LOCALIZED SVEP AND THE COMPONENTS OF QUASI-FREDHOLM RESOLVENT SET

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ABSTRACT. In this paper, new characterizations of the single valued extension property are given, for a bounded linear operator $T$ acting on a Banach space and its adjoint $T^*$, at $\lambda_0 \in \mathbb{C}$ in the case that $\lambda_0 I - T$ is quasi-Fredholm. With the help of a classical perturbation result concerning operators with eventual topological uniform descent, we show the constancy of certain subspace valued mappings on the components of quasi-Fredholm resolvent set. As a consequence, we obtain a classification of these components.

1. Introduction

Throughout this paper, $B(X)$ will denote the set of all bounded linear operators on an infinite-dimensional complex Banach space $X$. For an operator $T \in B(X)$, let $T^*$ denote its adjoint, $N(T)$ its kernel and $R(T)$ its range. Two important subspaces of $X$ are the hyperrange of $T$ defined by $R(T^\infty) = \cap_{n=1}^\infty R(T^n)$, and the hyperkernel of $T$ defined by $N(T^\infty) = \cup_{n=1}^\infty N(T^n)$, respectively. There are another two important subspaces of $X$, the analytical core $K(T)$ of $T$ defined by

$$K(T) = \{ x \in X : \text{there exist a sequence } \{ x_n \}_{n=0}^\infty \subseteq X \text{ and a constant } \delta > 0 \text{ such that } x_0 = x, T x_{n+1} = x_n \text{ and } \| x_n \| \leq \delta^n \| x \| \text{ for all } n \in \mathbb{N} \},$$

and the quasi-nilpotent part $H_0(T)$ of $T$ defined by

$$H_0(T) = \{ x \in X : \lim_{n \to \infty} \| T^n x \|^{1/n} = 0 \}.$$

It is well known that $K(T) \subseteq R(T^\infty)$ and $N(T^\infty) \subseteq H_0(T)$.

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Recall that $T \in \mathcal{B}(X)$ is called \textit{bounded below} if $T$ is injective and has closed range $R(T)$. An operator $T \in \mathcal{B}(X)$ is called \textit{semi-regular} if $R(T)$ is closed and $N(T) \subseteq R(T^\infty)$ (or equivalently, $N(T^\infty) \subseteq R(T)$). The concept of semi-regular was originated from Kato’s classical treatment \cite{11} of perturbation theory, even if originally these operators were not named in this way. Trivial examples of semi-regular operators are surjective operators and bounded below operators.

The lattice of invariant subspaces of an operator $T \in \mathcal{B}(X)$ is denoted as $\text{Lat}(T)$. A pair of closed subspace $(M, N)$ is said to reduce $T$ (denoted as $(M, N) \in \text{Red}(T)$), if $X = M \oplus N$ and $M, N \in \text{Lat}(T)$. For $M \in \text{Lat}(T)$, $T|_M$ denotes the restriction of $T$ to $M$. An operator $T \in \mathcal{B}(X)$ is said to be of \textit{Kato type} if there exists $(M, N) \in \text{Red}(T)$ such that $T|_M$ is semi-regular and $T|_N$ is nilpotent. If we assume in the definition above that $N$ is finite-dimensional, then $T$ is said to be \textit{essentially semi-regular}. Equivalently, essentially semi-regular operators can be characterized in such a way that $R(T)$ is closed and there exists a finite-dimensional subspace $F$ of $X$ for which $N(T) \subseteq R(T^\infty) + F$ (see \cite[Theorem 1.48]{1}).

For each $n \in \mathbb{N}$, we set $c_n(T) = \dim R(T^n)/R(T^{n+1})$ and $c_n'(T) = \dim N(T^{n+1})/N(T^n)$. It follows from \cite[Lemmas 3.1 and 3.2]{10} that, for every $n \in \mathbb{N}$,

$$c_n(T) = \dim X/(R(T) + N(T^n)), \quad c_n'(T) = \dim N(T) \cap R(T^n).$$

Hence, it is easy to see that the sequences $\{c_n(T)\}_{n=0}^\infty$ and $\{c_n'(T)\}_{n=0}^\infty$ are decreasing. Recall that the \textit{descent} and the \textit{ascent} of $T \in \mathcal{B}(X)$ are defined as $\text{dsc}(T) = \inf \{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}$ and $\text{asc}(T) = \inf \{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\}$, respectively (the infimum of an empty set is defined to be $\infty$). That is,

$$\text{dsc}(T) = \inf \{n \in \mathbb{N} : c_n(T) = 0\}$$

and

$$\text{asc}(T) = \inf \{n \in \mathbb{N} : c_n'(T) = 0\}.$$

Recall that an operator $T \in \mathcal{B}(X)$ is said to be \textit{left Drazin invertible} if $p := \text{asc}(T) < \infty$ and $R(T^{p+1})$ is closed.

