D(-1)-QUADRUPLES AND PRODUCTS OF TWO PRIMES

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Abstract. A D(-1)-quadruple is a set of positive integers \{a, b, c, d\}, with a < b < c < d, such that the product of any two elements from this set is of the form 1 + n^2 for some integer n. Dujella and Fuchs showed that any such D(-1)-quadruple satisfies a = 1. The D(-1) conjecture states that there is no D(-1)-quadruple. If b = 1 + r^2, c = 1 + s^2 and d = 1 + t^2, then it is known that r, s, t, b, c and d are not of the form p^k or 2p^k, where p is an odd prime and k is a positive integer. In the case of two primes, we prove that if r = pq and v and w are integers such that p^2v - q^2w = 1, then 4vw - 1 > r. A particular instance yields the result that if r = p(p + 2) is a product of twin primes, where p ≡ 1 (mod 4), then the D(-1)-pair \{1, 1 + r^2\} cannot be extended to a D(-1)-quadruple. Dujella’s conjecture states that there is at most one solution (x, y) in positive integers with y < k − 1 to the diophantine equation x^2 - (1 + k^2)y^2 = k^2. We show that the Dujella conjecture is true when k is a product of two odd primes. As a consequence it follows that if t is a product of two odd primes, then there is no D(-1)-quadruple \{1, b, c, d\} with d = 1 + t^2.

1. Introduction

Let n be a nonzero integer. A diophantine m-tuple with the property D(n), is a set of m positive integers, such that if a, b are any two elements from this set, then ab + n = k^2 for some integer k. We will look at the case n = -1. The cases n = 1 and n = 4 have been studied in great detail and still continue to be areas of active research. For more details on this subject the reader may consult [1], where a comprehensive and up to date list of references is available.

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In the case of $n = -1$, it has been conjectured that there is no $D(-1)$-quadruple. The first significant progress was made by Dujella and Fuchs ([2]), who showed that if $\{a, b, c, d\}$ is a $D(-1)$-quadruple with $a < b < c < d$, then $a = 1$. Subsequently, Dujella et al. ([3]) proved that there are only a finite number of such quadruples. Filipin and Fujita ([4]) showed that if $\{1, b, c\}$ is $D(-1)$-triple with $b < c$, then there exist at most two $d$’s such that $\{1, b, c, d\}$ is a $D(-1)$-quadruple.

Filipin, Fujita and Mignotte ([5]) showed that if $b = r^2 + 1$, then in each of the cases $r = p^k$, $r = 2p^k$, $b = p$ and $b = 2p^k$, where $p$ is an odd prime and $k$ is a positive integer, the $D(-1)$-pair $\{1, b\}$ cannot be extended to a $D(-1)$-quadruple $\{1, b, c, d\}$ with $b < c < d$. In [13] we showed that this also holds for $c = 1 + s^2$, that is, if $s = p^k$, $s = 2p^k$, $c = p$ or $c = 2p^k$, then the $D(-1)$-quadruple $\{1, b, c\}$ cannot be extended to a $D(-1)$-quadruple (one of the referees pointed out that this result was essentially proved in [5]). It is also known that the results mentioned above for $b$ and $c$ also hold for $d = 1 + t^2$ (see discussion following Conjecture 1.3). Note that $b, c$ and $d$ cannot be of the form $p^k$ with $k > 1$ and $p$ prime (see [8]). In the case of a product of two primes, we showed in [13] that if $r = pq$ then $p^4, q^4 > r$. The following result gives further conditions in this case.

**Theorem 1.1.** Let $\{1, b, c, d\}$ with $1 < b < c < d$ be a $D(-1)$-quadruple with $b = 1 + r^2$ where $r > 0$. Let $r = pq$, where $p$ and $q$ are distinct odd primes, and let $v$ and $w$ be integers such that $p^2v - q^2w = 1$. Then $4vw - 1 > r$.

**Corollary 1.2.** Let $b = 1 + r^2$ and $r = p(p + 2)$ where $p$ and $p + 2$ are both primes and $p \equiv 1 \pmod{4}$. Then the $D(-1)$-pair $\{1, b\}$ cannot be extended to a $D(-1)$-quadruple.

