SPANS OF CONTINUA RELATED TO INDENTED CIRCLES

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Abstract. Let $X$ be a special type of simple closed curve in the plane known as an indented circle. Let $Y$ be a continuum which is contained in $X \cup V$ where $V$ is the bounded component of $\mathbb{R}^2 - X$. We show that $\tau(Y) \leq \tau(X)$ where $\tau$ is the span $\sigma$, surjective span $\sigma^*$, semispan $\sigma_0$, surjective semispan $\sigma^*_0$, symmetric span $s$, or the surjective symmetric span $s^*$.

1. Introduction

The span of a metric continuum was originally defined by A. Lelek (see [L1], p. 209). Later variations of the span were defined (cf [L2, L3, D]). In general it is difficult to calculate the spans of a particular geometric object. Also, it is not clear how the various spans of related objects compare to each other. The following question on this topic was asked by H. Cook [C].

If $X_1$ is a plane simple closed curve and $X_2$ is a simple closed curve which is contained in the bounded component of $\mathbb{R}^2 - X_1$ then is $\sigma(X_2) < \sigma(X_1)$?

There have been various partial results on this question (cf [W1, W2, W3, T1, T2, DF]). In this paper we show the following:

If $X$ is a particular type of a simple closed curve known as an indented circle and $Y$ is any continuum contained in $X \cup V$ where $V$ is the bounded component of $\mathbb{R}^2 - X$, then $\tau(Y) \leq \tau(X)$ where $\tau$ is any of the various spans.

2. Preliminaries

The standard projections $p_1, p_2 : X \times X \to X$ are mappings defined by $p_1(x, y) = x$ and $p_2(x, y) = y$ for $(x, y) \in X \times X$.
Let $X$ be a nonempty metric space. The surjective span $\sigma^*(X)$ of $X$ is the least upper bound of real number $\alpha$ such that there exist nonempty connected sets $C_\alpha \subset X \times X$ with $d(x, y) \geq \alpha$ for $(x, y) \in C_\alpha$ and

$$(\sigma^*) \quad p_1(C_\alpha) = p_2(C_\alpha) = X.$$  

Relaxing condition $(\sigma^*)$ to the conditions

$$(\sigma) \quad p_1(C_\alpha) = p_2(C_\alpha),$$  

$$(\sigma_0) \quad p_2(C_\alpha) = X,$$  

$$(\sigma_0) \quad p_1(C_\alpha) \subset p_2(C_\alpha),$$  

we obtain the definitions of the span $\sigma(X)$, the surjective semispan $\sigma_0^*(X)$, and the semispan $\sigma_0(X)$ of $X$, respectively.

If to condition $(\sigma^*)$ we add the condition that $C^* = (C^*)^{-1}$ we get $s^*(X)$ the surjective symmetric span. If to condition $(\sigma)$ we add the condition that $C^* = (C^*)^{-1}$ we get $s(X)$ the symmetric span.

In [W1] we defined a particular type of closed curve which we called an indented circle. The construction is given below.

We start with a circle $S$ in the complex plane of radius $r$ and center the origin $O$. Also, we will consider $X$ as a subset of the real plane whenever this will simplify the exposition.

We choose angles $\theta_1, \ldots, \theta_n$ such that

$$0 < \theta_1 < \theta_2 < \cdots < \theta_n < \pi.$$  

We choose $4n$ more angles $\theta^1_j, \theta^2_j, \theta^3_j, \theta^4_j$, for $j = 1, 2, \ldots, n$ such that

$$0 \leq \theta^1_1 \leq \theta^2_1 \leq \cdots \leq \theta^4_1 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_n \leq \theta^4_n \leq 2\pi,$$

$$\pi \leq \theta^2_1 \leq \theta^3_1 \leq \cdots \leq \theta^4_1 \leq \theta_1 + \pi \leq \theta_2 + \pi \leq \cdots \leq \theta_n + \pi \leq \theta_1 + 2\pi \leq \theta_2 + 2\pi \leq \cdots \leq \theta_n + 2\pi,$$

either $\theta^1_j = \theta_j = \theta^2_j$ or $\theta^3_j < \theta_j < \theta^4_j$ for $j = 1, 2, \ldots, n$,

either $\theta^3_j = \theta_j + \pi = \theta^4_j$ or $\theta^3_j < \theta_j + \pi < \theta^4_j$ for $j = 1, 2, \ldots, n$,

$$\theta_j + \alpha^2_j \leq \theta_{j+1} - \alpha^1_{j+1} \text{ for } j = 1, 2, \ldots, n - 1,$$

where $\alpha^1_j = \text{Max}\{\theta_j - \theta^1_j, \theta_j + \pi - \theta^2_j\}$, $\alpha^2_j = \text{Max}\{\theta^3_j - (\theta_j + \pi), \theta^4_j - \theta_j\}$.

