ON A LEMMA OF THOMPSON

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Abstract. In Theorem 3 we improve [8, Lemma 5.41] (= Lemma 1, below) omitting one of its conditions. In Lemma 1 the structure of $T$, a Sylow 2-subgroup of $G$, is described only. In contrast to that lemma, we describe in detail the structure of the whole group $G$ and embedding of $T$ in $G$. In Theorem 4 we consider a similar, but more general, situation for groups of odd order.

In the first part [8] of his seminal N-paper Thompson considered, in particular, a number of special situations arising in the subsequent parts of that paper. He proved there the following

Lemma 1 ([8, Lemma 5.41]). Suppose that the following holds:
(a) $G$ is a finite nonnilpotent solvable group.
(b) $O_2(G) = \{1\}$.
(c) $G$ has a proper noncyclic abelian subgroup of order 8.
(d) If $K$ is any proper subgroup of $G$ of index a power of 2, then $K$ has no noncyclic abelian subgroup of order 8.

Let $T$ be a Sylow 2-subgroup of $G$. Then $T$ is normal in $G$ and one of the following holds:
(i) $T$ is abelian.
(ii) $T$ is an extraspecial group.
(iii) $T$ has a subgroup $T_0 \cong Q_8$ of index 2 and $T = T_0 Z(T)$.
(iv) $T$ is special and $Z(T) \cong E_4$.

In Theorem 3 we eliminate condition (c) from the hypothesis of Lemma 1 and, as a result, we obtain three additional non 2-closed groups; we also describe the structure of $G$ in some detail. Note also that our proof differs

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essentially from the original proof of Lemma 1. Theorem 4 is a stronger version of Theorem 3 for groups of odd order. In the proof of Theorem 4 we use \[3, \text{Theorem 4.1(i)}\] (= Lemma 2(e), below), a fairly deep result of finite p-group theory.

In what follows G is a finite group, p is a prime, \( \pi \) is a set of primes and \( \pi' \) is the set of primes not contained in \( \pi \), \( m,n \in \mathbb{N} \) and \( \pi(m) \) is the set of all prime divisors of \( m \). Next, \( C_m \) is the cyclic group of order \( m \); \( E_p^m \) is the elementary abelian group of order \( p^m \); \( SD_{2^n} \) (\( n > 3 \)), \( Q_{2^n} \) and \( D_{2^n} \) are the semidihedral, generalized quaternion group and dihedral groups of order \( 2^n \), respectively (these groups exhaust the groups of maximal class and order \( 2^n \)); \( \text{A}_4(\text{S}_4) \) are the alternating (symmetric) group of degree 4; \( \text{C}_G(M) \) (\( \text{N}_G(M) \)) is the centralizer (normalizer) of the subset \( M \in G \); \( Z(G) \), \( G' \) and \( \Phi(G) \) is the center, the derived subgroup and the Frattini subgroup of \( G \), respectively; \( \Omega_\pi(G) \) is the product of all normal \( \pi \)-subgroups of \( G \). If \( G \) is a p-group, then \( \Omega_1(G) = \{ x^p = 1 \mid x \in G \} \) and \( \Omega_1(G) = \{ x^p \mid x \in G \} \). By \( A \ast B \) we denote a central product of \( A \) and \( B \).

A p-group \( G \) is said to be special if \( G' = Z(G) = \Phi(G) > \{ 1 \} \) (in that case, \( \exp(G') \leq \exp(G/G') = p \) and \( G' \) is elementary abelian). A p-group \( G \) is said to be extraspecial if it is special with \( |G'| = p \).

Let \( G \) be a 2-group of maximal class. Then, if \( G \nmid Q_8 \), it contains a characteristic cyclic subgroup of index 2. In Lemma 2 we gathered some known facts used in what follows.

**Lemma 2.** (a) \([1, \text{Proposition 19(a)}]\) Let \( B \) be a nonabelian subgroup of order \( p^3 \) in a p-group \( G \). If \( G \) is not of maximal class, then \( C_G(B) \nsubseteq B \).

