SOME QUESTIONS OF EQUIVARIANT MOVABILITY

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Abstract. In this article some questions of equivariant movability, connected with the substitution of the acting group $G$ on closed subgroup $H$ and with transitions to spaces of $H$-orbits and $H$-fixed points spaces, are investigated. In a special case, the characterization of equivariantly movable $G$-spaces is given.

1. Introduction

This paper is devoted to equivariant movability of $G$-spaces, i.e., topological spaces endowed with an action of a given compact group $G$.

More precisely, in § 3 we define the notion of equivariant movability or $G$-movability and we prove several theorems, including the following ones. If $X$ is $p$-paracompact and $H \subseteq G$ is a closed subgroup, then $G$-movability of $X$ implies its $H$-movability (§ 3, Theorem 3.3). $G$-movability of $X$ also implies movability of the space $X[H]$ of $H$-fixed points in $X$ (§ 4, Theorem 4.1). In particular, equivariant movability of a $G$-space $X$ implies ordinary movability of the topological space $X$ (§ 3, Corollary 3.5). We construct a non-trivial example which shows, that the converse, in general, is not true, even if we take for $G$ the cyclic group $Z_2$ of order 2 (§ 5, Example 5.1). If $X$ is a metrizable $G$-movable space and $H$ is a closed normal subgroup of $G$, then the space $X[H]$ of its $H$-orbits is also $G$-movable (§ 6, Theorem 6.1). In the case $H = G$ we obtain that $G$-movability of a metrizable $G$-space implies ordinary movability of the orbit space $X[G]$ (§ 6, Corollary 6.2). The last assertion, in general, is not invertible (§ 6, Example 6.3). However, if $X$ is metrizable, $G$ is a compact Lie group and the action of $G$ on $X$ is free, then $X$ is $G$-movable if and only if the orbit space $X[G]$ is movable (§ 7, Theorem 7.2). Examples 6.3 (§ 6) and

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3 (§ 8) show that in the last theorem the assumption that the group $G$ is a Lie group and the assumption that the action is free cannot be omitted.

Some of the above listed results with an outline of proof were given in [9].

Let us denote the category of all topological spaces and continuous maps by $\text{Top}$, the category of all metrizable spaces and continuous maps by $M$ and the category of all $p$-paracompact spaces and continuous maps by $P$. Always in this article it is assumed that all topological spaces are $p$-paracompact spaces and the group $G$ is compact.

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2. Basic notions and conventions concerning equivariant topology

Let $G$ be a topological group. A topological space $X$ is called a $G$-space if there is a continuous map $\theta: G \times X \rightarrow X$ of the direct product $G \times X$ into $X$, $\theta(g, x) = gx$, such that

1) $g(hx) = (gh)x; \quad 2) \quad ex = x,$

for all $g, h \in G$, $x \in X$; here $e$ is the unity of $G$. Such a (continuous) map $\theta : G \times X \rightarrow X$ is called an (continuous) action of the group $G$ on the topological space $X$. An evident example is the so called trivial action of $G$ on $X$: $gx = x$, for all $g \in G$, $x \in X$. Another example is the action of the group $G$ on itself, defined by $(g, x) \rightarrow gx$ for all $g \in G$, $x \in G$.

If $X$ and $Y$ are $G$-spaces, then so is $X \times Y$, where $g(x, y) = (gx, gy)$, $g \in G$, $(x, y) \in X \times Y$.

A subset $A$ of a $G$-space $X$ is called invariant provided $g \in G$, $a \in A$ implies $ga \in A$. It is evident, that an invariant subset of a $G$-space is itself a $G$ space. If $A$ is an invariant subset of a $G$-space $X$, then every neighborhood of $A$ contains an open invariant neighborhood of $A$ (see [17], Proposition 1.1.14).

