t_j+t_l}{2}}(x).$$

Using the assumption that the function $t \mapsto [x_0, \ldots, x_k; \phi_t]$ is k-exponentially convex in the Jensen sense, we have

$$[x_0, \ldots, x_k, h] = \sum_{j,l=1}^{n} \xi_j \xi_l [x_0, \ldots, x_k; \phi_{\frac{t_j+t_l}{2}}] \ge 0,$$

which in turn implies that h is a k-convex function on J, so $\digamma_i(h) \ge 0$, $i = 1, 2$. Hence

$$\sum_{j,l=1}^{n} \xi_j \xi_l \digamma_i \left(\phi_{\frac{t_j+t_l}{2}}\right) \ge 0.$$

We conclude that the function $t \mapsto \digamma_i(\phi_t)$ is n-exponentially convex on J in the Jensen sense.

The remaining part follows from Proposition 4.3.

(ii) If the function $t \mapsto \digamma_i(\phi_t)$ is continuous on J, then it is n-exponentially convex on J by definition. □

The following corollaries is an immediate consequence of the above theorem

Corollary 4.7. *Let $\Omega = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $t \mapsto [x_0, \ldots, x_k; \phi_t]$ is exponentially convex in the Jensen sense on J for every $(k+1)$ mutually different points $x_0, \ldots, x_k \in I$. Then for the linear functionals $\digamma_i(\phi_t)$ $(i = 1, 2)$ as defined by (4.1) and (4.2), the following statements hold:*

(i) The function $t \mapsto \digamma_i(\phi_t)$ is exponentially convex in the Jensen sense on J and the matrix $[\digamma_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m$ is a positive semi-definite for all $m \in \mathbb{N}, m \le n$, $t_1, .., t_m \in J$. Particularly,

$$\det[\digamma_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m \ge 0 \text{ for all } m \in \mathbb{N},\ m = 1, 2, ..., n.$$

(ii) If the function $t \mapsto \digamma_i(\phi_t)$ is continuous on J, then it is exponentially convex on J.

COROLLARY 4.8. *Let $\Omega = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $t \mapsto [x_0, \dots, x_k; \phi_t]$ is 2-exponentially convex in the Jensen sense on J for every $(k+1)$ mutually different points $x_0, \dots, x_k \in I$. Let $\digamma_i$, $i = 1, 2$ be linear functionals defined by (4.1) and (4.2). Then the following statements hold:*

(i) If the function $t \mapsto \digamma_i(\phi_t)$ is continuous on J, then it is 2-exponentially convex function on J. If $t \mapsto \digamma_i(\phi_t)$ is additionally strictly positive, then it is also log-convex on J. Furthermore, the following inequality holds true:

$$[\digamma_i(\phi_s)]^{t-r} \leq [\digamma_i(\phi_r)]^{t-s} [\digamma_i(\phi_t)]^{s-r}, \quad i = 1, 2.$$

for every choice $r, s, t \in J$, such that $r < s < t$.

(ii) If the function $t \mapsto \digamma_i(\phi_t)$ is strictly positive and differentiable on J, then for every $p, q, u, v \in J$, such that $p \leq u$ and $q \leq v$, we have

$$\mu_{p,q}(\digamma_i, \Omega) \leq \mu_{u,v}(\digamma_i, \Omega), \tag{4.5}$$

where

$$\mu_{p,q}(\digamma_i, \Omega) = \begin{cases} \left(\frac{\digamma_i(\phi_p)}{\digamma_i(\phi_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{\frac{d}{dp}\digamma_i(\phi_p)}{\digamma_i(\phi_p)}\right), & p = q, \end{cases} \tag{4.6}$$

for $\phi_p, \phi_q \in \Omega$.

PROOF. (i) This is an immediate consequence of Theorem 4.6 and Remark 4.1.

(ii) Since $t \mapsto \digamma_i(\phi_t)$ is positive and continuous, by (i) we have that $t \mapsto \digamma_i(\phi_t)$ is log-convex on J, that is, the function $t \mapsto \log \digamma_i(\phi_t)$ is convex on J. Hence we get

$$\frac{\log \digamma_i(\phi_p) - \log \digamma_i(\phi_q)}{p - q} \leq \frac{\log \digamma_i(\phi_u) - \log \digamma_i(\phi_v)}{u - v}, \tag{4.7}$$

for $p \leq u, q \leq v, p \neq q, u \neq v$. So, we conclude that

$$\mu_{p,q}(\digamma_i, \Omega) \leq \mu_{u,v}(\digamma_i, \Omega).$$

Cases $p = q$ and $u = v$ follow from (4.7) as limit cases.

∎

REMARK 4.3. Note that the results from Theorem 4.6, Corollary 4.7 and Corollary 4.8 still hold when two of the points $x_0, ..., x_l \in [a, b]$ coincide, say $x_1 = x_0$, for a family of differentiable functions ϕ_s such that the function $s \mapsto$

$\phi_s\,[x_0,...,x_l]$ is an n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, they still hold when all $(l+1)$ points coincide for a family of l differentiable functions with the same property. The proofs are obtained by (1.15) and suitable characterization of convexity.