Let $r_j = r e^{i \theta_j}$, $q_j = r e^{i (\theta_j + \pi)}$, $x_j = r e^{i \theta^1_j}$, $y_j = r e^{i \theta^2_j}$, $s_j = r e^{i \theta^3_j}$, and $t_j = r e^{i \theta^4_j}$ for $j = 1, 2, \ldots, n$.

We represent the straight line interval in the plane with endpoints $a$ and $b$ by $\overline{ab}$. Pick points $v_j, w_j \neq O$ where $v_j \in \overline{Oq_j}$ and $w_j \in \overline{Oq_j}$ for $j = 1, 2, \ldots, n$. We must choose $v_j$ and $w_j$ such that the following restrictions are satisfied for $j = 1, 2, \ldots, n$. If $\theta^1_j = \theta^2_j$, then $v_j = r_j$. If $\theta^3_j = \theta^4_j$, then $w_j = q_j$.

Otherwise, we must choose $v_j$ and $w_j$ so that the following conditions are satisfied. If $\theta^1_j \neq \theta^2_j$, then the smaller angles formed by the following pairs of line intervals, the pair $\overline{x_jv_j}$ and $\overline{y_jv_j}$, and the pair $\overline{r_jv_j}$ and $\overline{t_jv_j}$ must be no greater than $90^\circ$. If $\theta^3_j \neq \theta^4_j$, then the smaller angles formed by the following...
pairs of line intervals, the pair $w_j q_j$ and $w_j t_j$ must be no greater than $90^\circ$. We will refer to these conditions as the angle conditions.

For each $j$, when $\theta_j^1 \neq \theta_j^2$, the shorter arc on $S$ with endpoints $x_j$ and $y_j$ is replaced by $\overline{x_j v_j} \cup \overline{v_j y_j}$ and when $\theta_j^3 \neq \theta_j^4$, the shorter arc on $S$ with endpoints $s_j$ and $t_j$ is replaced by $\overline{s_j w_j} \cup \overline{w_j t_j}$.

We refer to both $\overline{x_j v_j} \cup \overline{v_j y_j}$ and $\overline{s_j w_j} \cup \overline{w_j t_j}$ as indentations of $X$ for $j = 1, 2, \ldots, n$. We refer to $v_j$ and $w_j$ as the vertices of the corresponding indentations. The space $X$ consists of the remaining points of $S$ and the added indentations.

From the construction of $X$, we see that it is a simple closed curve. We call each such simple closed curve $X$ an indented circle (see Fig. 1).

Let $d_j$ be the point on $\overline{x_j v_j}$ closest to $t_j$, $c_j$ the point on $\overline{v_j y_j}$ closest to $s_j$, $b_j$ the point on $\overline{s_j w_j}$ closest to $y_j$, and $a_j$ be the point on $\overline{w_j t_j}$ closest to $x_j$, for $j = 1, 2, \ldots, n$.

Let $d'_j = d(d_j, t_j)$, $c'_j = d(c_j, s_j)$, $b'_j = d(b_j, y_j)$, and $a'_j = d(a_j, x_j)$, for $j = 1, 2, \ldots, n$. We call the number

$$s_X = \text{Min}\{\text{Max}\{a'_j, d'_j\}, \text{Min}\{b'_j, c'_j\} : j = 1, 2, \ldots, n\}$$

the indentation spread of the indented circle $X$.

In [W1] we proved the following:

**Theorem 2.1.** If $X$ is an indented circle and $s_X$ is the indentation spread of $X$, then

$$\sigma(X) = \sigma_0(X) = \sigma^*(X) = \sigma_0^*(X) = s_X.$$
Though it was not stated in this theorem, the proof also gives us that 
\( s(X) = s^*(X) \), since the continuum \( C \subset X \times X \) constructed in the proof of Theorem 2.1 is such that \( C = C^{-1} \) and \( p_1(C) = p_2(C) = X \).

3. Main Result

**Theorem 3.1.** If \( X \) is an indented circle, \( V \) is the bounded component of \( R^2 - X \), and \( Y \) is a continuum such that \( Y \subset X \cup V \) then \( \tau(Y) \leq \tau(X) \) where \( \tau = \sigma, \sigma_0, \sigma^*, \sigma^*_0, s, s^* \).

