ASYMPTOTIC BEHAVIOUR OF THE ITERATIVE PYTHAGOREAN MEANS

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Abstract. Asymptotic expansion and behaviour of the iterative combinations of the Pythagorean means (arithmetic, geometric and harmonic mean) is obtained and analyzed. Results are used for asymptotic comparison of means.

1. Introduction

The well-known Pythagorean means are three classical means: arithmetic, geometric and harmonic mean defined by

\begin{align}
  A(s, t) &= \frac{s + t}{2}, \quad G(s, t) = \sqrt{st}, \quad H(s, t) = \frac{2}{\frac{1}{s} + \frac{1}{t}},
\end{align}

where \( s \) and \( t \) are real positive numbers.

These means have following ordering

\begin{align}
  \min(s, t) \leq H(s, t) \leq G(s, t) \leq A(s, t) \leq \max(s, t),
\end{align}

and there are many results and papers concerning their properties, inequalities and comparison to other classical means.

In recent papers [5–7] authors studied asymptotic expansions of means and developed new technique of deriving and calculating coefficients in this expansions. They successfully used new method to establish various relations between classical means, see cited papers for details.

The asymptotic expansion of the mean \( M(s, t) \) is representation of this mean in the form

\begin{align}
  M(x + s, x + t) = x \sum_{n=0}^{\infty} c_n(s, t)x^{-n}, \quad x \to \infty,
\end{align}

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where \( c_n(s,t) \) are homogeneous polynomials of the degree \( n \) in variables \( s \) and \( t \). It is shown that simpler form of these coefficients is obtained under substitution

\begin{equation}
(1.4) \quad t = \alpha + \beta, \quad s = \alpha - \beta,
\end{equation}

so variables \( \alpha \) and \( \beta \) will be used in the rest of the paper.

Obviously, expansion of the arithmetic mean is

\begin{equation}
(1.5) \quad A(x + s, x + t) = x + \alpha
\end{equation}

and it has only these two terms. Asymptotic expansion of the geometric and harmonic mean is derived in [6] and it reads as

\begin{align}
(1.6) \quad G(x + s, x + t) & \sim x + \alpha - \frac{\beta^2}{4} x^{-1} + \frac{\alpha \beta^2}{2} x^{-2} - \frac{\beta^2}{8} (4\alpha^2 + \beta^2) x^{-3} + \ldots \\
(1.7) \quad H(x + s, x + t) & \sim x + \alpha - \frac{\beta^2}{4} x^{-1} + \alpha \beta^2 x^{-2} - \frac{\alpha^2 \beta^2}{2} x^{-3} + \ldots
\end{align}

Recall that arithmetic-geometric mean is an example of an interesting mean obtained by the iterative combination of arithmetic and geometric mean in the following way. Define \( a_0 = s, \; g_0 = t \) and

\begin{align}
(1.8) \quad a_{k+1} &= \frac{a_k + b_k}{2}, \quad g_{k+1} = \sqrt{a_k b_k}, \quad k \geq 0.
\end{align}

Then both of this sequences converge to the same limit \( AG(s, t) \) which is called arithmetic-geometric mean.

Same idea can be used to the other combinations of Pythagorean means as well. For an example, iteration of geometric and harmonic mean

\begin{align}
(1.9) \quad g_{k+1} &= \sqrt{g_k h_k}, \quad h_{k+1} = \frac{2}{g_k + h_k}, \quad k \geq 0.
\end{align}

leads to same limit geometric-harmonic mean and iteration of arithmetic and harmonic mean

\begin{align}
(1.10) \quad a_{k+1} &= \frac{a_k + h_k}{2}, \quad h_{k+1} = \frac{2}{a_k + h_k}, \quad k \geq 0.
\end{align}

defines arithmetic-harmonic mean. It is not hard to see that this mean is exactly equal to the geometric mean.

