Triangles with a given special figure

Daniela Ferrarello       Maria Flavia Mammana*

Abstract

A figure $F$ is said to be special for a triangle $T$ if $F$ can be determined from $T$ by a ruler and compass construction that does not depend on the order of vertices of $T$. In this paper, we deal with the problem of the existence of triangles with a given special figure that is a point, or a line, or a circle, or a triangle.

Keywords: triangles, ruler and compass constructions, similarities

Trokuti s danim posebnim likom

Sažetak

Kažemo da je lik $F$ poseban za trokut $T$ ukoliko $F$ može biti dobiven iz $T$ korištenjem ravnala i šestara neovisno o redoslijedu vrhova od $T$. U ovom radu razmatra se postojanje trokuta s danim posebnim likom koji je točka, pravac, kružnica ili trokut.

Ključne riječi: trokuti, konstrukcija ravnalom i šestarom, sličnosti

*Department of Mathematics and Computer Science, University of Catania. V. le A. Doria 6, 95125 Catania, Italy: fmammana@dmi.unict.it
1 The problem

A triangle $T$ is determined by its vertices, no matter what their order is, i.e., $T$ is determined by a non-ordered triple of non-collinear points, $\{A_1, A_2, A_3\}$. Consider a ruler and compass construction $\chi$ that associates a figure $F$ to any triangle $T = \{A_1, A_2, A_3\}$; precisely, $\chi$ is a construction that, starting from the vertices of $T$, determines $F$ no matter what the order of the vertices is, i.e.,:

$$\chi(A_i, A_j, A_k) = F$$

for any permutation $(i, j, k)$ of indices $1, 2, 3$.

We call such a construction $\chi$ a special construction and we say that $F = \chi(T)$ is a special figure for $T$. Any special construction is invariant with respect to similarities, that is, given any similarity $\sigma$ of the plane, for any triangle $T = A_1, A_2, A_3$, it is:

$$\chi(\sigma(T)) = \sigma(\chi(T)),$$

with $\sigma(T)$ being the correspondent triangle of $T$ with respect to $\sigma$.

We will consider the following types of special figures: a point, a line, a circle, and a triangle. For example, the centroid and the orthocenter of a triangle $T$ are special points for $T$; the Lemoine axis [1] and the Brocard axis [1] of a triangle $T$ are special lines for $T$; the circumcircle and the nine-point circle [2] of a triangle $T$ are special circles for $T$; the medial triangle [2] and the orthic triangle [2] of a triangle $T$ are special triangles for $T$.

Let $\chi$ be any special construction that associates a point (or a line, or a circle, or a triangle) to any triangle. We consider the following problem: given a point $P$ (or a line $l$, or a circle $c$, or a triangle $T$), do there exist triangles $T'$ such that $\chi(T') = P$ (or $\chi(T') = l$, or $\chi(T') = c$, or $c(T') = T$, respectively? And if they do, how many are there? Here we give a general procedure allowing us to solve the problem for special points, special lines, and special circles, and we will also show that for each of those cases there exist continuum many triangles $T'$, not similar to each other, that solve the problem. In the case of special triangles, the procedure cannot be used and the property does not hold.

2 A representative subset of the set of triangles

Let $T$ be the set of all triangles in the plane. In order to solve the problem posed in the previous section, we define a set $S$ of representatives for each class of similar triangles, i.e., a subset $S$ of $T$ such that any triangle in $T$ is similar to one and only one triangle in $S$. Fix a line $a$ and two distinct
points, $A_1$ and $A_2$, on $a$; let $d$ be one of the half-planes bounded by $a$. Consider any triangle $A'_1A'_2A'_3$ and suppose that $A'_1A'_2 \geq A'_1A'_3 \geq A'_2A'_3$. It is known that there exist only two similarities, one direct and one opposite, that transform $A'_1$ and $A'_2$ to $A_1$ and $A_2$, respectively [3, p. 73]. Let $\sigma$ be the one that transforms $A'_3$ to the point $A_3$ in the half-plane $d$. The similarity $\sigma$ determines the triangle $A_1A_2A_3$, completely contained in the half-plane $d$, such that $A_1A_2 \geq A_1A_3 \geq A_2A_3$. Denote by $S$ the set of all triangles $A_1A_2A_3$ that are obtained by varying the triangles $A'_1A'_2A'_3$ in $T$. Let $\gamma$ be the circle with center $A_1$ and radius $A_1A_2$. Let $b$ be the perpendicular bisector of the segment $A_1A_2$. Let $b$ intersect $\gamma$ at the point $B$ of $d$ and the segment $A_1A_2$ at $M$; thus $M$ is the midpoint of $A_1A_2$. We define the region $R$ as the subset of $d$ bounded by the arc $A_2B$ of $\gamma$ and the segments $BM$ and $MA_2$. Since $A_1A_2 \geq A_1A_3 \geq A_2A_3$, we see that the point $A_3$ lies in the region $R$. The triangle $A_1A_2A_3$ (see Figure 1) is scalene if $A_3$ is in the interior of $R$; it is isosceles if $A_3$ lies on the segment $BM$ or on the arc $A_2B$. The triangle $A_1A_2B$ is equilateral. Observe that since no two distinct triangles of $S$ are similar, then every triangle in $T$ is similar to exactly one triangle in $S$. Moreover, $S$ contains continuum many triangles.