**Proof.** Suppose that \( X^* \) is an indented circle that has \( n \) indentations. We know from [W1, Th2.1] that
\[
\sigma(X^*) = \sigma_0(X^*) = \sigma^*_0(X^*) = \sigma^*(X^*) = s_X,
\]
where \( s_{X^*} \) is the indentation spread of \( X^* \). For some \( j \),
\[
s_{X^*} = \max\{\min\{a_j', d_j'\}, \min\{c_j', b_j'\}\}.
\]
Let \( r : R^2 \to R^2 \) be the function that rotates the plane by an angle of \( \frac{\pi}{2} - \theta \) about the origin; so,
\[
r(v_j) = r_v e^{i\frac{\pi}{2}}
\]
where \( v_j = r_v e^{i\theta_j} \) and
\[
r(w_j) = r_w e^{i\Delta \theta}
\]
where \( w_j = r_w e^{i(\theta_j + \pi)} \).
Let
\[
r(x_j) = x, r(y_j) = y, r(s_j) = s, r(t_j) = t,
\]
\[
r(a_j) = a, r(b_j) = b, r(c_j) = c, \text{ and } r(d_j) = d.
\]
Let
\[
d(x, a) = a', d(t, d) = d', d(s, c) = c' \text{ and } d(y, d) = d'.
\]
Let
\[
X = \overline{\{x\}} \cup \overline{\{y\}} \cup \overline{\{s\}} \cup \overline{\{t\}} \cup \{r e^{i\theta} | \theta \in [0, \theta_1] \cup [\theta_y, \theta_2] \cup [\theta_t, 2\pi]\}
\]
where \( 0 \leq \theta_x \leq \theta_y \leq \theta_s \leq \theta_t \leq 2\pi \) and \( x = re^{i\theta_x}, y = re^{i\theta_y}, s = re^{i\theta_s} \) and \( t = re^{i\theta_t} \). From [W1, Th2.1] we see that
\[
s_{X^*} = \tau(X^*) = \tau(X) = s_X \text{ for } \tau = \sigma, \sigma_0, \sigma^*, \sigma^*_0.
\]
From the proof of the theorem we also see that
\[
s_{X^*} = \tau(X^*) = \tau(X) = s_X \text{ for } \tau = \sigma, \sigma_0, \sigma^*, \sigma^*_0
\]
where \( \tau = s \) or \( s^* \). Also, if \( Y \) is a continuum such that \( Y \subset X^* \cup V^* \) where \( V^* \) is the bounded component of \( R^2 - X^* \) then \( Y \subset X \cup V \) where \( V \) is the
bounded component of $R^2 - X$. So, without loss of generality we can assume that our indented circle is $X$ rather than $X^*$. Note that either

a) $0 < \theta_x < \frac{\pi}{2} < \theta_y < \pi < \theta_s < \frac{3\pi}{2} < \theta_t < 2\pi$,

b) $0 < \theta_x = \frac{\pi}{2} = \theta_y < \pi < \theta_s < \frac{3\pi}{2} < \theta_t < 2\pi$, or

c) $0 < \theta_x < \frac{\pi}{2} < \theta_y < \pi < \theta_s = \frac{3\pi}{2} = \theta_t < 2\pi$.

We first consider the situation in a) we have sixteen cases to consider.

A1

$$s_X = \max\{a', b'\}$$

$$a \neq w \neq b$$

A2

$$s_X = \max\{c', d'\}$$

$$c \neq v \neq d$$

If we rotate $X$ by $180^\circ$ in $R^2$ about the origin then case A2 is comparable to case A1.

B1

$$s_X = \max\{a', c'\}$$

$$a \neq w, c \neq v$$

B2

$$s_X = \max\{d', b'\}$$

$$d \neq v, b \neq w$$

If we rotate $X$ by $180^\circ$ in $R^2$ about the y-axis then case B2 is comparable to case B1.

C1

$$s_X = \max\{a', b'\}$$

$$a = b = w$$

C2

$$s_X = \max\{c, d\}$$

$$c = v = d$$

If we rotate $X$ by $180^\circ$ in $R^2$ about the origin then case C2 is comparable to case C1.

D1

$$s_X = \max\{a', c'\}$$

$$a = w, c = v$$

D2

$$s_X = \max\{d', b'\}$$

$$d = v, b = w$$

If we rotate $X$ by $180^\circ$ in $R^2$ about the y-axis then case D2 is comparable to case D1.

E1

$$s_X = \max\{a', b'\}$$

$$a = w, b \neq w$$
E2

\[ s_X = \max\{a', b'\} \]
\[ a \neq w, b = w \]

If we rotate \( X \) by 180° in \( R^2 \) about the y-axis then case E2 is comparable to case E1.