In a recent paper [4], authors studied arithmetic-geometric mean and derived its asymptotic expansion:

\begin{equation}
(1.11) \quad AG(x + t, x + s) \sim x + \alpha - \frac{\beta^2}{4} x^{-1} + \frac{\alpha \beta^2}{2} x^{-2} - \frac{\beta^2}{8} (16\alpha^2 + 5\beta^2) x^{-3} + \ldots
\end{equation}

By comparing coefficients in this expansion with expansion of \( A \) and \( G \) mean, one can see that \( AG \) clearly lies somewhere in the middle, but is a little bit closer to the geometric mean. This iterative process also showed interesting convergence and stationary properties of the coefficients in the asymptotic expansion of \( AG \) mean, for details see [4].
The main aim of this paper is to derive coefficients in asymptotic expansion of the arithmetic-harmonic and harmonic-geometric mean by analyzing their iterative processes in a similar way as it has been done for $AG$ mean. These coefficients will be used in the third section where we give relations and inequalities between this means.

We shall need the following fundamental lemma for the transformation of the asymptotic series. The proof is easy, see e.g. [6].

**Lemma 1.1.** Let function $f(x)$ have asymptotic expansion ($a_0 = 1$)

\[ f(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}, \quad x \to \infty. \]

Then for all real $p$ it holds

\[ [f(x)]^p \sim \sum_{n=0}^{\infty} c_n(p)x^{-n}, \]

where $c_0 = 1$ and

\[ c_n = \frac{1}{n} \sum_{k=1}^{n} [k(1 + p) - n] a_k c_{n-k}. \]  

(1.12)

We shall also need the following standard result:

**Lemma 1.2.** Let $f(x)$ and $g(x)$ have asymptotic expansions ($a_0 = b_0 = 1$):

\[ f(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}, \quad g(x) \sim \sum_{n=0}^{\infty} b_n x^{-n}. \]

Then their product $f(x)g(x)$ and quotient $f(x)/g(x)$ have asymptotic expansions

\[ f(x)g(x) \sim \sum_{n=0}^{\infty} c_n x^{-n}, \quad \frac{f(x)}{g(x)} \sim \sum_{n=0}^{\infty} d_n x^{-n}, \]

where $c_0 = d_0 = 1$ and

\[ c_n = \sum_{k=0}^{n} a_k b_{n-k}, \quad d_n = a_n - \sum_{k=1}^{n} b_k d_{n-k}. \]  

(1.13)
2. Asymptotic expansions of iterative Pythagorean means

To derive asymptotic expansion of the combination of two Pythagorean means, we shall follow its iterative process.

Let us start with arithmetic-harmonic mean. Let sequence \((a_n)\) be limiting sequence which converges above to the limit \(AH(s,t)\) and let \(A_n(s,t)\) be the value of \(n\)-th iteration. Similarly, let \(H_n(s,t)\) be defined through the sequence \((h_n)\). \(A_n\) and \(H_n\) are also means because they are obtained as the composition of arithmetic and harmonic mean to the previous members of these sequences.

Then, let

\[
A_n(s,t) = x \sum_{k=0}^{\infty} a_k^{(n)}(s,t)x^{-k},
\]

and

\[
H_n(s,t) = x \sum_{k=0}^{\infty} h_k^{(n)}(s,t)x^{-k},
\]

be asymptotic expansions of the \(n\)-th iteration.

We shall show that functions \(a_k^{(n)}\) and \(h_k^{(n)}\) converge to the same limit \(c_k(t,s)\) when \(n \to \infty\), and it holds

\[
AH(s,t) = x \sum_{k=0}^{\infty} c_k(s,t)x^{-k}.
\]

**Theorem 2.1.** Let \(n\) be arbitrary natural number. Then we have

\[
a_k^{(n)} = h_k^{(n)}
\]

for all \(k \leq 2n\).

In other words, for fixed \(k\), the sequence \(a_k^{(n)}\) is stationary sequence which defines the limiting value \(c_k\). Proof of the theorem follows from the next lemma by mathematical induction.

**Lemma 2.2.** Suppose that coefficients \(a_k^{(n)}\) and \(h_k^{(n)}\), for arbitrary \(n \geq 1\), satisfy

\[
a_k^{(n)} = h_k^{(n)}, \quad k = 0, 1, \ldots, K.
\]

Then it holds

\[
a_k^{(j)} = h_k^{(j)} = a_k^{(n)}, \quad j \geq n + 1, \quad k = 0, \ldots, K,
\]

and

\[
a_{K+1}^{(n+1)} = h_{K+1}^{(n+1)} = a_{K+2}^{(n+1)} = h_{K+2}^{(n+1)}.
\]

Therefore, in each following step, at least two new coefficients coincide and remain equal in the future.
Proof. For the arithmetic series the statement (2.5) is obvious. Let us examine \((h_n)\).