The following conjecture made by Andrej Dujella is closely related to the $D(-1)$ conjecture.

**Conjecture 1.3.** (Andrej Dujella) Let $k \geq 2$. Then there exists at most one solution $(x, y)$ in positive integers to the equation $x^2 - (k^2 + 1)y^2 = k^2$ with $y < k - 1$.

In [9] the authors studied the equation $x^2 - (k^2 + 1)y^2 = k^2$, calling it the Dujella equation and the conjecture above, which they called the unicity conjecture. They used a continued fraction approach and gave some interesting equivalent conjectures.

It is known that Dujella’s unicity conjecture implies the $D(-1)$ conjecture (see [9, Section 17]). Indeed the result [5] on the $D(-1)$ conjecture mentioned above, is based on [5, Lemma 6.1], which states that Conjecture 1.3 is true for the same cases, namely, when $k^2 + 1 = p, 2p^n$, or $k = p^n, 2p^n$, where $p$ is an odd prime and $n$ is a positive integer. It follows, also from [5, Lemma 6.1], that the $D(-1)$ conjecture holds in the case when $t$ or $d = 1 + t^2$ is of the form $p^n$ or $2p^n$, where $p$ is an odd prime and $k$ is a positive integer.
K. Matthews communicated to the author an unpublished short proof (along with J. Robertson) of Conjecture 1.3 in the case when \( k^2 + 1 \) is divisible by exactly two odd primes. We show that Conjecture 1.3 is true when \( k \) is a product of two odd primes.

**Theorem 1.4.** Let \( k = pq \) where \( p \) and \( q \) are distinct odd primes. Then the equation \( x^2 - (1 + k^2)y^2 = k^2 \) has at most one solution \((x, y)\) in positive integers with \( y < k - 1 \).

An immediate corollary is the following.

**Corollary 1.5.** If \( x \) is a product of two distinct odd primes and \( d = 1 + x^2 \), then there is no \( D(-1)\)-quadruple \( \{1, b, c, d\} \) with \( 1 < b < c < d \).

2. Binary quadratic forms

In this section we present the basic theory of binary quadratic forms. An excellent reference is [11], where Sections 4 to 7 and Section 11 of Chapter 6 pertain to the matter at hand.

A primitive binary quadratic form \( f = (a, b, c) \) of discriminant \( d \) is a function \( f(x, y) = ax^2 + bxy + cy^2 \), where \( a, b, c \) are integers with \( b^2 - 4ac = d \) and \( \gcd(a, b, c) = 1 \). Note that the integers \( b \) and \( d \) have the same parity. All forms considered here are primitive binary quadratic forms and henceforth we shall refer to them simply as forms.

Two forms \( f \) and \( f' \) are said to be equivalent, written as \( f \sim f' \), if for some \( A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z}) \) (called a transformation matrix), we have \( f'(x, y) = f(\alpha x + \beta y, \gamma x + \delta y) = (a', b', c') \), where \( a', b', c' \) are given by

\[
(2.1) \quad a' = f(\alpha, \gamma), \quad b' = 2(\alpha \beta \gamma + \alpha c \gamma + \alpha \delta \gamma + \gamma \beta \gamma) \quad c' = f(\beta, \delta).
\]

It is easy to see that \( \sim \) is an equivalence relation on the set of forms of discriminant \( d \). The equivalence classes form an abelian group called the class group with group law given by composition of forms. The identity form is defined as the form \((1, 0, -d)\) or \((1, 1, -d)\), depending on whether \( d \) is even or odd respectively. The inverse of \( f = (a, b, c) \) denoted by \( f^{-1} \), is given by \((a, -b, c)\).

A form \( f \) is said to represent an integer \( m \) if there exist integers \( x \) and \( y \) such that \( f(x, y) = m \). If \( \gcd(x, y) = 1 \), we call the representation a primitive one. Observe that equivalent forms primitively represent the same set of integers, as do a form and its inverse. Hence, sometimes we will refer to a class of forms that represents an integer.