E3

\[ s_X = \max\{d', c'\} \]
\[ d = v, c \neq v \]

If we rotate \( X \) by 180° in \( R^2 \) about the x-axis then case E3 is comparable to case E1.

E4

\[ s_X = \max\{d', c'\} \]
\[ d \neq v, c = v \]

If we rotate \( X \) by 180° in \( R^2 \) about the origin then case E4 is comparable to case E1.

F1

\[ s_X = \max\{a', c'\} \]
\[ a = w, c \neq v \]

F2

\[ s_X = \max\{a', c'\} \]
\[ a \neq w, c = v \]

If we rotate \( X \) by 180° in \( R^2 \) about the origin then case F2 is comparable to case F1.

F3

\[ s_X = \max\{d', b'\} \]
\[ d = v, b \neq w \]

If we rotate \( X \) by 180° in \( R^2 \) about the x-axis then case F3 is comparable to case F1.

F4

\[ s_X = \max\{d', b'\} \]
\[ d \neq v, b = w \]

If we rotate \( X \) by 180° in \( R^4 \) about the y-axis then case F4 is comparable to case F1.

Now we consider the situations in b) and c).

G1

\[ s_X = d(v, w) \]
\[ v = re^{i\frac{\pi}{4}} \]

G2

\[ s_X = d(v, w) \]
\[ w = re^{i\frac{3\pi}{4}} \]
If we rotate $X$ by $180^\circ$ in $R^2$ about the origin then case $G2$ is comparable to case $G1$. So, in order to prove the theorem we just need to examine cases $A1, B1, C1, D1, E1, F1$ and $G1$.

In order to do this we first define functions $p_\varepsilon$ and $q_\varepsilon$ under various conditions. We define continuous functions $p_\varepsilon$ and $q_\varepsilon$ where

$$p_\varepsilon : R \to \overrightarrow{wt}$$

$$q_\varepsilon : L \to \overrightarrow{ws}$$

$$R = \{(x_1, y_1) \in X \cup V|x_1 \geq 0\},$$

and

$$L = \{(x_1, y_1) \in X \cup V|x_1 \leq 0\}.$$

First we define $p_\varepsilon$ in two different cases.

$p_\varepsilon$ Case 1: $a \neq w$

We define $p_\varepsilon$ for $\varepsilon$ where $0 < \varepsilon < \frac{1}{4} \min\{d(w, a), d(w, v)\}$. Pick $m \in \overrightarrow{ws}$ such that $0 < d(w, m) < \varepsilon$. Let $n \in \overrightarrow{wt}$ such that $\overrightarrow{mn}$ is perpendicular to $\overrightarrow{wt}$.

Let $P_1$ be the portion of the plane which is bound by

$$B_1 = \overrightarrow{tm} \cup \overrightarrow{mn} \cup \overrightarrow{nX} \cup \{re^{i\theta}|0 \leq \theta \leq \theta_x, \theta_t \leq \theta \leq 2\pi\}$$

together with its boundary $B_1$.

For $0 \leq t \leq 1$, let $n_t = tn + (1 - t)v$, $m_t = tm + (1 - t)w$, and $R_t = \overrightarrow{mn_t}$.

We define $p_\varepsilon : R \to \overrightarrow{wt}$ as follows:

a) $p_\varepsilon/P_1$ is the perpendicular projection of $P_1$ into $\overrightarrow{wt}$.

b) $p_\varepsilon/R_t$ is the constant function which sends each point of $R_t$ to $m_t$ for $0 \leq t \leq 1$.

Observation 1: If $x_1$ and $x_2 \in P_1$ where $\overrightarrow{x_1x_2}$ is perpendicular to $\overrightarrow{wt}$ then $d(x_1, x_2) \leq a'$.

Proof. To see this, let $L_x$ be the line through $x$ which is parallel to $\overrightarrow{wt}$. Note that

$$P_1 \subseteq L_x \cup \overrightarrow{wt} \cup V(L_x, \overrightarrow{wt})$$

where $V(L_x, \overrightarrow{wt})$ is the portion of the plane bound by $L_x$ and $\overrightarrow{wt}$. Consequently, if $x_1$ and $x_2 \in P_1$ where $\overrightarrow{x_1x_2}$ is perpendicular to $\overrightarrow{wt}$ then $d(x_1, x_2) \leq d(a, x) = a'$. \[\square\]