Denote
\[
A_{n,k} = \sum_{j=0}^{k} a_j^{(n)} x^{-j+1}, \quad AR_{n,k} = \sum_{j=k+1}^{\infty} a_j^{(n)} x^{-j+1},
\]
and similarly for \(H_{n,k}, HR_{n,k}\).

Then
\[
H_{n+1} = \frac{2}{A_{n+1} + H_n} = \frac{2A_n H_n}{A_n + H_n} = \frac{2(A_{n,k} + AR_{n,k})(H_{n,k} + HR_{n,k})}{A_{n,k} + AR_{n,k} + H_{n,k} + HR_{n,k}}.
\]

Since \(A_{n,k} = H_{n,k}\), this can be written as
\[
H_{n+1} = A_{n,k} \left(1 + \frac{AR_{n,k}}{A_{n,k}} \right) \left(1 + \frac{HR_{n,k}}{A_{n,k}} \right)
\]
and clearly (2.5) follows.

Now, to prove (2.6), let us find the first two coefficients in the expansion of the quotient \(AR_{n,k}/A_{n,k}\). Using Lemma 1.2 (recall that \(a_0 = g_0 = 1\), we have
\[
AR_{n,k} = \frac{a_k^{(n)}}{x^{k+1}} + \frac{a_{k+2}^{(n)} - a_1^{(n)} a_{k+1}^{(n)}}{x^{k+2}} + \ldots
\]

In a same way,
\[
HR_{n,k} = \frac{h_k^{(n)}}{x^{k+1}} + \frac{h_{k+2}^{(n)} - a_1^{(n)} h_{k+1}^{(n)}}{x^{k+2}} + \ldots
\]

Now, from (2.7) it follows
\[
H_{n+1} = x \left(1 + \frac{a_k^{(n)}}{x} + \ldots \right) \left(1 + \frac{a_{k+1}^{(n)}}{x^{k+1}} + \frac{a_{k+2}^{(n)} - a_1^{(n)} a_{k+1}^{(n)}}{x^{k+2}} + \ldots \right)
\]
\[
\cdot \left(1 + \frac{h_{k+1}^{(n)}}{x^{k+1}} + \frac{h_{k+2}^{(n)} - a_1^{(n)} h_{k+1}^{(n)}}{x^{k+2}} + \ldots \right) \left(1 + \frac{u_{k+1}^{(n)}}{x^{k+1}} + \frac{u_{k+2}^{(n)} - a_1^{(n)} u_{k+1}^{(n)}}{x^{k+2}} + \ldots \right)^{-1},
\]
where
\[
\frac{a_k^{(n)}}{x} = \frac{a_{k+1}^{(n)} + h_{k+1}^{(n)}}{2},
\]
and therefore
\[
h_{k+1}^{(n+1)} = a_k^{(n)} + h_{k+1}^{(n)} - u_{k+1}^{(n)} = a_{k+1}^{(n+1)},
\]
\[
h_{k+2}^{(n+1)} = a_k^{(n)} a_{k+1}^{(n)} + a_1^{(n)} h_{k+1}^{(n)} - a_1^{(n)} u_{k+1}^{(n)} + \left( a_{k+2}^{(n)} - a_1^{(n)} a_{k+1}^{(n)} \right) + \]
\[
+ \left( h_{k+2}^{(n)} - a_1^{(n)} h_{k+1}^{(n)} \right) - \left( u_{k+2}^{(n)} - a_1^{(n)} u_{k+1}^{(n)} \right) = a_{k+2}^{(n+1)}.
\]
Now we can calculate coefficients $c_k$ of the asymptotic expansion (2.3). In each iteration, coefficients $a_k^{(n)}$ are easily obtained and for calculating sequence

$$h^{(n)} = \frac{2a^{(n-1)}h^{(n-1)}}{a^{(n-1)} + h^{(n-1)}},$$

we use Lemma 1.2 for multiplication and quotient of asymptotic series.