(b) Let \( G \) be a p-group generated by two elements. Then \( \pi(|\text{Aut}(G)|) \subseteq \pi(p(p - 1)(p + 1)) \). In particular, \( p \) is the maximal prime divisor of \( |\text{Aut}(G)| \), unless \( p = 2 \). If, in addition, \( G \) has a characteristic subgroup of index \( p \), then \( \pi(|\text{Aut}(G)|) \subseteq \pi(p(p - 1)) \). In particular, if \( G \) is a 2-group of maximal class and \( \text{Aut}(G) \) is not a 2-group, then \( G \cong Q_8 \).

(b1) \( \text{Aut}(Q_8) \cong S_4 \).

(c) Let \( \alpha \) be a \( p' \)-automorphism of a p-group \( G \) acting trivially on \( \Omega_1(G) \). If \( p > 2 \) or \( G \) is abelian, then \( \alpha = \text{id}_G \).

(d) If a p-group \( G \) has no noncyclic abelian subgroup of order \( p^3 \), then one and only one of the following holds: (i) \( G \) is cyclic, (ii) \( G \cong E_{p^2} \), (iii) \( G \) is a 2-group of maximal class, (iv) \( G \) is nonabelian of order \( p^3 \), \( p > 2 \).

(e) \([3, \text{Theorem 4.1(i)}]\) Let \( G \) be a p-group, \( p > 2 \). Suppose that \( G \) has no elementary abelian subgroup of order \( p^3 \). Then one of the following holds: (i) \( G \) is metacyclic, (ii) \( G \) is an irregular 3-group of maximal class, (iii) \( G = EC \), where \( E = \Omega_1(G) \) is nonabelian of order \( p^3 \) and exponent \( p \) and \( C \) is cyclic.
(f) Let $A$ be a $\pi'$-group acting on a $\pi$-group $G$. Let $C: G = G_0 > G_1 > \cdots > G_n = \{1\}$ be a chain of $A$-invariant normal subgroups of $G$. If $A$ centralizes all factors $G_i/G_{i+1}$ of the chain $C$ (i.e., $A$ stabilizes $C$), then $A$ centralizes $G$.

(g) (Transfer Theorem) Suppose that a Sylow $p$-subgroup of a group $G$ is abelian. If $p$ divides $|Z(G)|$, then $G$ has a normal subgroup of index $p$.

According to Hall-Burnside, if $\alpha$ is a $p'$-automorphism of a $p$-group $G$ inducing identity on $G/\Phi(G)$, then $\alpha = \text{id}_G$. Indeed, assuming, without loss of generality, that $o(\alpha) = q$, a prime, we see that $\alpha$ fixes an element of every coset $x\Phi(G)$. Since these fixed elements generate $G$, our claim follows.

If $d$ is a minimal number of generators of a $p$-group $G$, then (Hall) $|\text{Aut}(G)|$ divides the number $(p^d-1)(p^d-p) \cdots (p^d-p^{d-1})|\Phi(G)|^d$ (indeed, that number is the cardinality of the set $\mathcal{B}$ of minimal bases of $G$, and $G$ has no fixed points on the set $\mathcal{B}$), and this justifies the main assertion of Lemma 2(b). If a two-generator $p$-group $G$ has a characteristic subgroup $H$ of index $p$ and $\alpha \in \text{Aut}(G)$ has prime order $q \notin \pi(p(p-1))$, then $\alpha$ stabilizes the chain $\{1\} < H/\Phi(G) < G/\Phi(G)$ so $\alpha = \text{id}_G$, by the previous paragraph and (f), a contradiction. In (c), the partial holomorph $(\alpha) \cdot G$ has no minimal nonnilpotent subgroup so it is nilpotent, by Frobenius’ Normal $p$-Complement Theorem [5, Theorem 9.18] (here we use the structure of minimal nonnilpotent groups; see [4, Satz 3.5.2]). Lemma 2(d) follows from Roquette’s Lemma [4, Satz 3.7.6], in which the $p$-groups without normal abelian subgroups of type $(p,p)$ are classified. Lemma 2(g) follows from Wielandt’s Theorem [4, Satz 4.8.1] and Fitting’s Lemma [2, Corollary 1.18]. As to Lemma 2(f), assume that $A$ does not centralize $G$ and $|AG|$ is as small as possible. Then $AG$ is minimal nonnilpotent. Since all nilpotent images of $AG$ must be $\pi'$-groups, we get a contradiction with hypothesis.