If $T \in \mathcal{B}(X)$, for each $n \in \mathbb{N}$, $T$ induces a linear transformation from the vector space $R(T^n)/R(T^{n+1})$ to the space $R(T^{n+1})/R(T^{n+2})$. Let $k_n(T)$ be the dimension of the kernel of the induced map. From \cite[Lemma 2.3]{9} it follows that, for every $n \in \mathbb{N}$,

$$k_n(T) = \dim (N(T) \cap R(T^n))/(N(T) \cap R(T^{n+1})) = \dim (R(T) + N(T^{n+1}))/ (R(T) + N(T^n)).$$

We remark that the sequence $\{k_n(T)\}_{n=0}^\infty$ is not always decreasing. For this, see the following simple example.
Example 1.1. An operator $T \in \mathcal{B}(l_2^{(1)} \oplus l_2^{(2)})$ is defined as follows:

$$T = \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix} : l_2^{(1)} \oplus l_2^{(2)} \to l_2^{(1)} \oplus l_2^{(2)},$$

where $S : l_2^{(2)} \to l_2^{(1)}$ is an isomorphism. It is easy to know that $N(T) = l_2^{(1)} \oplus \{0\}$, $R(T) = l_2^{(1)} \oplus \{0\}$, and $R(T^n) = \{0\} \oplus \{0\}$ for all $n \geq 2$. Then we have that

$$k_0(T) = \dim \frac{N(T)}{N(T) \cap R(T)} = 0, \quad k_1(T) = \dim \frac{N(T) \cap R(T)}{N(T) \cap R(T^2)} = \infty,$$

$$k_n(T) = \dim \frac{N(T) \cap R(T^n)}{N(T) \cap R(T^{n+1})} = 0, \quad \text{for all } n \geq 2.$$


Definition 1.2. Let $d \in \mathbb{N}$. An operator $T \in \mathcal{B}(X)$ is said to be quasi-Fredholm of degree $d$ if $k_n(T) = 0$ for $n \geq d$, and the subspaces $N(T^d) + R(T)$ and $N(T) \cap R(T^d)$ are closed.

An operator $T \in \mathcal{B}(X)$ is said to be quasi-Fredholm if it is quasi-Fredholm of some degree $d$.

Discussions of quasi-Fredholm operators may be found in [2, 4, 13, 15, 18]. The following lemma describes some equivalent conditions of the assumption that the subspaces $N(T^d) + R(T)$ and $N(T) \cap R(T^d)$ are closed.

Lemma 1.3 ([18, Proposition 3]). Let $T \in \mathcal{B}(X)$, $d \in \mathbb{N}$ and let $k_n(T) = 0$ for all $n \geq d$. The following statements are equivalent:

1. $T$ is quasi-Fredholm, i.e. $N(T^d) + R(T)$ and $N(T) \cap R(T^d)$ are closed.
2. $R(T^{d+1})$ is closed.
3. $R(T^n)$ is closed for all $n \geq d$.
4. $R(T^i) + N(T^j)$ is closed for all $i, j$ with $i + j \geq d$.

The next definition, which was introduced by S. Grabiner ([9]), is closely related to that of quasi-Fredholm operators.

Definition 1.4. Let $d \in \mathbb{N}$. An operator $T \in \mathcal{B}(X)$ is said to have topological uniform descent for $n \geq d$ if $k_n(T) = 0$ for $n \geq d$, and the subspace $N(T^d) + R(T)$ is closed.

An operator $T \in \mathcal{B}(X)$ is said to have eventual topological uniform descent if there exists $d \in \mathbb{N}$ such that it has topological uniform descent for $n \geq d$.