Here are the first few coefficients in terms of variables $\alpha$ and $\beta$:

$$
\begin{align*}
  c_0 &= 1, \\
  c_1 &= \alpha, \\
  c_2 &= \frac{-\beta^2}{2}, \\
  c_3 &= \frac{\alpha\beta^2}{2}, \\
  c_4 &= \frac{-\beta^2}{8}(4\alpha^2 + \beta^2), \\
  c_5 &= \frac{\alpha\beta^2}{8}(4\alpha^2 + 3\beta^2), \\
  c_6 &= \frac{-\beta^2}{16}(8\alpha^4 + 12\alpha^2\beta^2 + \beta^4), \\
  c_7 &= \frac{\alpha\beta^2}{16}(8\alpha^4 + 20\alpha^2\beta^2 + 5\beta^4), \\
  c_8 &= \frac{-\beta^2}{128}(64\alpha^6 + 240\alpha^4\beta^2 + 120\alpha^2\beta^4 + 5\beta^6).
\end{align*}
$$

As expected, since arithmetic-harmonic mean is equal to the geometric mean, we have obtained the same coefficients as in expansion (1.6), but this way we have shown that iterative process for arithmetic-harmonic means has same stationary and convergence properties as arithmetic-geometric mean analyzed in paper [4].

We will now show that this properties are also valid for the geometric-harmonic mean.

As before, let sequence $(g_n)$ be limiting sequence which converges above to the limit $GH(s, t)$ and let $G_n(s, t)$ be the value of $n$-th iteration. $H_n(s, t)$ is defined as before.

Then,

$$G_n(s, t) = x \sum_{k=0}^{\infty} g_k^{(n)}(s, t)x^{-k},$$

(2.8)
and

\begin{equation}
H_n(s,t) = x \sum_{k=0}^{\infty} h_k^{(n)}(s,t)x^{-k},
\end{equation}

are asymptotic expansions of the \(n\)-th iteration.

Functions \(g_k^{(n)}\) and \(h_k^{(n)}\) converge to the same limit \(d_k(t,s)\) when \(n \to \infty\), and it holds

\begin{equation}
GH(s,t) = x \sum_{k=0}^{\infty} d_k(s,t)x^{-k}.
\end{equation}

**Theorem 2.3.** Let \(n\) be arbitrary natural number. Then we have

\begin{equation}
g_k^{(n)} = h_k^{(n)}
\end{equation}

for all \(k \leq 2n\).

Again, for fixed \(k\), the sequence \(g_k^{(n)}\) is stationary sequence which defines the limiting value \(d_k\). Proof follows from the next lemma.

**Lemma 2.4.** Suppose that coefficients \(g_k^{(n)}\) and \(h_k^{(n)}\), for arbitrary \(n \geq 1\) satisfy

\begin{equation}
g_k^{(n)} = h_k^{(n)}, \quad k = 0, 1, \ldots, K.
\end{equation}

Then it holds

\begin{equation}
g_k^{(j)} = h_k^{(j)} = g_k^{(n)}, \quad j \geq n + 1, \quad k = 0, \ldots, K,
\end{equation}

and

\begin{equation}
g_{K+1}^{(n+1)} = h_{K+1}^{(n+1)}, \quad g_{K+2}^{(n+1)} = h_{K+2}^{(n+1)}.
\end{equation}

**Proof.** Let us first prove (2.5).

Denote

\begin{equation}
G_{n,k} = \sum_{j=0}^{k} g_j^{(n)}x^{-j+1}, \quad GR_{n,k} = \sum_{j=k+1}^{\infty} g_j^{(n)}x^{-j+1},
\end{equation}

and same for \(H_{n,k}\), \(HR_{n,k}\).

Then

\begin{equation}
H_{n+1} = \frac{2}{G_{n+1} + H_{n+1}} = \frac{2G_nH_{n+1}}{G_{n} + H_{n}} = \frac{2(G_{n,k} + GR_{n,k})(H_{n,k} + HR_{n,k})}{G_{n,k} + GR_{n,k} + H_{n,k} + HR_{n,k}}.
\end{equation}

Since \(G_{n,k} = H_{n,k}\), this can be written as

\begin{equation}
H_{n+1} = G_{n,k} \left(1 + \frac{GR_{n,k}}{G_{n,k}}\right) \left(1 + \frac{HR_{n,k}}{G_{n,k}}\right)
\end{equation}

(2.14)
and clearly (2.12) is valid for the harmonic series.