Recall that there are two representation groups of the symmetric group $S_4$, their orders are equal to 48, Sylow 2-subgroups of these groups are generalized quaternion and semidihedral, respectively; see [7, Theorem 3.2.21].

Now we are ready to prove our main results.

**Theorem 3.** Suppose that the following holds:

(a) $G$ is a nonnilpotent solvable group with a Sylow 2-subgroup $T$ and $2'$-Hall subgroup $H$.

(b) $O_2(G) = \{1\}$.

(c) Whenever $K$ is a proper subgroup of $G$ such that $|G:K|$ is a power of 2, then $K$ has no noncyclic abelian subgroup of order 8 (or, what is the same, every maximal subgroup of $G$ containing $H$, has no noncyclic abelian subgroup of order 8).

Then one and only one of the following assertions is true:

A. If $T$ is not normal in $G$, then either $G \cong S_4$ or $G$ is one of two representation groups of $S_4$. 
B If $T$ is normal in $G$, then one of the following holds:

(B1) If $T$ is abelian, then $T \in \{E_{2m}, C_4 \times C_4\}$.

(B1.1) If $T \cong C_4 \times C_4$, then $G$ is a Frobenius group with $|G:T|=3$.

(B1.2) Let $T \cong E_{2m}$ be not a minimal normal subgroup of $G$. Then either $G \in \{A_4 \times C_2, A_4 \times A_4\}$ or $m=4$ and $G$ is a Frobenius group with $|G:T|=3$.

(B2) $T$ is extraspecial of order $2^{2m+1}$, $m \geq 1$. If $m=1$, then $G \cong SL(2,3)$. Next assume that $m > 1$.

(B2.1) If $m > 2$, then $T/Z(T)$ is a minimal normal subgroup of $G/Z(T)$.

(B2.2) If $T/Z(T)$ is not a minimal normal subgroup of $G/Z(T)$, then $T = U \ast V$, where $U \cong V \cong Q_8$, $U \lhd G$; in that case, $G/T \cong H$ is isomorphic to a subgroup of $E_{2^m}$. Moreover, if $H \cong E_2$, then $G = A \times B$, where $A \cong B \cong SL(2,3)$, and $A \cap B = Z(A) = Z(B)$. If $|H|=3$, then $UH \cong SL(2,3) \cong VH$.

(B3) $T$ has a $G$-invariant subgroup $T_0 \cong Q_8$ of index 2 and $T = T_0 Z(T)$. In that case, $G/T' \cong A_4 \times C_2$, $G' = T_0$ and, if $D/T_0 < G/T_0$ is of order 3, then $D \cong SL(2,3)$.

(B4) $T$ is special with $Z(T) = Z(G) \cong E_4$ and $T/Z(T)$ is a minimal normal subgroup of $G/Z(T)$.

**Proof.** The solvable group $G$ contains a $2'$-Hall subgroup $H$. Since $O_{2'}(G) = \{1\}$, $T \in \text{Syl}_2(G)$ is noncyclic (Lemma 2(b)), $C_G(O_{2'}(G)) \leq O_2(G)$ (Hall-Higman) so, if $T$ is abelian, it is normal in $G$.

Suppose that $T$ is abelian and $\exp(T) > 2$. Then $\Omega_1(T)$ is normal in $G$ since $T \lhd G$. Next, $|G : \Omega_1(T)H| > 1$ is a power of 2 so $\Omega_1(T) \cong E_4$ since $T$ is noncyclic. The number $|G : H\Omega_2(T)|$ is a power of 2 and $H\Omega_2(T)$ contains a noncyclic abelian subgroup of order 8, so we get $\Omega_2(T)H = G$ and $\exp(T) = 4$. Since $G$ has no normal 2-complement, $T$ is abelian of type (4, 4) (Lemma 2(b)). Then $\Omega_1(T)H$ is a Frobenius group (otherwise, by Lemma 2(c), $\{1\} < H \lhd G$) so $|H|=3$; in that case, $G$ is also a Frobenius group.