Let $X$ be any $G$-space and let $H$ be a closed and normal subgroup of the group $G$. The set $Hx = \{hx; h \in H\}$ is called the $H$-orbit of the point $x \in X$. Clearly the $H$-orbits of any two points in $X$ are either equal or disjoint, in other words $X$ is partitioned by its $H$-orbits. We denote the set of all $H$-orbits of the $G$-space $X$ by $X|_H$. The set $X|_H$ endowed with the quotient topology is called the $H$-orbit space of $X$. There is a continuous action of the group $G$ on the space $X|_H$ defined by the formula $gHx = Hgx$, $g \in G, x \in X$. So, $X|_H$ is a $G$-space. In case $H = G$ the $G$-orbit of the point $x \in X$ is called the orbit of the point $x$ and the $G$-orbit space is called the orbit space of the $G$-space $X$. 
We denote by $X[H]$ the subspace of fixed points of $H$ on $X$, or the $H$-fixed point subspace of the $G$-space $X$. Let us recall that $X[H] = \{ x \in X; h x = x, \text{ for any } h \in H}\}.$

The set $G_x = \{ g \in G; g(x) = x \}$ is a closed subgroup of the group $G$, for every $x \in X$. $G_x$ is called the stationary subgroup (or stabilizer) at the point $x$. The action of the group $G$ on $X$ (or the $G$-space $X$) is called free if the stationary subgroup $G_x$ is trivial, for every $x \in X$. It is clear that $G_{gx} = gG_x g^{-1}$, i.e., the stationary subgroups at any two points of the same orbit are conjugate. The orbits $Gx$ and $Gy$ of points $x$ and $y$, respectively, are said to have the same type if the stationary subgroups $G_x$ and $G_y$ are conjugate.

Let $X, Y$ be $G$-spaces. A (continuous) map $f : X \to Y$ is called a $G$-map, or an equivariant map, if $f(gx) = gf(x)$ for every $g \in G, x \in X$. Note that the identity map $i : X \to X$ is equivariant and the composition of equivariant maps is equivariant. Therefore, all $G$-spaces and equivariant maps form a category. Let us denote the category of all topological $G$-spaces and equivariant maps by $Top_G$, the category of all metrizable $G$-spaces and equivariant maps by $M_G$ and the category of all $p$-para-compact $G$-spaces and equivariant maps by $P_G$.

Let $Z$ be a $G$-space and let $Y \subseteq Z$ be an invariant subset. A $G$-retraction of $Z$ to $Y$ is a $G$-map $r : Z \to Y$ such that $r|_Y = 1_Y$.

Let $K_G$ be class of $G$-spaces. A $G$-space $Y$ is called a $G$-absolute neighborhood retract for the class $K_G$ or a $G-ANR(K_G)$ ($G$-absolute retract for the class $K_G$ or a $G-A\mathcal{R}(K_G)$), provided $Y \in K_G$ and whenever $Y$ is a closed invariant subset of a $G$-space $Z \in K_G$, then there exist an invariant neighborhood $U$ of $Y$ and a $G$-retraction $r : U \to Y$ (there exists a $G$-retraction $r : Z \to Y$).

A $G$-space $Y$ is called a $G$-absolute neighborhood extensor for the class $K_G$ or a $G-ANE(K_G)$ ($G$-absolute extensor for the class $K_G$ or a $G-AE(K_G)$), provided for any $G$-space $X \in K_G$ and any closed invariant subset $A \subseteq X$, every equivariant map $f : A \to Y$ admits an equivariant extension $\tilde{f} : U \to Y$, where $U$ is an invariant neighborhood of $A$ in $X$ ($\tilde{f} : X \to Y$).

### 3. Movability and equivariant movability


Following Mardešić and Segal [14], let us define the notion of equivariant movability or $G$-movability:
Definition 3.1. An inverse $G$-system $\mathcal{X} = \{X_\alpha, p_{\alpha\alpha'}, A\}$ where each $X_\alpha$, $\alpha \in A$, is a $G$-space and every $p_{\alpha\alpha'} : X_\alpha' \to X_\alpha$, $\alpha \leq \alpha'$, is a $G$-homotopy class, is called equivariantly movable or $G$-movable if for every $\alpha \in A$, there exists an $\alpha' \in A$, $\alpha' \geq \alpha$ such that for all $\alpha'' \in A$, $\alpha'' \geq \alpha$ there exists a $G$-homotopy class $r^{\alpha''\alpha'} : X_\alpha' \to X_\alpha''$ such that

$$p_{\alpha''\alpha'} \circ r^{\alpha''\alpha'} = p_{\alpha\alpha'}.$$  

It is known (see [1], Theorem 2) that every $G$-space $X$ admits a $G$-ANR-expansion in the sense of Mardešić (see [15], I, § 2.1), which is the same as saying that there is an inverse $G$-ANR-system ($G$-system consisting of $G$-ANR’s) $\mathcal{X} = \{X_\alpha, p_{\alpha\alpha'}, A\}$ associated with $X$ in the sense of Morita [16].