From Definition 1.4 we see easily that $T \in \mathcal{B}(X)$ is semi-regular if and only if $T$ has topological uniform descent for $n \geq 0$. By Lemma 1.3, we
know that quasi-Fredholm operators of degree $d$ are precisely all operators $T \in \mathcal{B}(X)$ that have topological uniform descent for $n \geq d$ and closed range $R(T^{d+1})$.

The single valued extension property was introduced by N. Dunford in [6,7] and plays an important role in local spectral theory and Fredholm theory, see the recent monographs [1] by P. Aiena and [14] by K. B. Laursen and M. M. Neumann.

**Definition 1.5.** An operator $T \in \mathcal{B}(X)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at $\lambda_0$), if for every open disc $U$ of $\lambda_0$, the only analytic function $f : U \rightarrow X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ is the constant function $f \equiv 0$.

An operator $T \in \mathcal{B}(X)$ is said to have the SVEP if $T$ has the SVEP at every point $\lambda \in \mathbb{C}$.

The notion of localized SVEP at a point dates back to J. Finch ([8]). Some characterizations of the SVEP were given by P. Aiena ([2]), for an operator $T \in \mathcal{B}(X)$ and its adjoint $T^\ast$, at $\lambda_0 \in \mathbb{C}$ in the case that $\lambda_0 I - T$ is quasi-Fredholm.

This paper is organized as follows. In section 2, as a continuation of [2], we give new characterizations of the SVEP, for $T$ and $T^\ast$, at $\lambda_0 \in \mathbb{C}$ in the case that $\lambda_0 I - T$ is quasi-Fredholm. In section 3, with the help of a classical perturbation result concerning operators with eventual topological uniform descent, we show the constancy of certain subspace valued mappings on the components of quasi-Fredholm resolvent set. As a consequence, a classification of these components is obtained. This generalizes the corresponding results of P. Aiena and F. Villafañe ([3]).

2. New characterizations of the localized SVEP

V. Müller in [18] proved that if $T \in \mathcal{B}(X)$ is quasi-Fredholm of degree $d$ then $T^\ast \in \mathcal{B}(X^\ast)$ is also quasi-Fredholm of the same degree $d$. The following result shows that the reverse is also true.

For a subspace $M$ of $X$, let $M^\perp \subseteq X^\ast$ denote the annihilator of $M$. For a subspace $N$ of $X^\ast$, let $^\perp N \subseteq X$ denote the pre-annihilator of $N$.

**Theorem 2.1.** Let $d \in \mathbb{N}$. Then $T \in \mathcal{B}(X)$ is quasi-Fredholm of degree $d$ if and only if $T^\ast \in \mathcal{B}(X^\ast)$ is quasi-Fredholm of degree $d$.

**Proof.** For the “only if” part, see [18, Lemma 4].

For the “if” part, suppose that $T^\ast$ is quasi-Fredholm of degree $d$. From Lemma 1.3, $R(T^{j+1})$ is closed for all $j \geq d$. By the closed range theorem we know that $R(T^j)$ is closed for all $j \geq d$ and, we can get the following equation

\begin{equation}
R(T^{j+1}) \cap N(T^\ast) = N(T^j)^\perp \cap R(T)^\perp = (N(T^j) + R(T))^\perp
\end{equation}
for all \( j \geq d \). Since \( T(-j)(R(T(j+1))) = N(T^j) + R(T) \) for all \( j \geq d \), \( N(T^j) + R(T) \) is closed for all \( j \geq d \). From the fact that \( k_j(T^*) = 0 \) for all \( j \geq d \) and by equation (2.1), we can obtain that

\[
N(T^j) + R(T) = \frac{1}{j}((N(T^j) + R(T))^\perp) = \frac{1}{j}((N(T^d) + R(T))^\perp) = N(T^d) + R(T)
\]

for all \( j \geq d \). Therefore \( k_j(T) = 0 \) for all \( j \geq d \). By Lemma 1.3 again, it follows that \( T \) is quasi-Fredholm of degree \( d \).

P. Aiena in [2] gave some characterizations of the SVEP, for \( T \), at \( \lambda_0 \in \mathbb{C} \) in the case that \( \lambda_0 I - T \) is quasi-Fredholm.