Now suppose that $T \cong E_{2m}$; then $m > 1$. If $m = 2$, then $G \cong A_4$. Now we let $m > 2$ and suppose that $T$ is not a minimal normal subgroup of $G$. Then $T = R \times R_1$, where $R, R_1 > \{1\}$, are normal in $G$ (Maschke) and, since $|G : RH| > 1$ and $|G : R_1 H| > 1$ are powers of 2, we conclude that $|R| \leq 4$, $|R_1| \leq 4$ so $m \in \{3, 4\}$. If $m=3$, then $G \cong A_4 \times C_2$. If $m=4$, then $G$ is either a Frobenius group with kernel $T \cong E_{24}$, of index 3 in $G$ or $G \cong A_4 \times A_4$. Indeed, assume that $G$ is not a Frobenius group; then $|H| > 3$. Setting $Z = C_H(R)$, $Z_1 = C_H(R_1)$, we have $|H : Z| = 3 = |H : Z_1|$ and $Z \cap Z_1 \leq O_{2'}(G) = \{1\}$ so $H = Z_1 \times Z_2$, $RZ_1 \cong A_4 \cong R_1 Z$ and $G = (RZ_1) \times (R_1 Z)$.

In what follows we assume that $T$ is nonabelian.

A. Suppose that $T$ is normal in $G$. 


(i) Suppose that $T$ has no noncyclic abelian subgroup of order 8. Then, by Lemma 2(d), $T$ is of maximal class, and, by Lemma 2(b), $T \cong Q_8$, which is extraspecial (in that case, $G \cong SL(2,3)$).

In what follows we assume that $T$ has a noncyclic abelian subgroup of order 8 so $T$ is not of maximal class; then $|T| > 8$.

(ii) Suppose that $K < G$ and $|G : K| = 2$; then $K$ has no noncyclic abelian subgroup of order 8, by hypothesis. We get $O_{2'}(K) \leq O_{2'}(G) = \{1\}$ so $T \cap K$ is noncyclic and is not of maximal class and order $> 8$, by Lemma 2(b). It follows from Lemma 2(d), that $T \cap K \cong Q_8$ and, since $T$ is not of maximal class, $T = (T \cap K)C_T(K \cap T) = (T \cap K)Z(T)$ since $|T| = 16$ (Lemma 2(a)). The subgroup $T \cap K < G$. Then, in view of Lemma 2(b1) and (a) (see the theorem), we conclude that

$$|H| = |G : T| = 3, \ (T \cap K)H \cong SL(2,3), \ G' = T \cap K, \ G/G' \cong C_6$$

and so $G$ is as in part (B3).

Next we assume that $G$ has no subgroup of index 2; then $T \leq G'$.

(iii) Let $R$ be a $G$-invariant subgroup of $T$ such that $T/R$ is a minimal normal subgroup of $G/R$; then $R > \{1\}$ since $T$ is nonabelian. Since $|G : RH| > 1$ is a power of 2, $R$ has no noncyclic abelian subgroup of order 8, by hypothesis (see (c)), so we have for $R$ the following possibilities listed in Lemma 2(d): either $R \leq 4$ or $R \cong Q_8$ (here we also use Lemma 2(b)).

(iv) Suppose that $H$ centralizes $R$. Then $G/C_G(R)$ is a 2-group, so $C_G(R) = G$, by (ii). Thus, $R \leq Z(G)$. By hypothesis (see (a)), $Z(G)$ is a 2-subgroup and, in view of the maximal choice of $R$, we get $Z(T) = R = Z(G)$. Assume that $T' < R$; then $|R/T'| = 2$. In that case, by Lemma 2(g), applied to the pair $T/T' < G/T'$, the group $G/T'$ has a normal subgroup of index 2, contrary to (ii). Thus, $T' = R = \Phi(T)$ so $T$ is special since $\exp(T') \leq \exp(T/T') = 2$, and $R \in \{C_2, E_4\}$. Therefore, we are done if $|R| = 2$.