Observation 2: If $x_1, x_2 \in R_t$ then $d(x_1, x_2) \leq a' + 2\varepsilon.$
PROOF. Note that
\[
\begin{align*}
  d(x_1, x_2) &\leq d(m_t, n_t) \\
  &\leq d(m_t, m) + d(m, n_t) \\
  &\leq \varepsilon + d(m, n_t) \\
  &\leq \varepsilon + \max\{d(m, v), d(m, n)\} \\
  &\leq \varepsilon + \max\{a', d(v, w) + \varepsilon\} \\
  &\leq \varepsilon + \max\{a', a' + \varepsilon\} \\
  &= a' + 2\varepsilon.
\end{align*}
\]

(*) So we see that for \(y' \in p_\varepsilon(R)\), \(\text{diam}(p_\varepsilon^{-1}\{y'\}) \leq a' + 2\varepsilon\).

\(p_\varepsilon\) CASE 2: \(a = w\)

We define \(p_\varepsilon\) for \(\varepsilon\) where \(0 \leq \varepsilon \leq \frac{1}{2} \min\{d(v, w), d(w, t)\}\). Pick \(m \in \overline{wt}\) such that \(0 < d(w, m) < \varepsilon\). Pick \(m_1 \in \overline{m1}\) such that \(0 < d(m_1, m_2) < \varepsilon\). Let \(m_2 \in X\) such that \(\overline{m1m2}\) is perpendicular to \(\overline{wt}\). Let \(m_2 = re^{i\theta m_2}\). Either \(0 \leq \theta m_2 < \theta x\) or \(\theta t < \theta m_2 < 2\pi\).

Let

\[
B_1 = \overline{m1t} \cup \overline{m1m2} \cup X_{tm2}
\]

where \(X_{tm2} = \{re^{i\theta} | \theta \in [\theta t, \theta m_2] \text{ if } \frac{\pi}{2} < \theta m_2 < 2\pi, \text{ or } \theta \in [\theta t, 2\pi) \cup [0, \theta m_2] \text{ if } 0 \leq \theta m_2 < \frac{\pi}{2}\}\).

Let \(P_1\) be the portion of the plane bound by \(B_1\) together with its boundary \(B_1\). Let

\[
X_{m2x} = \{re^{i\theta} | \theta \in [\theta m_2, \theta x] \text{ if } 0 \leq \theta m_2 < \frac{\pi}{2} \text{ or } \theta \in [\theta m_2, 2\pi) \cup [0, \theta x] \text{ if } \frac{3\pi}{2} < \theta m_2 < 2\pi\}.
\]

Let \(r : [0, 1] \to X_{m2x}\) be a continuous surjective function where \(r(0) = m_2\) and \(r(1) = x\). Let

\[
m_{1t} = (1 - t)m_1 + tm \text{ and } M_t = m_{1t}r(t).
\]

For \(0 \leq t \leq 1\) let

\[
n_t = tx + (1 - t)v, \\
m_t = tm + (1 - t)w, \text{ and } R_t = m_tn_t.
\]

We define \(p_\varepsilon : R \to \overline{wt}\) as follows:

a) \(p_\varepsilon / P_1\) is the perpendicular projection of \(P_1\) into \(\overline{m1t}\),

b) \(p_\varepsilon / M_t\) is the constant function which sends each point of \(M_t\) to the point \(m_{1t}\),
c) $p \times R_l$ is the constant function which sends each point of $R_l$ to the point $m_l$.

Observation 3: If $x_1, x_2 \in M_l$ then $d(x_1, x_2) \leq a' + 2\varepsilon$.

Proof. First we observed that the function $d^* : [0, 1] \to R^+$ given by
$d^*(t) = d(m_1, r(t))$ is increasing. To see this compare the two triangles
$\triangle Om_1r(0)$ and $\triangle Om_1r(t)$ where $0 < t \leq 1$. Let $\alpha$ be the smaller angle
between $Om_1$ and $Or(t)$ for $0 \leq t \leq 1$. Note that $Om_1$ is of fixed length, $r$ is
the length of $Or(t)$ for each $0 \leq t \leq 1$, and $\alpha' > \alpha$ for $0 \leq t' < t'' \leq 1$. So,
$d^*(m_1, r(t))$ increases as $t$ increases. Hence, $d(m_1, m_2) < d(m_1, x) < a' + 2\varepsilon$.

In this case as in case 1, we see that for $y' \in p \times (R),$
$$diam(p^{-1}\{y'\}) \leq a' + 2\varepsilon.$$ Now we define $q_\varepsilon$ in four different cases.