Similarly,

\[ G_{n+1} = \sqrt{G_n H_n} = \sqrt{(G_{n,k} + GR_{n,k})(H_{n,k} + HR_{n,k})}, \]

which can be written as

\[ G_{n+1} = G_{n,k} \sqrt{\left(1 + \frac{GR_{n,k}}{G_{n,k}}\right) \left(1 + \frac{HR_{n,k}}{G_{n,k}}\right)} \]

and (2.12) follows.

To prove (2.13), we again start with the first two coefficients in the expansion of the quotients

\[ GR_{n,k} = \frac{g_{k+1}^{(n)}}{x^{k+1}} + \frac{g_{k+2}^{(n)} - g_1^{(n)} g_{k+1}^{(n)}}{x^{k+2}} + \ldots, \]

and

\[ HR_{n,k} = \frac{h_{k+1}^{(n)}}{x^{k+1}} + \frac{h_{k+2}^{(n)} - g_1^{(n)} h_{k+1}^{(n)}}{x^{k+2}} + \ldots. \]

Now, applying binomial formula to (2.15) it follows

\[ G_{n+1} = x \left(1 + \frac{g_1^{(n)}}{x} + \ldots\right) \left(1 + \left(\frac{1}{2}\right) \frac{g_{k+1}^{(n)}}{x^{k+1}} + \left(\frac{1}{2}\right) \frac{g_{k+2}^{(n)} - g_1^{(n)} g_{k+1}^{(n)}}{x^{k+2}} + \ldots\right). \]

and from (2.14) we have

\[ H_{n+1} = x \left(1 + \frac{g_1^{(n)}}{x} + \ldots\right) \left(1 + \frac{h_{k+1}^{(n)}}{x^{k+1}} + \frac{h_{k+2}^{(n)} - g_1^{(n)} h_{k+1}^{(n)}}{x^{k+2}} + \ldots\right) \left(1 + \frac{v_{k+1}^{(n)}}{x^{k+1}} + \frac{v_{k+2}^{(n)} - g_1^{(n)} v_{k+1}^{(n)}}{x^{k+2}} + \ldots\right)^{-1}, \]

where

\[ v_{k+1}^{(n)} = \frac{g_{k+1}^{(n)} + h_{k+1}^{(n)}}{2}. \]

Therefore,

\[ h_{k+1}^{(n+1)} = g_{k+1}^{(n)} + h_{k+1}^{(n)} - v_{k+1}^{(n)} = \frac{g_{k+1}^{(n)} + h_{k+1}^{(n)}}{2} = g_{k+1}^{(n)}; \]

\[ h_{k+2}^{(n+1)} = g_1^{(n)} g_{k+1}^{(n)} + g_1^{(n)} h_{k+1}^{(n)} - g_1^{(n)} v_{k+1}^{(n)} + \left(g_{(n)k+2}^{(n)} - g_1^{(n)} g_{k+1}^{(n)}\right) + \left(h_{k+2}^{(n)} - g_1^{(n)} h_{k+1}^{(n)}\right) - \left(v_{k+2}^{(n)} - g_1^{(n)} v_{k+1}^{(n)}\right). \]
\[ g_{n+1} \frac{g(n)_{k+1}}{2} + \frac{g(n)_{k+1}}{2} + \frac{h(n)_{k+2} - g(n)_{k+1}}{2} + \frac{h(n)_{k+2} - g(n)_{k+1}}{2} = g_{k+2}^{(n+1)}, \]

and the proof is complete.

Finally, we shall derive coefficients \( d_k \) of the asymptotic expansion (2.10). In each iteration, for calculating \( h(n) \) we use Lemma 1.2 for multiplication and quotient of asymptotic series, and for \( g(n) \) we use Lemma 1.1 with \( p = \frac{1}{2} \).