5. Examples

In this section, we present some families of functions which fulfil the conditions of Theorem 4.6, Corollary 4.7 and Corollary 4.8. This enables us to construct a large families of functions which are exponentially convex. Explicit form of this functions is obtained after we calculate explicit action of functionals on a given family.

Example 5.1. Let us consider a family of functions

$$\Omega_1 = \{\phi_t : \mathbb{R} \to \mathbb{R} : t \in \mathbb{R}\}$$

defined by

$$\phi_t(x) = \begin{cases} \frac{e^{tx}}{t^n}, & t \neq 0, \\ \frac{x^n}{n!}, & t = 0. \end{cases}$$

Since $\frac{d^n\phi_t}{dx^n}(x) = e^{tx} > 0$, the function ϕ_t is n-convex on $\mathbb{R}$ for every $t \in \mathbb{R}$ and $t \mapsto \frac{d^n\phi_t}{dx^n}(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 4.6 we also have that $t \mapsto [x_0, \ldots, x_n; \phi_t]$ is exponentially convex (and so exponentially convex in the Jensen sense). Now, using Corollary 4.7 we conclude that $t \mapsto \digamma_i(\phi_t), i = 1, 2$, are exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous (although the mapping $t \mapsto \phi_t$ is not continuous for $t = 0$), so it is exponentially convex. For this family of functions, $\mu_{p,q}(\digamma_i, \Omega_1)$, $i = 1, 2$, from (4.6), becomes

$$\mu_{p,q}(\digamma_i, \Omega_1) = \begin{cases} \left(\frac{\digamma_i(\phi_p)}{\digamma_i(\phi_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{\digamma_i(id\cdot\phi_p)}{\digamma_i(\phi_p)} - \frac{n}{p}\right), & p = q \neq 0, \\ \exp\left(\frac{1}{n+1}\frac{\digamma_i(id\cdot\phi_0)}{\digamma_i(\phi_0)}\right), & p = q = 0, \end{cases}$$

where id is the identity function. By Corollary 4.8 $\mu_{p,q}(\digamma_i, \Omega_1)$ is a monotonic function in parameters p and q.

Since

$$\left(\frac{\frac{d^n\phi_p}{dx^n}}{\frac{d^n\phi_q}{dx^n}}\right)^{\frac{1}{p-q}}(\log x) = x,$$

using Theorem 4.5 it follows that:

$$M_{p,q}(\digamma_i, \Omega_1) = \log \mu_{p,q}(\digamma_i, \Omega_1), \quad i = 1, 2$$

satisfies

$$\alpha \leq M_{p,q}(\digamma_i, \Omega_1) \leq \beta, \quad i = 1, 2.$$

So, $M_{p,q}(\digamma_i, \Omega_1)$ is a monotonic mean.

EXAMPLE 5.2. Let us consider a family of functions

$$\Omega_2 = \{g_t : (0, \infty) \to \mathbb{R} : t \in \mathbb{R}\}$$

defined by

$$g_t(x) = \begin{cases} \frac{x^t}{t(t-1)\cdots(t-n+1)}, & t \notin \{0, 1, \dots, n-1\}, \\ \frac{x^j \log x}{(-1)^{n-1-j} j!(n-1-j)!}, & t = j \in \{0, 1, \dots, n-1\}. \end{cases}$$

Since $\frac{d^n g_t}{dx^n}(x) = x^{t-n} > 0$, the function g_t is $n-$convex for $x > 0$ and $t \mapsto \frac{d^n g_t}{dx^n}(x)$ is exponentially convex by definition. Arguing as in Example 5.1 we get that the mappings $t \mapsto \digamma_i(g_t), i = 1, 2$ are exponentially convex. Hence, for this family of functions $\mu_{p,q}(\digamma_i, \Omega_2)$, $i = 1, 2$, from (4.6), is equal to

$$\mu_{p,q}(\digamma_i, \Omega_2) = \begin{cases} \left(\frac{\digamma_i(g_p)}{\digamma_i(g_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left((-1)^{n-1}(n-1)!\frac{\digamma_i(g_0 g_p)}{\digamma_i(g_p)} + \sum\limits_{k=0}^{n-1} \frac{1}{k-p}\right), & p = q \notin \{0, 1, \dots, n-1\}, \\ \exp\left((-1)^{n-1}(n-1)!\frac{\digamma_i(g_0 g_p)}{2\digamma_i(g_p)} + \sum\limits_{\substack{k=0 \\ k \neq p}}^{n-1} \frac{1}{k-p}\right), & p = q \in \{0, 1, \dots, n-1\}. \end{cases}$$

Again, using Theorem 4.5 we conclude that

$$\alpha \leq \left(\frac{\digamma_i(g_p)}{\digamma_i(g_q)}\right)^{\frac{1}{p-q}} \leq \beta, \quad i = 1, 2. \tag{5.1}$$

So, $\mu_{p,q}(\digamma_i, \Omega_2)$, $i = 1, 2$ is a mean and by (4.5) it is monotonic.