$q_\varepsilon$ Case 1: $b \neq w$
We define $q_\varepsilon$ for $\varepsilon$ where $0 < \varepsilon < \frac{1}{4} \min\{d(w, b), d(w, v)\}$. Pick $p \in \overline{sw}$
such that $0 < d(w, p) < \varepsilon$. Let $u \in \overline{wv}$ such that $\overline{pu}$ is perpendicular to $\overline{wv}$.
For $0 \leq t \leq 1$, let
$$p_t = tp + (1 - t)w,$$
$$u_t = tu + (1 - t)v,$$ and
$$L_t = \overline{pu_t}.$$ Let $P_2$ be the portion of the plane which is bound by
$$B_2 = \overline{sp} \cup \overline{pt} \cup uy \cup \{re^{i\theta} \theta_y < \theta \leq \theta_s\}$$
together with its boundary $B_2$. We define $q_\varepsilon : L \to \overline{ws}$ as follows
a) $q_\varepsilon / L_t$ is the constant function which sends each point of $L_t$ to $P_t$,
b) $q_\varepsilon / P_2$ is the perpendicular projection of $P_2$ into $\overline{ws}$.

From previous observations we can see that for $y' \in q_\varepsilon(L),$
$$diam(q_\varepsilon^{-1}\{y'\}) \leq b' + 2\varepsilon.$$
Let $X_{p_2y} = \{re^{i\theta}| \theta \in [\theta_y, \theta_{p_2}]\}$. Let $l : [0, 1] \to X_{p_2y}$ be a continuous surjective function where $l(0) = p_2$, $l(1) = y$. Let $p_{1t} = (1 - t)p_1 + tp$ and $U_t = p_{1t}l(t)$. Let $P_2$ be the portion of the plane which is bound by $B_2 = \overline{\text{sp}} \cup \overline{\text{sp}_2} \cup X_{p_2s}$ where $X_{p_2s} = \{re^{i\theta}| \theta_{p_2} \leq \theta \leq \theta_s\}$ together with its boundary $B_2$. We define $q_x : L \to \overline{w}\overline{s}$ as follows

a) $q_{c/L}$ is the constant function which takes each point of $L_t$ to the point $p_t$.
b) $q_{c/U}$ is the constant function which sends each point of $U_t$ to the point $p_{1t}$.
c) $q_{c/p_2}$ is the perpendicular projection of $P_2$ into $\overline{w}\overline{s}$.

From previous observations we can see that for $y' \in q_x(L)$,
\[
\text{diam}(q_x^{-1}(y')) \leq b' + 2\varepsilon.
\]

$q_x$. Case 3: $c \neq v$

We define $q_x$ for $\varepsilon$ where $0 < \varepsilon < \frac{1}{4}\min\{d(v, w), d(v, c)\}$. Pick $u \in \overline{vw}$ such that $0 < d(u, v) < \varepsilon$. Let $p \in \overline{vw}$ such that $\overline{pu}$ is perpendicular to $\overline{vy}$. Let
\[
 u_t = tu + (1 - t)v,
 p_t = tp + (1 - t)w, \quad \text{and}
 L_t = \overline{u_tp}.
\]

Let $P_2$ be the portion of the plane bound by
\[
 B_2 = \overline{vy} \cup \overline{vw} \cup \overline{vy} \cup X_{ys}
\]
where $X_{ys} = \{e^{i\theta}| \theta_{y} \leq \theta \leq \theta_s\}$ together with its boundary $B_2$. Let $q : L \to \overline{vy}$ be defined as follows

a) $q/L_t$ is the constant function that sends each point of $L_t$ to $u_t$.
b) $q/P_2$ is the perpendicular projection of $P_2$ into $\overline{vy}$.

Let $q(L) = \overline{vy}$. Let $q^* : \overline{vy} \to \overline{vw}$ be a surjective continuous map such that $q^*(v) = w$, $q^*(u) = p$, $q^*(y') = s$. Let $q_x = q^* \circ q$. From previous observations it is clear that if $y' \in \overline{vw}$, $\text{diam}q_x^{-1}(y') \leq c' + 2\varepsilon$.