So, the first few coefficients in the expansion of the geometric-harmonic mean, in terms of variables \( \alpha \) and \( \beta \), are:

\[ d_0 = 1, \]
\[ d_1 = \alpha, \]
\[ d_2 = -\frac{3\beta^2}{4}, \]
\[ d_3 = \frac{3\alpha\beta^2}{4}, \]
\[ d_4 = -\frac{\beta^2}{64}(48\alpha^2 + 7\beta^2), \]
\[ d_5 = \frac{3\alpha\beta^2}{64}(16\alpha^2 + 7\beta^2), \]
\[ d_6 = -\frac{\beta^2}{256}(192\alpha^4 + 168\alpha^2\beta^2 + 11\beta^4), \]
\[ d_7 = \frac{\alpha\beta^2}{256}(192\alpha^4 + 280\alpha^2\beta^2 + 55\beta^4). \]
\[ d_8 = -\frac{3\beta^2}{16384}(4096\alpha^6 + 8960\alpha^4\beta^2 + 3520\alpha^2\beta^4 + 125\beta^6). \]

**Remark 2.5.** Asymptotic expansion of the geometric-harmonic mean can be also obtained through its relation to the arithmetic-geometric mean. It holds

\[ GH(s, t) = \frac{1}{AG(s, t^t)} \]

Therefore, we can apply Lemma 1.1 \((p = -1)\) with asymptotic expansion of \( AG \) mean derived in paper [4], but it was also interesting to examine stationary properties of \( GH \) mean through its iterative process.

### 3. Asymptotic comparison of Pythagorean means

In [5, 7], authors developed techniques for comparison of means through their asymptotic expansions.
Definition 3.1. Let $M_1$ and $M_2$ be any two means and

$$M_1(x + s, x + t) - M_2(x + s, x + t) = c_k(s, t)x^{-k+1} + O(x^{-k}).$$

If $c_k(s, t) > 0$ for all $s$ and $t$ then we say that mean $M_1$ is asymptotically greater than mean $M_2$ and write

$$M_1 \gg M_2.$$

Of course, this is equivalent to

$$M_1 \ll M_2.$$

Theorem 3.2. If $M_1 \geq M_2$, then $M_1 \gg M_2$.

In other words, asymptotic inequalities can be considered as a necessary relation between comparable means, see cited papers.

We will now present asymptotic relation between Pythagorean means and their iterative combinations. It is shown in [5] that for the comparison of means, it is sufficient to consider the case $\alpha = 0$. In this case $c_{2n+1} = 0$, so in the next table we will show only even coefficients in the asymptotic expansions obtained in the previous section.

Table 1. Expansions of the iterative Pythagorean means

<table>
<thead>
<tr>
<th>$M$</th>
<th>$x$</th>
<th>$t^2/x$</th>
<th>$t^4/x^3$</th>
<th>$t^6/x^5$</th>
<th>$t^8/x^7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$AG$</td>
<td>1</td>
<td>$-\frac{1}{4}$</td>
<td>$-\frac{5}{64}$</td>
<td>$-\frac{11}{256}$</td>
<td>$\frac{469}{16384}$</td>
</tr>
<tr>
<td>$G = AH$</td>
<td>1</td>
<td>$-\frac{1}{2}$</td>
<td>$-\frac{1}{8}$</td>
<td>$\frac{1}{16}$</td>
<td>$-\frac{5}{128}$</td>
</tr>
<tr>
<td>$GH$</td>
<td>1</td>
<td>$-\frac{3}{4}$</td>
<td>$-\frac{7}{64}$</td>
<td>$-\frac{11}{256}$</td>
<td>$\frac{375}{16384}$</td>
</tr>
<tr>
<td>$H$</td>
<td>1</td>
<td>$-1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

As we can see, coefficients coincide with the known inequality between Pythagorean means:

(3.1) \[ H \leq GH \leq G = AH \leq AG \leq A. \]

Iterative means $AG$ and $GH$ obviously lie in the middle of their starting means, but they are both closer to the geometric mean. According to coefficients next to $x^{-3}$, it is also interesting to see that $GH$ is a little bit closer
to the geometric mean than \( AG \). Using this method, one can easily compare \( GH \) with other classical means as well, see cited papers for details about this concept.

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References


Asimptotsko ponašanje iterativnih pitagorejskih sredina

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Sažetak. Dobiveni su i analizirani asimptotski razvoji i ponašanje iterativnih kombinacija pitagorejskih sredina (aritmetičke, geometrijske i harmonijske sredine). Rezultati se koriste za asimptotsku usporedbu sredina.

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