**Proposition 2.2 ([2, Theorem 2.7]).** Let \( T \in \mathcal{B}(X) \) be quasi-Fredholm of degree \( d \). Then the following statements are equivalent:

(i) \( T \) has SVEP at 0;

(ii) \( \text{asc}(T) < \infty \);

(iii) \( \sigma_{\text{op}}(T) \) does not cluster at 0;

(iv) there exists \( n \in \mathbb{N} \) such that \( R(T^n) \) is closed and \( T|_{R(T^n)} \) is bounded below;

(v) \( T \) is left Drazin invertible;

(vi) there exists \( m \in \mathbb{N} \) such that \( H_0(T) = N(T^m) \);

(vii) \( H_0(T) \) is closed;

(viii) \( H_0(T) \cap K(T) = \{0\} \).

Dually, P. Aiena ([2]) gave some characterizations of the SVEP, for \( T^* \), at \( \lambda_0 \in \mathbb{C} \) in the case that \( \lambda_0 I - T \) is quasi-Fredholm.

**Proposition 2.3 ([2, Theorem 2.11]).** Let \( T \in \mathcal{B}(X) \) be quasi-Fredholm of degree \( d \). Then the following statements are equivalent:

(i) \( T^* \) has SVEP at 0;

(ii) \( \text{dsc}(T) < \infty \);

(iii) \( \sigma_{\text{sa}}(T) \) does not cluster at 0;

(iv) there exists \( n \in \mathbb{N} \) such that \( R(T^n) \) is closed and \( T|_{R(T^n)} \) is onto;

(v) \( X = H_0(T) + K(T) \);

(vi) there exists \( m \in \mathbb{N} \) such that \( K(T) = R(T^m) \).

We give new characterizations of the SVEP, for \( T \), at \( \lambda_0 \in \mathbb{C} \) in the case that \( \lambda_0 I - T \) is quasi-Fredholm.

**Theorem 2.4.** Let \( T \in \mathcal{B}(X) \) be quasi-Fredholm of degree \( d \). Then the conditions (i)-(viii) of Proposition 2.2 are equivalent to the following assertions:

1. \( N(T^\infty) \cap R(T^\infty) = \{0\} \);
2. \( N((T^\infty)^+) + R((T^\infty)^+) = X^* \);
3. \( N((T^\infty)^*) \cap R((T^\infty)^*) \) is weak*-dense in \( X^* \);
4. \( H_0(T^*) + K(T^*) \) is weak*-dense in \( X^* \);  
5. \( H_0(T^*) + R(T^*) \) is weak*-dense in \( X^* \).
Thus for all $n$. Since a classical theorem of T. Kato, $N(T^\infty)$ is quasi-Fredholm of degree $d$. Hence, by a classical theorem of T. Kato, $N(T^\infty) + R(T^\infty) = (N(T^\infty) \cap R(T^\infty))^\perp = X^*$. By the assumption of (2), we have $\cap_{n=1}^\infty \overline{R((T^*)^n)} = R((T^*)^\infty)$. Since $T$ is quasi-Fredholm of degree $d$, by Lemma 1.3 again, $R(T^n)$ is closed for all $n \geq d$. Hence $\cap_{n=1}^\infty R(T^n) = R(T^\infty)$. Thus $\cap_{n=1}^\infty \overline{R((T^*)^n)} = R((T^*)^\infty)$. By the assumption of (2), we have $X^* = N(T^\infty)^\perp + R(T^\infty)^\perp \subseteq R((T^*)^\infty)$. Therefore, $N((T^*)^\infty) + R((T^*)^\infty)$ is weak*-dense in $X^*$. (3) \Rightarrow (4) Since $T$ is quasi-Fredholm of degree $d$, by Theorem 2.1, $T^*$ is quasi-Fredholm of degree $d$. Hence, by [2, Lemma 2.6], $R((T^*)^\infty) = K(T^*)$ and the desired conclusion follows.

(4) \Rightarrow (5) Since $K(T^*) \subseteq R(T^*)$, the desired conclusion follows.

(5) \Rightarrow (i) See [1, Theorem 2.36].

The next result, which is dual to Theorem 2.4, give new characterizations of the SVEP, for $T^*$, at $\lambda_0 \in \mathbb{C}$ in the case that $\lambda_0 I - T$ is quasi-Fredholm.