Definition 3.2. A $G$-space $X$ is called equivariantly movable or $G$-movable if there is an equivariantly movable inverse $G$-ANR-system $\mathcal{X} = \{X_\alpha, p_{\alpha\alpha'}, A\}$ associated with $X$.

Note that the last definition of equivariant movability coincides with the notion of ordinary movability if $G = \{e\}$ is the trivial group.

Let $X$ be an equivariantly movable $G$-space. The evident question arises: does movability of the space $X$ follows from its equivariant movability? The following, more general theorem gives a positive answer (Corollary 3.5) to the above question.

Theorem 3.3. Let $H$ be a closed subgroup of a group $G$. Every $G$-movable $G$-space is $H$-movable.

To prove this theorem the next result is important.

Theorem 3.4. Let $H$ be a closed subgroup of a group $G$. Every $G - AR(P_G)$ ($G - ANR(P_G)$)-space is an $H - AR(P_H)(H - ANR(P_H))$-space.

Proof. According to a theorem of de Vries ([7], Theorem 4.4), it is sufficient to show that if $X$ is a $p$-paracompact $H$-space, then the twisted product $G \times_H X$ is also $p$-paracompact. Indeed, since $X$ is $p$-paracompact and $G$ is compact, $G \times X$ is $p$-paracompact. Therefore, the twisted product $G \times_H X$ is $p$-paracompact.

Proof of Theorem 3.3. Let $X$ be any equivariantly movable $G$-space. With respect to the theorem of Smirnov ([18], Theorem 1.3), there is a closed and equivariant embedding of the $G$-space $X$ to some $G - AR(P_G)$-space $Y$. Let us consider all open $G$-invariant neighborhoods of type $F_\alpha$ of the $G$-space $X$ in $Y$. By a result of R. Palais ([17], Proposition 1.1.14), these neighborhoods form a cofinal family in the set of all open neighborhoods of $X$ in $Y$, in particular, in the set of all open and $H$-invariant neighborhoods of the $H$-space $X$ in the $H$-space $Y$, which, by Theorem 3.3 is an $H - AR(P_H)$-space. Hence, from the $G$-movability of the above mentioned family follows
its $H$-movability, i.e. from the $G$-movability of the $G$-space $X$ follows the $H$-movability of the $H$-space $X$.

From Theorem 3.3 we obtain the following corollary if we consider the trivial subgroup $H = \{e\}$ of the group $G$.

**Corollary 3.5.** Every equivariantly movable $G$-space $X$ is movable.

The converse, in general, is not true, even if one takes for $G$ the cyclic group $Z_2$ of order 2 (see Example 5.1).

4. **Movability of the $H$-fixed point space**

**Theorem 4.1.** Let $H$ be a closed subgroup of a group $G$. If a $G$-space $X$ is equivariantly movable, then the $H$-fixed point space $X[H]$ is movable.

The proof requires the use of the following theorem.

**Theorem 4.2.** Let $H$ be a closed subgroup of a group $G$. Let $X$ be a $G - AR(P_G)(G - ANR(P_G))$-space. Then the $H$-fixed point space $X[H]$ is an $AR(P)(ANR(P))$-space.

**Proof.** Let $X$ be a $G - AR(P_G)(G - ANR(P_G))$-space. By Theorem 3.4, it is sufficient to prove the theorem in the case $H = G$. I.e., we must prove that $X[G]$ is $AR(P)$-space. By a theorem of Smirnov ([18], Theorem 1.3), we can consider $X$ as a closed $G$-subspace of a $G - AR(P_G)$-space $C(G, V) \times \prod D_\lambda$ where $V$ is a normed vector space and thus an $AE(M)$-space, $C(G, V)$ is the space of continuous maps from $G$ to $V$ with the compact-open topology and with the action $(gf)(g) = f(gg'), g, g' \in G, f \in C(G, V)$ of the group $G$ and $D_\lambda$ is a closed ball of a finite-dimensional Euclidean space $E_\lambda$ with the orthogonal action of the group $G$.