We end this section with two elementary observations about forms. Firstly, if a form \( f \) represents primitively an integer \( n \), then \( f \sim (n, b, c) \) for some integers \( b, c \). This follows simply by noting that if \( f(\alpha, \gamma) = n \) with \( \gcd(\alpha, \gamma) = 1 \), then there exists a transformation matrix \( A \) as given above such
that (2.1) holds. Secondly, if \( b \equiv b' \pmod{2n} \), then the forms \((n, b, c)\) and \((n, b', c')\) are equivalent. This equivalence follows using the transformation matrix \( A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) where \( b' = b + 2n\delta \).

3. THE DIOPHANTINE EQUATION \( x^2 - dy^2 = n \)

For any positive integer \( d \) that is not a square, all representations \((x, y)\) of an integer \( n \) by the form \((1, 0, -d)\) may be put into equivalence classes using the following notion of equivalence.

**Definition 3.1.** Two solutions \((x, y)\) and \((x', y')\) of \( X^2 - dY^2 = n \) are said to be equivalent, written as \((x, y) \sim (x', y')\) if the following congruences
\[
xx' \equiv dyy' \pmod{n}, \quad xy' \equiv yx' \pmod{n}
\]
are satisfied.

The result given below is used at several places, and hence we isolate it as a lemma.

**Lemma 3.2.** Let \( k \) be an odd integer. If a solution \((x, y)\) of the equation \( x^2 - (1 + k^2)y^2 = k^2 \) satisfies \((x, y) \sim (x, -y)\), then \( k \) divides \( x \) and \( y \).

**Proof.** If \((x, y) \sim (x, -y)\), then (3.1) gives \( x^2 \equiv -y^2 \pmod{k^2} \). Moreover, from the Dujella equation, \( x^2 \equiv y^2 \pmod{k^2} \), hence \( k \) divides \( x \) and \( y \).

The following lemma connects primitive representations of \( x^2 - dy^2 = n \) and forms that represent \( n \) and is crucial for our proofs.

**Lemma 3.3.** Let \( n \) be a positive integer such that \( \gcd(n, 2\Delta) = 1 \) and suppose that \( n \) is primitively represented by some form of discriminant \( \Delta \). Then the following claims hold.

1. If \( A = \{(n, b, c); 0 < b < 2n\} \) and \( w(n) \) is the number of distinct primes dividing \( n \), then \( |A| = 2^{w(n)} \).

2. There is a one-to-one correspondence between the set of equivalence classes of primitive solutions \((x, y)\) of the equation \( X^2 - dY^2 = n \) and the set \( A_0 = \{(n, b, c) \sim (1, 0, -d); 0 < b < 2n\} \) of forms in \( A \) equivalent to the identity form.

**Proof.** As \( n \) is primitively represented by some form of discriminant \( \Delta \), there is a solution to the congruence \( \Delta \equiv x^2 \pmod{4n} \) ([11, Solution of problem 1]). It follows from a classical result (see for instance [14, Chapter V, §4] or [7, Theorem 122]) that there are \( 2^{w(n)} \) solutions modulo \( 4n \). As \( x \) and \( -x \) are both solutions to \( \Delta \equiv x^2 \pmod{4n} \), there are \( 2^{w(n)} \) solutions to the congruence \( \Delta \equiv x^2 \pmod{4n} \) with \( 0 < x < 2n \). The first part of the lemma now follows from [11, Solution of problem 2], where it is shown that
there is a one-to-one correspondence between the set $A$ and solutions to the congruence $\Delta \equiv x^2 \pmod{4n}$ with $0 < x < 2n$.

The second part of the lemma follows from the following facts that are given in [11, Solution of problem 3]. Each primitive representation $(x, y)$ of $X^2 - dY^2 = n$ corresponds to a unique form $(n, b, c)$, where $0 < b < 2n$. If two such representations correspond to the same form, then the representations are equivalent. Moreover, each form in set $A_0$ corresponds to a unique equivalence class of primitive representations $(x, y)$ of $X^2 - dY^2 = n$, and hence the correspondence in part 2 of the lemma follows.

The next lemma has been used by several authors in the study of the current problem, such as [5, Lemma 6.2] and [13, Lemma 3.2].

**Lemma 3.4** ([6, Lemma 2.3]). Let $n$ be an integer such that $1 < |n| \leq k$. Then there are no primitive solutions $(x, y)$ such that $x^2 - (k^2 + 1)y^2 = n$.