EXAMPLE 5.3. Let

$$\Omega_3 = \{\zeta_t : (0, \infty) \to (0, \infty) : \, t \in (0, \infty)\}$$

be a family of functions defined by

$$\zeta_t(x) \; = \begin{cases} \frac{t^{-x}}{(\log t)^n}, & \text{t} \neq 1; \\ \\ \frac{x^n}{n!}, & \text{t} = 1. \end{cases}$$

Since $\frac{d^n \zeta_t}{dx^n}(x) = t^{-x}$ is the Laplace transform of a non-negative function (see [25]) it is exponentially convex. Obviously ζ_t are n-convex functions for every $t > 0$.

For this family of functions, $\mu_{t,q}(\digamma_i,\Omega_3)$, $i=1,2$, in this case for $[\alpha,\beta]\subset\mathbb{R}^+$, from (4.6) becomes

$$\mu_{t,q}(\digamma_i,\Omega_3) = \begin{cases} \left(\frac{\digamma_i(\zeta_t)}{\digamma_i(\zeta_q)}\right)^{\frac{1}{t-q}}, & t\neq q; \\ \exp\left(-\frac{\digamma_i(id.\zeta_t)}{t\digamma_i(\zeta_t)}-\frac{n}{t\log t}\right), & t=q\neq 1; \\ \exp\left(-\frac{1}{n+1}\frac{\digamma_i(id.\zeta_1)}{\digamma_i(\zeta_1)}\right), & t=q=1. \end{cases}$$

This is monotonous function in parameters t and q by (4.5).

Using Theorem 4.5 it follows that

$$M_{t,q}(\digamma_i,\Omega_3) = -L(t,q)\log\mu_{t,q}(\digamma_i,\Omega_3), \quad i=1,2.$$

satisfy

$$\alpha\leq M_{t,q}(\digamma_i,\Omega_3)\leq\beta, \quad i=1,2.$$

This shows that $M_{t,q}(\digamma_i,\Omega_3)$ is mean for $i=1,2$. Because of the above inequality (4.5), this mean is also monotonic. $L(t,q)$ is logarithmic mean defined by

$$L(t,q) = \begin{cases} \frac{t-q}{\log t-\log q}, & \text{t} \neq \text{q}; \\ t, & \text{t} = \text{q}. \end{cases}$$

Example 5.4. Let

$$\Omega_4=\{\gamma_t:(0,\infty)\to(0,\infty):\, t\in(0,\infty)\}$$

be a family of functions defined by

$$\gamma_t(x)=\frac{e^{-x\sqrt{t}}}{t^n}.$$

Since $\frac{d^n\gamma_t}{dx^n}(x)=e^{-x\sqrt{t}}$ is the Laplace transform of a non-negative function (see [25]) it is exponentially convex. Obviously γ_t are n-convex function for every $t>0$.

For this family of functions, $\mu_{t,q}(\digamma_i,\Omega_4)$, $i=1,2$, in this case for $[\alpha,\beta]\subset\mathbb{R}^+$, from (4.6) becomes

$$\mu_{t,q}(\digamma_i,\Omega_4) = \begin{cases} \left(\frac{\digamma_i(\gamma_t)}{\digamma_i(\gamma_q)}\right)^{\frac{1}{t-q}}, & t\neq q; \\ \exp\left(-\frac{\digamma_i(id.\gamma_t)}{2\sqrt{t}\digamma_i(\gamma_t)}-\frac{n}{t}\right), & t=q. \end{cases}$$

This is monotonous function in parameters t and q by (4.5).

Using Theorem 4.5 it follows that

$$M_{t,q}(\digamma_i,\Omega_4) = -\left(\sqrt{t}+\sqrt{q}\right) ln\log\mu_{t,q}(\digamma_i,\Omega_4), \quad i=1,2.$$

satisfy

$$\alpha\leq M_{t,q}(\digamma_i,\Omega_4)\leq\beta, \quad i=1,2.$$

This shows that $M_{t,q}(\digamma_i, \Omega_4)$ is mean for $i = 1, 2$. Because of the above inequality (4.5), this mean is also monotonic.

REMARK 5.1. The results of this Section 5 are similar to related results from [2, Section 5], [3, Section 5] and [10, Section 6].

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Poopćenje teorema o majorizaciji preko Abel-Gontscharoffovih interpolacijskih polinoma

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Sažetak. U radu su dana poopćenja teorema o majorizaciji za klasu n-konveksnih funkcija korištenjem Abel-Gontscharoffovih interpolacijskih polinoma. Također su dobiveni i neki srodni rezultati.

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