First, let us prove that the set $(C(G, V) \times \prod D_\lambda)[G]$ of all fixed points of the $G$-space $C(G, V) \times \prod D_\lambda$ is an $AR(P)$-space. The spaces $C(G, V)$ and $E_\lambda$ are normed spaces. Since the actions of the group $G$ on $C(G, V)$ and $E_\lambda$ are linear, the sets $C(G, V)[G]$ and $E_\lambda[G]$ will be closed convex sets of locally convex spaces $C(G, V)$ and $E_\lambda$, respectively. Therefore, by a well-known theorem of Kuratowski and Dugundji [3], $C(G, V)$ and $E_\lambda$ are absolute retracts for metrizable spaces. By a theorem of Lisica [12], they are also absolute retracts for $p$-paracompact spaces. For a closed ball $D_\lambda \subset E_\lambda$ the last conclusion is true since the set $D_\lambda[G] = D_\lambda \cap E_\lambda[G]$ is closed and convex in $E_\lambda$.

Since the group $G$ acts on the product $C(G, V) \times \prod D_\lambda$ coordinate-wise,

$$(C(G, V) \times \prod D_\lambda)[G] = C(G, V)[G] \times \left(\prod D_\lambda\right)[G].$$

Hence, $(C(G, V) \times \prod D_\lambda)[G]$ is an $AR(P)$-space, because it is a product of two $AR(P)$-spaces.
Now let us prove that $X[G]$ is an $AR(P)$-space. Since $X$ is a $G-AR(P_G)$-space, it is a $G$-retract of the product $C(G, V) \times \prod D\lambda$. Therefore, $X[G]$ is a retract of the $AR(P)$-space $(C(G, V) \times \prod D\lambda)[G]$, hence, it is an $AR(P)$-space.

The absolute neighborhood retract case is proved similarly. \hfill \Box

**Proof of Theorem 4.1.** Let $X$ be a $G$-movable space. By Theorem 3.3, it is sufficient to prove the theorem in the case $H = G$. So, we must prove movability of the space $X[G]$ of all $G$-fixed points. We consider the $G$-space $X$ as a closed and $G$-invariant space of some $G-AR(P_G)$-space $Y$ ([18], Theorem 1.3). The family of all open, $G$-invariant $F_\sigma$-type neighborhoods $U_\alpha$ of the $G$-space $X$ in $Y$, is cofinal in the set of all open neighborhoods of $X$ in $Y$ ([17], Proposition 1.1.14). It consists of $G-ANR(P_G)$-spaces. The intersections $U_\alpha \cap Y[G] = U_\alpha[G]$ are $ANR(P)$-spaces (Theorem 4.2). They form a cofinal family of neighborhoods of the space $X[G]$ in $Y[G]$. Indeed, for any neighborhood $U$ of the set $X[G]$ in $Y[G]$ there is a neighborhood $V$ of the set $X[G]$ in $Y$ such that $V \cap Y[G] = U$. Then the set $W = (Y \setminus Y[G]) \cup V$ is a neighborhood of the set $X$ in $Y$, moreover, $W \cap Y[G] = U$. There is an $\alpha$ such that $U_\alpha \subset W$ and therefore $U_\alpha[G] \subset U$. So the family of neighborhoods $U_\alpha[G]$ is cofinal.