(v) Suppose that $T$ is extraspecial of order $2^{2m+1}$, $m > 1$, and $|R| > 2$; then, by (iv), $H$ does not centralize $R$. If $|R| = 4$, then $|T : C_T(R)| = 2$ and $C_T(R)H$ has index 2 in $G$, contrary to (ii) (note that $C_T(R)$ is normal in $G$ since $T$ and $R$ are). Thus, $|R| > 4$ so $R \cong Q_8$ (Lemma 2(d,b)). Let $R_1 = C_T(R)$; then $R_1 \cong R \cong Q_8$, by what has just been said. In that case, $T = R \ast R_1$ is extraspecial of order $2^5$. Suppose that $|H|$ is not a prime. Setting $C_H(R) = Z$ and $C_H(R_1) = Z_1$, we get, by Lemma 2(h1), $|H/Z| = 3 = |H/Z_1|$, $Z \cap Z_1 = \{1\}$ and so $H = Z \times Z_1$, $RZ_1 \cong SL(2,3) \cong R_1Z$, and we conclude that $G = (RZ_1) \ast (R_1Z)$ with $(RZ_1) \cap R_1Z = Z(RZ_1)$. If $|H|$ is a prime, then $|G : T| = 3$ and, as above, $RH \cong SL(2,3) \cong R_1H$. Thus, $G$ as in part (B2).

In what follows we assume that $T$ is not extraspecial.

(vi) Suppose that $T$ has a maximal $G$-invariant cyclic subgroup $Z$ of order $\geq 4$. One may choose $R$ so that it contains $Z$. Then $H$ centralizes $Z$ (Lemma
2(b)) so, by (iv), \( Z \leq Z(G) \). By Lemma 2(d), \( R \) must be cyclic, contrary to (iv).

Thus, \( T \) has no \( G \)-invariant cyclic subgroup of order 4 and so \( R \) is noncyclic. Therefore, by (iii), \( R \in \{ E_4, Q_8 \} \).

(vii) Let \( R \cong E_4 \). In that case, \( C_T(R) \) is normal in \( G \) and \( |T : C_T(R)| \leq 2 \). Since \( |T : C_T(R)H| \leq 2 \), we get \( C_T(R) = T \), by (ii). Since \( T \) is nonabelian, we get \( R = Z(T) \), by the maximal choice of \( R \). As in (iv), we get \( T' = R \) so \( \Phi(T) = R \) and \( T \) is special since, by the above, \( R = Z(G) \).

(viii) Now let \( R \cong Q_8 \). By (iv), \( |R, H| > \{1 \} \). By Lemma 2(b1), \( G/C_T(R) \) is a subgroup of \( S_4 \) containing a subgroup isomorphic to \( R/Z(R) \cong E_4 \) (Lemma 2(b1)). Since \( T \) is normal in \( G \), we get \( G/C_T(R) \not\cong S_4 \). Thus, \( |T : C_T(R)| = 4 = |R : Z(R)| \) so \( |H| = 3 \) and \( T = R \ast C_T(R) \), by the product formula. Thus, \( T/C_T(R) \cong E_4 \). By (ii), \( |T : R| > 2 \) so \( C_T(R) \) is noncyclic of order > 4. Then, by Lemma 2(d), \( C_T(R) \cong Q_8 \) so \( T \cong Q_8 \ast Q_8 \) is extraspecial of order \( 2^5 \).

The case where \( T \) is normal in \( G \), is complete.