A useful consequence of the above lemma is the following result.

**Lemma 3.5** ([13, Lemma 3.3]). Let $k = ff'$ be a positive integer such that $1 < f < k$. If $x^2 - (k^2 + 1)y^2 = f'^2$ for some coprime integers $x$ and $y$, then $f'$ is not an odd prime power.

4. **Proofs**

Throughout this section the following terminology will be used.

Let $\{1, b, c, d\}$ be a $D(-1)$-quadruple with $1 < b < c < d$. Set

$$b = 1 + r^2, \quad c = 1 + s^2, \quad d = 1 + x^2$$

and

$$bd = 1 + y^2, \quad cd = 1 + z^2, \quad bc = 1 + t^2.$$  

Then

$$t^2 - (1 + r^2)s^2 = r^2$$

and

$$t^2 - (1 + s^2)r^2 = s^2.$$  

It is easy to see (using (3.1)) that the equation $X^2 - (r^2 + 1)Y^2 = r^2$ has the inequivalent solutions $(r, 0)$ and $(r^2 + 1 - r, \pm(r - 1))$. In [5], solutions equivalent to these three solutions were called regular solutions and it was shown that $(t, s)$ is not a regular solution.

**Lemma 4.1** ([5, Corollary 1.2]). The solution $(t, s)$ of $X^2 - bY^2 = r^2$ is not equivalent to any of the solutions $(b - r, \pm(r - 1))$ and $(r, 0)$.

**Lemma 4.2.** Let $r = pq$ where $p$ and $q$ are distinct odd primes. Then there are exactly four inequivalent classes of primitive representations of $r^2$ by the form $(1, 0, -(1 + r^2))$, namely, $(b - r, \pm(r - 1))$ and $(t, \pm s)$. Moreover, $r^2$ is primitively represented only by the identity class.
Proof. Let \( \gcd(t, s) = n \). As \( r = pq \), from (4.1) we have \( n = 1, r, p \) or \( q \). Observe that by Lemma 3.5 the cases \( n = p \) and \( n = q \) are not possible. If \( n = r \), then \( t \) and \( s \) are divisible by \( r \). It follows by equivalence of solutions (Definition 3.1) that \((t, s) \sim (r, 0)\), which is not possible by Lemma 4.1. Hence \( \gcd(t, s) = 1 \), and it follows from Lemma 3.2 and Lemma 4.1 that \((b - r, \pm(r - 1))\) and \((t, \pm s)\) are inequivalent primitive representations. By Lemma 3.3, the set \( A_0 \) (given therein) has at least 4 elements. Moreover, by the same lemma, the set \( A \) has exactly 4 elements and therefore \( A = A_0 \) as \( A_0 \subseteq A \) and hence there are exactly four inequivalent classes of primitive representations of \( r^2 \) by \((1, 0, -b)\), namely the ones given above.

The second part of the following lemma follows on application of [12, Theorem 1] (a converse to Nagell's theorem). However, the article mentioned above only provides an outline of the proof and we are grateful to a referee for the details given below.

Lemma 4.3. Let \( k = pq \), where \( p \) and \( q \) are distinct odd primes. Then the following hold.

1. Any solution \((\alpha, \beta)\) of \( X^2 - (1 + k^2)Y^2 = k^2 \) with \( 0 < \beta < k \) satisfies \( \gcd(\alpha, \beta) = 1 \).
2. Let \((x, y)\) and \((x', y')\) be two equivalent solutions in positive integers to \( X^2 - (1 + k^2)Y^2 = k^2 \) that satisfy \( y, y' < k - 1 \). Then \( x = x' \) and \( y = y' \).

Proof. As seen in the beginning of the proof of Lemma 4.2, either \( \gcd(\alpha, \beta) = 1 \) or \( k \) divides both \( \alpha \) and \( \beta \), the latter of which is not possible as \( 0 < \beta < k \) and hence \( \gcd(\alpha, \beta) = 1 \).