**Theorem 2.5.** Let $T \in B(X)$ be quasi-Fredholm of degree $d$. Then the conditions (i)-(viii) of Proposition 2.3 are equivalent to the following assertions:

1. $N(T^\infty) + R(T^\infty) = X$;
2. $N(T^\infty)^\perp \cap R(T^\infty)^\perp = \{0\}$;
3. $N((T^*)^\infty) \cap R((T^*)^\infty) = \{0\}$;
(4) \( N(T^*) \cap R(T^*)^\infty = \{0\} \).

**Proof.** (ii) \( \Rightarrow \) (1) Let \( dsc(T) = q < \infty \). Then \( R(T^\infty) = R(T^q) \) and, by [1, Lemma 3.2], \( N(T^\infty) + R(T^\infty) = N(T^\infty) + R(T^q) = \{0\} \) and, hence, \( N(T^\infty) = X \).

Therefore, \( N(T^\infty) + R(T^\infty) = X \).

(1) \( \Rightarrow \) (2) Since \( N(T^\infty) + R(T^\infty) = X \), it follows that \( N(T^\infty)^\perp \cap R(T^\infty)^\perp = \{0\} \).

(2) \( \Rightarrow \) (3) Since \( T \) is quasi-Fredholm of degree \( d \), by Lemma 1.3, \( R(T^n) \) is closed for all \( n \geq d \). Hence

\[
R((T^*)^\infty) = \bigcap_{n=1}^{\infty} R((T^*)^n) = \bigcap_{n=1}^{\infty} N(T^n)^\perp = \left( \bigcup_{n=d}^{\infty} N(T^n)^\perp \right) = N(T^\infty)^\perp.
\]

(2.2)

Since \( N(T^\infty)^\perp \cap R(T^\infty)^\perp = \{0\} \), it follows that \( (N(T^\infty) + R(T^\infty))^\perp = N(T^\infty)^\perp \cap R(T^\infty)^\perp = \{0\} \), hence \( N(T^\infty) + R(T^\infty) = X \). Since \( T \) is quasi-Fredholm, by [2, Lemma 2.6], \( R(T^\infty) = K(T) \). Therefore \( X = N(T^\infty) + R(T^\infty) \subseteq H_0(T) + K(T) \subseteq X \), so \( H_0(T) + K(T) = X \). By Proposition 2.3, \( dsc(T) < \infty \). Hence \( asc(T^* \leq dsc(T) < \infty \). Let \( dsc(T) = q < \infty \). It is easy to see that

\[
N(T^*) = N(T^\infty) = R(T^\infty)^\perp = R(T^\infty)^\perp.
\]

Thus \( N(T^*) \subseteq N(T^\infty) \), the desired conclusion follows. (4) \( \Rightarrow \) (i) See [1, Theorem 2.22].

3. Components of quasi-Fredholm resolvent set

The following proposition, which was due to S. Grabiner, is a classical perturbation result concerning operators with eventual topological uniform descent.

**Proposition 3.1** ([9, Theorem 4.7]). Suppose that \( T \in B(X) \) has topological uniform descent for \( n \geq d \), and that \( S \in B(X) \) commutes with \( T \). If \( S \) is sufficiently small and invertible, then

(a) \( T + S \) is semi-regular;
(b) \( R((T + S)^\infty) = N(T^\infty) + R(T^\infty) \);
(c) \( N((T + S)^\infty) = N(T^\infty) \cap R(T^\infty) \).

For \( T \in B(X) \), the Kato type spectrum and the quasi-Fredholm spectrum are defined as \( \sigma_{kt}(T) = \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not of Kato type} \} \) and \( \sigma_{qf}(T) = \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not quasi-Fredholm} \} \), respectively. From [1, Theorem 1.42] it follows that \( \sigma_{qf}(T) \subseteq \sigma_{kt}(T) \). It is known that \( \sigma_{kt}(T) \) is closed, see [1, Corollary 1.45]. According to Proposition 3.1, it follows easily that \( \sigma_{qf}(T) \) is also closed.
The Kato type resolvent set and the quasi-Fredholm resolvent set are defined as \( \rho_{kt}(T) = \mathbb{C} \setminus \sigma_{kt}(T) \) and \( \rho_{qf}(T) = \mathbb{C} \setminus \sigma_{qf}(T) \), respectively. The sets \( \rho_{kt}(T) \) and \( \rho_{qf}(T) \) are open subsets of \( \mathbb{C} \), so they can be decomposed into connected disjoint open non-empty components.