B. Now suppose that \( T \) is not normal in \( G \). Then \( T_0 = O_2(G) \notin \{1\} \) since \( O_2(G) = \{1\} \) and \( G \) is solvable. Since \( |G : T_0H| > 1 \) is a power of \( 2 \), \( T_0 \) is a group of Lemma 2(d). It follows from \( C_G(T_0) \leq T_0 \) that \( T_0 \) is noncyclic and, if \( T_0 \) is of maximal class, then \( T_0 \cong Q_8 \) (Lemma 2(b2)). If \( T_0 \cong E_4 \), then \( G \cong S_4 \) since \( \text{Aut}(E_4) \cong S_3 \). Now let \( T_0 \cong Q_8 \). Since \( \text{Aut}(T_0) \cong S_4 \) (Lemma 2(b1)), we conclude that \( G/Z(T_0) \) is isomorphic to a nonnilpotent subgroup of \( S_4 \) containing the subgroup \( T_0/Z(T_0) \cong E_4 \) of even index (by assumption, \( T_0 < T \)). We conclude that \( C_T(T_0) < T_0 \) so \( T \) is of maximal class, namely, \( T \) is generalized quaternion of semidihedral of order 16 (Lemma 2(a)). It follows that \( G/Z(T_0) \cong S_4 \) so \( G \) is a representation group of \( S_4 \).

Since all groups listed in the conclusion of the theorem, satisfy the hypothesis, the proof is complete. \( \square \)

Next we expand Theorem 3 to groups of odd order.

**Theorem 4.** Let \( G \) be a non-nilpotent group and let \( p > 2 \) be the least prime divisor of \( |G| \). Suppose that the following holds:

(a) \( O_p'(G) = \{1\} \).
(b) Whenever \( K \) is a proper subgroup of \( G \) such that \( |G : K| \) is a power of \( p \), then \( K \) has no elementary abelian subgroup of order \( p^3 \).

Let \( T \) be a Sylow \( p \)-subgroup of \( G \). Then \( T \) is normal in \( G \) and one and only one of the following assertions takes place:

A \( T \) is a minimal normal subgroup of \( G \), \( d(T) > 2 \).
B \( T \) is special of exponent \( p \) with \( Z(T) = Z(G) \) is of order at most \( p^2 \), \( T/Z(T) \) is a minimal normal subgroup of \( G/Z(T) \).
PROOF. Since \( G \) has odd order, it is solvable hence, in view of (a), \( C_G(O_p(G)) \leq O_p(G) \) and so, if \( T \) is abelian, it is normal in \( G \). By Lemma 2(b), \( O_p(G) \) is not two-generator. Let \( H \) be a \( p' \)-Hall subgroup of \( G \).

(\ast) Let \( M < T \) be \( G \)-invariant. We contend that \( H \) centralizes \( M \). Indeed, since \( |G : MH| > 1 \) is a power of \( p \), \( M \) is a group of Lemma 2(e), by hypothesis (see (b)). Then, by Lemma 2(b), \( H \) centralizes \( M \) if \( d(M) \leq 2 \). Now let \( d(M) > 2 \). Then, by Lemma 2(e), \( M = \Omega_1(M)C \), where \( \Omega_1(M) \) is nonabelian of order \( p^3 \) and exponent \( p \) and \( C \) is cyclic. Note, that \( \Omega_1(M) \) is normal in \( G \).

By Lemma 2(b), \( H \) centralizes \( \Omega_1(M) \) so \( H \) centralizes \( M \), by Lemma 2(c).

1. Let \( T \) be normal in \( G \).

(i) Assume that \( T \) is a group of Lemma 2(e). Then, as in (\ast), \( H \) centralizes \( T \) so \( H \) is normal in \( G \), which is a contradiction. Thus, \( T \) possesses a subgroup \( \cong E_{p^3} \); then, by Lemma 2(e), \( T \) has a normal subgroup \( \cong E_{p^3} \).

(ii) Suppose that \( T \) is abelian. Since \( |G : H\Omega_1(T)| \) is a power of \( p \) and, by (i), \( \Omega_1(T) \) has a subgroup \( \cong E_{p^3} \), we get \( T = \Omega_1(T) \) so \( T \) is elementary abelian. Assume that \( T = V_1 \times V_2 \), where \( V_1 \) is normal in \( G \).

Then, by (\ast), \( H \) centralizes \( V_i \), \( i = 1, 2 \) (Lemma 2(b)) so \( H \) centralizes \( T \), which is not the case. Thus, \( T \) is a minimal normal subgroup of \( G \) (Maschke).