For the second part, observe that \((2k^2 + 1, 2k)\) is the fundamental solution of the Pell equation \( X^2 - (1 + k^2)Y^2 = 1 \). It is well known (see for example [12]) that if \((x, y)\) and \((x', y')\) are equivalent, then

\[
(4.3) \quad x' + y'\sqrt{k} = \pm(x + y\sqrt{k})(2k^2 + 1 + 2k\sqrt{k})^n,
\]

for some integer \( n \). Since \( x^2 - dy^2 = k^2 \), we may rewrite (4.3) as

\[
(4.4) \quad (x' + y'\sqrt{k})(x - y\sqrt{k}) = \pm k^2(2k^2 + 1 + 2k\sqrt{k})^n = A + B\sqrt{k}.
\]

It is easy to see that \( 2k^3 \) divides \( B \) in the above equation and hence it also divides \( xy' - yx' \). Observe that since \( y \) and \( y' \) are positive integers less than \( k - 1 \), it follows from the Dujella equation that \( x \) and \( x' \) are less than \( k^2 - k + 1 \). Hence, as \( xy' - yx' \) is divisible by \( 2k^3 \), we have \( xy' = yx' \), which gives \( x = x' \) and \( y = y' \), since from part one of the lemma \( \gcd(x, y) = \gcd(x', y') = 1 \).

Proof of Theorem 1.1. Let \( v \) and \( w \) be integers such that \( vp^2 - wq^2 = 1 \) and let \( h \) be the form \((r^2, 4q^2w + 2, 4vw - 1)\), where \( r = pq \). It is straightforward to see that \( h \) is a form of discriminant 40 and that \( 4vw - 1 > 0 \). Moreover, \( h \) primitively represents \( r^2 \) and thus, by Lemma 4.2, we have \( h \sim (1, 0, -b) \).
Furthermore, $h$ also primitively represents $4vw - 1$ and hence, by Lemma 3.4, we have $4vw - 1 > r$.

**Proof of Corollary 1.2.** Note that if $v = \frac{p+3}{4}$ and $w = \frac{v-1}{4}$, then we have $vp^2 - w(p+2)^2 = 1$. Moreover, $4vw - 1 = (p + 3)\frac{v-1}{4} - 1 < p(p+2)$ and the corollary follows from Theorem 1.1.

**Proof of Theorem 1.4.** Let $(x, y)$ be a solution of the Dujella equation $x^2 - (1 + k^2)y^2 = k^2$, with $x, y > 0$ and $y < k-1$. Then $x = |x| < k^2 - k + 1$ and $0 < x + y < k^2$. Now suppose that $(x, y) \sim (1 + k^2 - k, \pm(k - 1))$. Then (3.1) gives

$$x \equiv \pm y \pmod{k^2},$$

which is not possible, as we have shown above that $0 < x + y < k^2$. Therefore $(x, y)$ is not equivalent to either of the solutions $(1 + k^2 - k, \pm(k - 1))$. Furthermore, using Lemma 3.2 and Lemma 4.3, part 1, it follows that the solutions $(x, \pm y)$ and $(1 + k^2 - k, \pm(k - 1))$ are inequivalent primitive solutions. Therefore $|A_0| \geq 4$, where $A_0$ is as given in Lemma 3.3. From the same lemma we have $|A| = 4$ and as $A_0 \subseteq A$ it follows that $A_0 = A$. Thus there are exactly four inequivalent classes of primitive solutions, namely the classes represented by $(x, \pm y)$ and $(1 + k^2 - k, \pm(k - 1))$. Now, if $(x', y')$ is another solution in positive integers to the Dujella equation satisfying $y' < k - 1$, then it must be equivalent to one of $(x, \pm y)$ (since we have shown above that any such solution is not equivalent to $(1 + k^2 - k, \pm(k - 1))$). From Lemma 4.3 part 2, we have $(x, y) = (x', y')$, and hence there is at most one solution in positive integers $(x, y)$ with $y < k - 1$ to the equation $X^2 - (1 + k^2)Y^2 = k^2$, and the theorem is proved.

**Proof of Corollary 1.5.** By Theorem 1.4, if $x$ is a product of two distinct odd primes, then the equation $\alpha^2 - (1 + x^2)\beta^2 = x^2$ has at most one positive solution $(\alpha, \beta)$ with $\beta < x - 1$. In other words, the Dujella conjecture holds for this equation and as shown in [9, Section 17], this implies that the $D(-1)$ conjecture is true.

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