M. Mbekhta and A. Ouahab ([16]) showed that the mappings

\[
(3.1) \quad \lambda \mapsto H_0(\lambda I - T) + K(\lambda I - T), \quad \lambda \mapsto \overline{H_0(\lambda I - T) \cap K(\lambda I - T)}
\]

are constant on the components of \( \rho_{kt}(T) \). P. Aiena and F. Villafañe ([3]) proved that the mappings (3.1) and the mappings

\[
(3.2) \quad \lambda \mapsto N((\lambda I - T)\infty) + R((\lambda I - T)\infty), \quad \lambda \mapsto N((\lambda I - T)\infty) \cap R((\lambda I - T)\infty)
\]

coincide, respectively, on the components of \( \rho_{kt}(T) \).

We generalize these results to the components of \( \rho_{qf}(T) \). We first show the constancy of the mappings (3.2) on the components of \( \rho_{qf}(T) \).

**Lemma 3.2.** Let \( T \in B(X) \) be quasi-Fredholm of degree \( d \). Then there exists an \( \varepsilon > 0 \) such that:

1. \( N((\lambda I - T)\infty) + R((\lambda I - T)\infty) = N(T\infty) + R(T\infty) \) for all \( 0 < |\lambda| < \varepsilon \);
2. \( N((\lambda I - T)\infty) \cap R((\lambda I - T)\infty) = N(T\infty) \cap R(T\infty) \) for all \( 0 < |\lambda| < \varepsilon \).

**Proof.** Since \( T \) is quasi-Fredholm of degree \( d \), \( T \) has topological uniform descent for \( n \geq d \). By Proposition 3.1, there exists an \( \varepsilon > 0 \) such that \( \lambda I - T \) is semi-regular, \( R((\lambda I - T)\infty) = N(T\infty) + R(T\infty) \) and

\[
N((\lambda I - T)\infty) = N(T\infty) \cap R(T\infty)
\]

for all \( 0 < |\lambda| < \varepsilon \). By [17, Theorem 1.2], \( N((\lambda I - T)\infty) \subseteq R((\lambda I - T)\infty) \). Moreover, by [1, Theorem 1.24] \( R((\lambda I - T)\infty) \) is closed, consequently, \( N((\lambda I - T)\infty) \subseteq R((\lambda I - T)\infty) \). Hence

\[
N((\lambda I - T)\infty) + R((\lambda I - T)\infty) = R((\lambda I - T)\infty) = N(T\infty) + R(T\infty)
\]

and

\[
N((\lambda I - T)\infty) \cap R((\lambda I - T)\infty) = N((\lambda I - T)\infty) \cap N(T\infty) \cap R(T\infty)
\]

\[\overset{[9, Lemma 3.6(d)]}{=} N(T\infty) \cap R(T\infty) \]

\[\overset{[2, Lemma 2.6]}{=} N(T\infty) \cap R(T\infty) \]

for all \( 0 < |\lambda| < \varepsilon \).

By using the classical Heine-Borel theorem, we obtain the following result.
Corollary 3.3. Let $T \in B(X)$. If $\Omega$ is a component of $\rho_{qf}(T)$ and $\lambda_0 \in \Omega$, then
\[ N((\lambda I - T)^\infty) + R((\lambda I - T)^\infty) = N((\lambda_0 I - T)^\infty) + R((\lambda_0 I - T)^\infty) \]
and
\[ N((\lambda I - T)^\infty) \cap R((\lambda I - T)^\infty) = N((\lambda_0 I - T)^\infty) \cap R((\lambda_0 I - T)^\infty) \]
for all $\lambda \in \Omega$.

Therefore, the mappings
\[ \lambda \mapsto N((\lambda I - T)^\infty) + R((\lambda I - T)^\infty) \]
and
\[ \lambda \mapsto N((\lambda I - T)^\infty) \cap R((\lambda I - T)^\infty) \]
are constant on the components of $\rho_{qf}(T)$.

The following theorem extends [3, Theorem 2.1].

Theorem 3.4. Let $\lambda I - T$ be quasi-Fredholm. Then
\begin{enumerate}
\item $N((\lambda I - T)^\infty) + R((\lambda I - T)^\infty) = H_0(\lambda I - T) + K(\lambda I - T)$.
\item $N((\lambda I - T)^\infty) \cap R((\lambda I - T)^\infty) = H_0(\lambda I - T) \cap K(\lambda I - T)$.
\end{enumerate}

Proof. Without loss of generality, we may assume that $\lambda = 0$.