Next we assume that \( T \) is nonabelian; then \( |T| \geq p^4 \), by (i).

(iii) Assume that \( p \) divides \( |G : G'| \). Then, by (\ast), \( H \) stabilizes the chain \( 1 < T \cap G' < T \) so \( H \) is normal in \( G \) (Lemma 2(f)), a contradiction. Thus, \( p \) does not divide \( |G : G'| \).

(iv) Let \( A < T \) be a \( G \)-invariant subgroup. We claim that \( A \leq Z(T) \). Assume that this is false. Since \( H \) centralizes \( A \), by (\ast), \( C_G(A) \) is normal in \( G \) and \( G/C_G(A) \) is a \( p \)-group > \( \{1\} \), contrary to (iii). Thus, \( A \leq Z(T) \); moreover, \( A \leq Z(G) \).

(v) Let \( R < T \) be \( G \)-invariant and such that \( T/R \) is minimal normal in \( G/R \). Then, by (iv), \( R \leq Z(T) \); moreover, \( R = Z(T) \), by the maximal choice of \( R \). It follows that the class of \( T \) equals 2 so, since \( p > 2 \), we get \( \exp(\Omega_1(T)) = p \). By (i), \( T \) possesses a subgroup \( E \cong E_{p^3} \). Since \( E \leq \Omega_1(T) \) and \( |G : H\Omega_1(T)| \) is a power of \( p \), we get \( G = H\Omega_1(T) \) so \( T = \Omega_1(T) \) is of exponent \( p \). It remains to show that \( T \) is special. Since \( |G : RH| > 1 \) is a power of \( p \), \( R \) is elementary abelian of order at most \( p^2 \). If \( T' < R \), then, by Lemma 2(g), applied to the pair \( T/M < G/M \), the group \( G/M \) has a normal subgroup of index \( p \), contrary to (iii). Thus, \( T' = R \). Since \( T \) is of exponent \( p \), we have \( T' = \Phi(T) \). Thus, \( Z(G) = R = T' = \Phi(T) \) so \( T \) is special.

We see that if \( T \) is nonabelian, it is special of exponent \( p \) with \( R = T' = Z(T) = \Phi(T) \) of order \( \leq p^2 \). By the maximal choice of \( R \), \( T/R \) is a minimal normal subgroup of \( G/R \) so the case where \( T \) is normal in \( G \), is complete.

It remains to show that \( T \) is normal in \( G \) always.

2. Now assume that \( T \) is not normal in \( G \). Since \( O_{p'}(G) = \{1\} \) and \( G \) is solvable, we get \( T > T_0 = O_p(G) \). Therefore, we have \( C_G(T_0) \leq T_0 \) so \( H \) acts faithfully on \( T_0 \). Since \( |G : T_0H| > 1 \) is a power of \( p \), \( T_0 \) has no
elementary abelian subgroup of order $p^3$. It follows that $T_0$ is a group of Lemma 2(e). However, as shows the argument in (i), $H$ centralizes $T_0$, a final contradiction.

Since groups from parts A and B satisfy the hypothesis, the proof is complete. \[\Box\]

Note that if $G$ is a 2-group without normal elementary abelian subgroup of order 8, then it possesses a normal metacyclic subgroup $M$ such that $G/M$ is isomorphic to a subgroup of $D_8$ [6]. Therefore, it is natural to classify the non-nilpotent solvable groups $G$, satisfying (i) $O_{2^k}(G) = \{1\}$ and (ii) if $K < G$ is such that $|G : K|$ is a power of 2, then $K$ has no elementary abelian subgroup of order 8. However, the proof of such result would be very long since the groups appearing in [6] are not so small as groups of Lemma 2(e).

Theorem 4 also holds for each odd prime divisor $p$ of $|G|$ such that $|G|$ and $p^2 - 1$ are coprime (in that case, $|G|$ is odd so solvable). To prove this, we have to repeat, word for word, the proof of Theorem 4.

References


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