![Figure 1: $A_1A_2A_3$ construction](image)

### 3 Solution of the problem

Denote by $\chi_i, i = 1, 2, 3, 4$, a special construction that associates to any triangle a point if $i = 1$, a line if $i = 2$, a circle if $i = 3$, a triangle if $i = 4$. Denote by $F_i$ a point if $i = 1$, a line if $i = 2$, a circle if $i = 3$, and a triangle if $i = 4$. 
Theorem 1. Given a special construction $\chi_i$, $i = 1, 2, 3$, for every figure $F_i$ there exist continuum many triangles $T'$, not similar to each other, such that $\chi_i(T') = F_i$.

Proof. Fix a special construction $\chi_i$, $i = 1, 2, 3$, and a figure $F_i$. Let $T$ be any triangle in $S$ and let $\Omega_i(T)$ be the set of similarities that transform $\chi_i(T)$ to $F_i$. It is easy to see that for any $i \in \Omega_i(T)$ is non-empty. Let $\omega \in \Omega_i(T)$ and let $T' = \omega(T)$. Since $\chi_i$ is invariant under similarities, we have

$$\chi_i(T') = \chi_i(\omega(T)) = \omega(\chi_i(T)) = F_i.$$  

Then, since $\Omega_i(T)$ is non-empty, there exists at least one triangle $T'$ such that $\chi_i(T') = F_i$. Moreover, if $\omega' \in \Omega_i(T)$ and $\omega'(T) = T''$, then the triangles $T'$ and $T''$ are similar, because $T''$ corresponds to $T'$ by means of the similarity $\omega^{-1} \cdot \omega'$.

In order to see how many non-similar triangles solve the problem, we can observe that each triangle $T$ in $S$ determines a family $S(T)$ of similar triangles. Precisely, $S(T) = \{T' = \omega(T) : \omega \in \Omega_i(T)\}$. These families are disjoint. In fact, suppose that $T_1, T_2 \in S$, with $T_1 \neq T_2$. If $\omega_1 \in \Omega_i(T_1)$, $\omega_2 \in \Omega_i(T_2)$, $\omega_1(T_1) = T'_1$ and $\omega_2(T_2) = T'_2$, then $T'_1$ and $T'_2$ are triangles of the families $S(T_1)$ and $S(T_2)$, respectively. $T'_1$ and $T'_2$ are non-similar because if there exists a similarity $\omega^*$ such that $\omega^*(T'_1) = T'_2$, then $T_2$ corresponds to $T_1$ through the similarity $\omega_1 \cdot \omega^* \cdot \omega_2^{-1}$, which cannot happen because distinct triangles in $S$ are not similar to each other. Since $S$ contains continuum many triangles, there are continuum many non-similar triangles that solve the problem, one per each family $S(T)$. \hfill $\square$

Fix now a special construction $\chi_4$ and any figure $F_4$. Consider the problem: do there exist triangles $T'$ such that $\chi_4(T') = F_4$? The general method that we have just illustrated cannot be used to solve such problem, because given two triples of non-aligned points, $(H_1, H_2, H_3)$ and $(H'_1, H'_2, H'_3)$, generally there exists no similarity $\omega$ such that $\omega(H_i) = H'_i$ for $i = 1, 2, 3$ (unless the triangles $H_1H_2H_3$ and $H'_1H'_2H'_3$ are similar). Nevertheless, it is known that, given a triangle $T$, there exists one and only one triangle whose medial triangle is $T$, the anticomplementary triangle of $T$ [1]. Then we can state that for $i = 4$ the theorem does not hold. Moreover, it is still an open problem to see for which special constructions $\chi_4$ it happens that, for any figure $F_4$, there exist triangles $T'$ such that $\chi_4(T') = F_4$. \hfill 16
References