Since $T$ is quasi-Fredholm of degree $d$, by Theorem 2.1, $T^*$ is also quasi-Fredholm of degree $d$. Then by [2, Lemma 2.6], $R(T^\infty) = K(T)$ and $R((T^*)^\infty) = K(T)$. By [1, Theorem 1.70], $N(T^\infty) \subseteq H_0(T) \subseteq K(T^*)$. By equation (2.2), $\frac{N(T^\infty)}{T^*} = R((T^*)^\infty) = K(T^*)$. So, $\overline{N(T^\infty)} = \frac{1}{K(T^*)}$. Hence, $H_0(T^\infty) = H_0(T)$. Consequently, $N(T^\infty) \cap R(T^\infty) = H_0(T) \cap K(T)$. This shows (2).

On one hand, $N(T^\infty) + R(T^\infty) \subseteq H_0(T) + R(T^\infty) = H_0(T) + K(T)$. On the other hand,
\[ H_0(T) + K(T) \subseteq \overline{H_0(T)} + K(T) = \overline{N(T^\infty)} + R(T^\infty) \]
by [9, Lemma 3.6(a)]. Therefore, $N(T^\infty) + R(T^\infty) = H_0(T) + K(T)$. This shows (1).

By Corollary 3.3 and Theorem 3.4, we obtain the next result which generalizes the corresponding result of M. Mbekhta and A. Ouahab ([16]).

Corollary 3.5. The mappings
\[ \lambda \mapsto H_0(\lambda I - T) + K(\lambda I - T) \]
and
\[ \lambda \mapsto H_0(\lambda I - T) \cap K(\lambda I - T) \]
are constant on the components of $\rho_{qf}(T)$. 

Combining Theorem 2.4 with Corollary 3.3, the following classification is obtained.

**Theorem 3.6.** Let $T \in \mathcal{B}(X)$ and $\Omega$ a component of $p_{qf}(T)$. Then the following alternative holds:

1. $T$ has the SVEP at every point of $\Omega$. In this case, $\text{asc}(\lambda I - T) < \infty$ for all $\lambda \in \Omega$. Moreover, $\sigma_{ap}(T)$ does not have limit points in $\Omega$; every point of $\Omega$, except possibly for at most countably many isolated points, is not an eigenvalue of $T$.

2. $T$ has the SVEP at no point of $\Omega$. In this case, $\text{asc}(\lambda I - T) = \infty$ for all $\lambda \in \Omega$. Every point of $\Omega$ is an eigenvalue of $T$.

**Proof.** (1) Suppose that $T$ has the SVEP at $\lambda_0 \in \Omega$. Then by Proposition 2.2, $\text{asc}(\lambda_0 I - T) < \infty$, so $N((\lambda_0 I - T)^\infty)$ is closed. By Theorem 2.4, $N((\lambda_0 I - T)^\infty) \cap R((\lambda_0 I - T)^\infty) = N((\lambda_0 I - T)^\infty) \cap R((\lambda_0 I - T)^\infty) = \{0\}$. By Corollary 3.3 the mapping

$$\lambda \mapsto N((\lambda I - T)^\infty) \cap R((\lambda I - T)^\infty)$$

is constant on $\Omega$, so $N((\lambda I - T)^\infty) \cap R((\lambda I - T)^\infty) = \{0\}$ for all $\lambda \in \Omega$. Thus, $N((\lambda I - T)^\infty) \cap R((\lambda I - T)^\infty) = \{0\}$ for all $\lambda \in \Omega$. Therefore, again by Theorem 2.4, $T$ has the SVEP at every $\lambda \in \Omega$. This is equivalent, by Proposition 2.2, to saying that $\text{asc}(\lambda I - T) < \infty$ for all $\lambda \in \Omega$. Moreover, from Proposition 2.2, $\sigma_{ap}(T)$ does not have limit points in $\Omega$ and, consequently, every point of $\Omega$ is not an eigenvalue of $T$, except possibly for at most countably many isolated points.