$q_x$. Case 4: $v = c$

We define $q_x$ for $\varepsilon$ where $0 < \varepsilon < \frac{1}{4}\min\{d(v, w), d(v, y)\}$. Pick $u \in \overline{vy}$ such that $0 < d(v, u) < \varepsilon$. Pick $u_1 \in \overline{vy}$ such that $0 < d(u, u_1) < \varepsilon$. Let $u_2 \in X$ such that $\overline{u_1u_2}$ is perpendicular to $\overline{vy}$. Let $u_2 = re^{i\theta_2}$. Let
\[
 u_t = tu + (1 - t)v,
 p_t = ts + (1 - t)w.
\]

Let $L_t = \overline{u_t}$. Let $X_{u_2s} = \{re^{i\theta}| \theta_{u_2} \leq \theta \leq \theta_s\}$. Let $l : [0, 1] \to X_{u_2s}$ be a continuous surjective function where $l(0) = u_2$ and $l(1) = s$. Let $u_{1t} = (1 - t)u_1 + tu$ and $U_t = u_{1t}l(t)$. Let $B_2 = \overline{uy} \cup \overline{u_2y} \cup X_{ys}$ where $X_{ys} = \{re^{i\theta}| \theta_{y} \leq \theta \leq \theta_s\}$.
\[ \{ r e^{i \theta} | \theta_y \leq \theta \leq \theta_{u_2} \} \]. Let \( P_2 \) be the portion of the plane bound by \( B_2 \) together with its boundary \( B_2 \). We define a function \( q : L \rightarrow \mathbb{R} \) as follows

a) \( q/L \) is the constant function that sends each point of \( L \) to \( U \),

b) \( q/U \) sends each point of \( U \) to \( u_1 \),

c) \( q/P_2 \) is the perpendicular projection of \( P_2 \) into \( u_1 \).

Let \( q(L) = \mathbb{R} \). Let \( q^* : \mathbb{R} \rightarrow \mathbb{R} \) be a surjective continuous map such that \( q^*(v) = w, q^*(y') = s \). Let \( q_e = q^* \circ q \). From previous observations it is clear that if \( x' \in \mathbb{R} \) then \( \text{diam} \{ x' \} \leq c + 2\varepsilon \).

Let \( Y \) be the continuum as given above. We consider 7 cases as given below:

Case A1: \( Y \) = max\{a', b'\} \( a \neq w \neq b \)

Case B1: \( Y \) = max\{a', c'\} \( a \neq w, c \neq v \)

Case C1: \( Y \) = max\{a', b'\} \( a = w = b \)

Case D1: \( Y \) = max\{a', c'\} \( a = w, c = v \)

Case E1: \( Y \) = max\{a', b'\} \( a = w, b \neq w \)

Case F1: \( Y \) = max\{a', c'\} \( a = w, c \neq v \)

Case G1: \( Y = v = re^{i \frac{\pi}{2}} \) \( s_x = d(v, w) \)

Let \( C \subseteq Y \times Y \) be a continuum such that \( p_1[C] \subseteq p_2[C] \subseteq Y \).

Case A1: \( Y = \max\{a', b'\} \) \( a \neq w \neq b \)

Let \( p : L \cup R \rightarrow \mathbb{R} \) be given by \( p/R = p_e \) as defined in case 1 for \( p_e \).

Case B1: \( Y = \max\{a', c'\} \) \( a \neq w, c \neq v \)

In this case we define \( p : L \cup R \rightarrow \mathbb{R} \) by \( p/R = p_e \) as in case 1 for \( p_e \) and \( p/L = q_e \) as in case 3 for \( q_e \).

The rest of this case is handled as in case A1. Our conclusion now is that

\[ \tau(Y) \leq \max\{a', b'\} = s_X = \tau(X) \] where \( \tau \) is any of the spans.

Case C1: \( Y = \max\{a', b'\} \) \( a = w = b \)

In this case we define \( p : L \cup R \rightarrow \mathbb{R} \) by \( p/R = p_e \) as defined in case 2 for \( p_e \) and \( p/L = q_e \) as defined in case 2 for \( q_e \).

In a manner similar to the previous cases we can conclude that

\[ \tau(Y) \leq \max\{a', b'\} = s_X = \tau(X) \] where \( \tau \) is any of the spans.

Case D1: \( Y = \max\{a', c'\} \) \( a = w, c = v \)
In this case we define \( p : L \cup R \to \overline{sw} \cup \overline{vt} \) by \( p/R = p_\varepsilon \) as in case 2 for \( p_\varepsilon \) and \( p/L = q_\varepsilon \) as in case 4 for \( q_\varepsilon \).

As in the previous cases we can conclude that

\[
\tau(Y) \leq \max\{a', c'\} = s_X = \tau(X) \text{ where } \tau \text{ is any of the spans.}
\]

**Case E1:** \( s_X = \max\{a', b'\} \), \( a = w, b \neq w \)

In this case we define \( p : L \cup R \to \overline{sw} \cup \overline{vt} \) by \( p/R = p_\varepsilon \) as defined in case 2 for \( p_\varepsilon \) and \( p/L = q_\varepsilon \) as defined in case 2 for \( q_\varepsilon \).