(2) Suppose that $T$ has the SVEP at no point of $\Omega$. Then by Proposition 2.2, $\text{asc}(\lambda I - T) = \infty$ for all $\lambda \in \Omega$ and, consequently, every point of $\Omega$ is an eigenvalue of $T$. $\blacksquare$

Recall that $\lambda \in \mathbb{C}$ is said to be a deficiency value for if $\lambda I - T$ is not surjective. Combining Theorem 2.5 with Corollary 3.3, the following classification is obtained.

**Theorem 3.7.** Let $T \in \mathcal{B}(X)$ and $\Omega$ a component of $p_{qf}(T)$. Then the following alternative holds:

1. $T^*$ has the SVEP at every point of $\Omega$. In this case, $\text{dsc}(\lambda I - T) < \infty$ for all $\lambda \in \Omega$. Moreover, $\sigma_{mn}(T)$ does not have limit points in $\Omega$; every point of $\Omega$, except possibly for at most countably many isolated points, is not a deficiency value of $T$.

2. $T^*$ has the SVEP at no point of $\Omega$. In this case, $\text{dsc}(\lambda I - T) = \infty$ for all $\lambda \in \Omega$. Every point of $\Omega$ is a deficiency value of $T$.

**Proof.** (1) Suppose that $T^*$ has the SVEP at $\lambda_0 \in \Omega$. Then, by Theorem 2.5, $N((\lambda_0 I - T)^\infty) + R((\lambda_0 I - T)^\infty) = X$. By Corollary 3.3 the mapping

$$\lambda \mapsto R((\lambda I - T)^\infty) + N((\lambda I - T)^\infty)$$

is constant on $\Omega$, so $R((\lambda I - T)^\infty) + N((\lambda I - T)^\infty) = X$ for all $\lambda \in \Omega$. Thus, $R((\lambda I - T)^\infty) + N((\lambda I - T)^\infty) = X$ for all $\lambda \in \Omega$. Therefore, again by Theorem 2.5, $T^*$ has the SVEP at every $\lambda \in \Omega$. This is equivalent, by Proposition 2.2, to saying that $\text{dsc}(\lambda I - T) < \infty$ for all $\lambda \in \Omega$. Moreover, from Proposition 2.2, $\sigma_{mn}(T)$ does not have limit points in $\Omega$ and, consequently, every point of $\Omega$ is not a deficiency value of $T$, except possibly for at most countably many isolated points.

(2) Suppose that $T^*$ has the SVEP at no point of $\Omega$. Then by Proposition 2.2, $\text{dsc}(\lambda I - T) = \infty$ for all $\lambda \in \Omega$ and, consequently, every point of $\Omega$ is a deficiency value of $T$. $\blacksquare$
is constant on $\Omega$, so $R((\lambda I - T)^\infty) + N((\lambda I - T)^\infty) = X$ for all $\lambda \in \Omega$. Therefore, again by Theorem 2.5, $T^*$ has the SVEP at every $\lambda \in \Omega$. This is equivalent, by Proposition 2.3, to saying that $dsc(\lambda I - T) < \infty$ for all $\lambda \in \Omega$. Moreover, from Proposition 2.3, $\sigma_{su}(T)$ does not have limit points in $\Omega$ and, consequently, every point of $\Omega$ is not a deficiency value of $T$, except possibly for at most countably many isolated points.

(2) Suppose that $T^*$ has the SVEP at no point of $\Omega$. Then by Proposition 2.3, $dsc(\lambda I - T) = \infty$ for all $\lambda \in \Omega$ and, consequently, every point of $\Omega$ is a deficiency value of $T$.

At last, as an application, we give a characterization of finite-dimensional Banach spaces.

**Corollary 3.8.** Let $X$ be a Banach space. The following assertions are equivalent:

1. $X$ is finite-dimensional;
2. $\sigma_{qf}(T) = \emptyset$ for every $T \in B(X)$.

**Proof.** (1) $\implies$ (2) Clear.

(2) $\implies$ (1) For every $T \in B(X)$, since $\sigma_{qf}(T) = \emptyset$, $\rho_{qf}(T)$ has only one component $\Omega = \mathbb{C}$. Then by Theorem 3.7, $\sigma_{dsc}(T) := \{\lambda \in \mathbb{C} : dsc(T - \lambda) = \infty\} = \emptyset$. Consequently, by [5, Corollary 1.10], $X$ is finite-dimensional.

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