In this case we can conclude that

\[
\tau(Y) \leq \max\{a', b'\} = s_X = \tau(X) \text{ where } \tau \text{ is any of the spans.}
\]

**Case F1:** \( s_X = \max\{a', c'\} \), \( a = w, c \neq v \)

In this case we define \( p : R \cup L \to \overline{sw} \cup \overline{vt} \) by \( p/R = p_\varepsilon \) as in case 2 for \( p_\varepsilon \) and \( p/L = q_\varepsilon \) as in case 3 for \( q_\varepsilon \).

Our conclusion in this case is that

\[
\tau(Y) \leq \max\{a', c'\} = s_X = \tau(X) \text{ where } \tau \text{ is any of the spans.}
\]

**Case G1:**

We define \( p : R \cup L \to \overline{vt} \cup \overline{ws} \) when \( v = re^{i\theta} \). In this case \( s_X = d(v, w) \).

Pick \( \varepsilon \) where \( 0 < \varepsilon < \frac{1}{2} \min\{d(w, t), d(w, s)\} \). Pick \( m \in \overline{vt} \) such that \( 0 < d(w, m) < \varepsilon \). Let \( n \in X \) such that \( mnr \) is perpendicular to \( \overline{vt} \). Pick \( u \in \overline{ws} \) such that \( 0 < d(w, u) < \varepsilon \). Let \( p = re^{i\theta}p \in X \) such that \( mnr \) is perpendicular to \( \overline{ws} \). Let \( B_1 = mnr \cup \overline{vt} \cup X_{tn} \) where \( X_{tn} = \{re^{i\theta}| \theta \in \{0, \theta_n\} \cup \{\theta_t, 2\pi\} \) if \( 0 \leq \theta_n < \frac{\pi}{2}, \theta \in [\theta_t, \theta_n] \) if \( \frac{\pi}{2} \leq \theta_n < 2\pi \} \). Let \( P_1 \) be the portion of the plane bound by \( B_1 \) together with its boundary \( B_1 \). Let \( r : [0, 1] \to X_{tn} \) where \( X_{en} = \{re^{i\theta}| \theta \in \theta_n \} \) if \( 0 \leq \theta_n < \frac{\pi}{2}, \theta \in [\theta_n, 2\pi) \cup [0, \frac{\pi}{2}) \) if \( \frac{\pi}{2} < \theta_n < 2\pi \) be a continuous, surjective function such that \( r(0) = v \) and \( r(1) = n \). Let \( m_t = (1-t)w + tm \). Let \( L_t = m_t r(t) \). Let \( l : [0, 1] \to X_{en} \) where \( X_{en} = \{re^{i\theta}| \theta \in \theta_n \} \) be a continuous surjective function such that \( l(0) = v, l(1) = p \). Let \( u_t = (1-t)u + tu \). Let \( L_t = u_t l(t) \). Let \( B_2 = mnr \cup \overline{vt} \cup X_{ps} \) where \( X_{ps} = \{re^{i\theta}| \theta \leq \theta_p \} \). Let \( P_2 \) be the portion of the plane bound by \( B_2 \) together with its boundary \( B_2 \). We define \( p : R \cup L \to \overline{vt} \cup \overline{ws} \) as follows:

\[
p/P_1 \text{ is the perpendicular projection of } P_1 \text{ into } \overline{vt}
\]
\[
p/R_1 \text{ is the constant function which sends each point in } R_t \text{ to } m_t.
\]
\[
p/L_t \text{ is the constant function which sends each point in } L_t \text{ to } u_t.
\]
\[
p/P_2 \text{ is the perpendicular projection of } P_2 \text{ into } \overline{ws}.
\]

**Observation 4:**

Note that the continuous function \( d^* : [0, 1] \to R^+ \) given by \( d(w, r(t)) \) is decreasing. So, for each \( t \in [0, 1] \), \( d(m_t, r(t)) \leq d(v, w) + \varepsilon \). Similarly, it follows that \( d(u_t, d(t)) \leq d(v, w) + \varepsilon \) for \( t \in [0, 1] \). Using this observation together with observation 1, we see that for \( y' \in p(R \cup L) \), \( d^{-1}(y') \leq d(v, w) + \varepsilon \). Since
this is true for all \( \varepsilon > 0 \), we can conclude that
\[
\tau(Y) \leq d(v, w) = s_X = \tau(X) \text{ where } \tau \text{ is any of the spans.}
\]

References