Since $X$ is $G$-movable, for every $U_\alpha$ there is a neighborhood $U_\alpha' \subset U_\alpha$ such that, for any other neighborhood $U_{\alpha'} \subset U_\alpha'$, there exists a $G$-equivariant homotopy $F: U_{\alpha'} \times I \to U_\alpha$ such that $F(y, 0) = y$ and $F(y, 1) \in U_{\alpha'}$, for any $y \in U_{\alpha'}$. It is not difficult to verify that the homotopy $F[G]: U_{\alpha'}[G] \times I \to U_\alpha[G], \text{induced by } F$, satisfies the condition of movability of $X[G]$. \hfill \Box

5. **Example of a movable, but not equivariantly movable space**

**Example 5.1.** We will use the idea of S. Mardešić [13]. Let us consider the unit circle $S = \{z \in C; |z| = 1\}$. Let us denote $B = [S \times \{1\}] \cup [\{1\} \times S]$. $B$ is the wedge of two copies of the unit circle $S$ with base point $\{1\}$. Let us define a continuous map $f: B \to B$ by the formulas:

\[
\begin{align*}
f(z,1) &= \begin{cases} 
(z^4,1), & 0 \leq \arg(z) \leq \frac{\pi}{2} \\
(1,z^4), & \frac{\pi}{2} \leq \arg(z) \leq \pi \\
(z^{-4},1), & \pi \leq \arg(z) \leq \frac{3\pi}{2} \\
(1,z^{-4}), & \frac{3\pi}{2} \leq \arg(z) \leq 2\pi 
\end{cases} \\
\end{align*}
\]

\[
\begin{align*}
f(1,t) &= \begin{cases} 
(t^{-4},1), & 0 \leq \arg(t) \leq \frac{\pi}{2} \\
(1,t^{-4}), & \frac{\pi}{2} \leq \arg(t) \leq \pi \\
(t^4,1), & \pi \leq \arg(t) \leq \frac{3\pi}{2} \\
(1,t^4), & \frac{3\pi}{2} \leq \arg(t) \leq 2\pi 
\end{cases} \\
\end{align*}
\]
for every $z$ and $t$ from $S$. Let us consider the ANR-sequences

$$B \xleftarrow{f} B \xleftarrow{f} B \xleftarrow{f} \cdots$$

and

$$\Sigma B \xleftarrow{\Sigma f} \Sigma B \xleftarrow{\Sigma f} \Sigma B \xleftarrow{\Sigma f} \cdots$$

where $\Sigma$ is the operation of suspension. Let us denote

$$P = \lim \{B, f\}.$$ 

Then

$$\Sigma P = \lim \{\Sigma B, \Sigma f\}.$$ 

Let us define an action of the group $Z_2 = \{e, g\}$ on $\Sigma B$ by the formulas

$$e[x, t] = [x, t]; \quad g[x, t] = [x, -t].$$

for every $[x, t] \in \Sigma B$, $-1 \leq t \leq 1$. It induces an action on $\Sigma P$.

**Proposition 5.2.** The space $\Sigma P$ has trivial shape, but it is not $Z_2$-movable.

**Proof.** The triviality of shape of the space $\Sigma P$ is proved by the method of Mardešić [13]. Let us prove that the space $\Sigma P$ is not $Z_2$-movable. Consider the set $\Sigma P[Z_2]$ of all fixed-points of $Z_2$-space $\Sigma P$. It is obvious that $\Sigma P[Z_2] = P$. Hence, by Theorem 4.1, it is sufficient to prove the following proposition.

**Proposition 5.3.** The space $P$ is not movable.

**Proof.** Since the movability of an inverse system remains unchanged under the action of a functor, it is sufficient to prove non-movability of the inverse sequence of groups

$$\pi_1(B) \xrightarrow{f_*} \pi_1(B) \xrightarrow{f_*} \pi_1(B) \xrightarrow{f_*} \cdots,$$

where $\pi_1(B)$ is the fundamental group of the space $B$ and $f_*$ is the homomorphism induced by the mapping $f : B \to B$.

It is known that for sequences of groups movability implies the following condition of Mittag-Leffler, abbreviated as $ML$ ([15], p. 166, Corollary 4):

*The inverse system $\{G_\alpha, p_{\alpha\alpha'}, A\}$ of the pro–GROUP category is said to be $ML$ provided for every $\alpha \in A$, there exist $\alpha' \in A, \alpha' \geq \alpha$, such that $p_{\alpha\alpha'}(G_{\alpha'}) = p_{\alpha\alpha''}(G_{\alpha''})$, for any $\alpha'' \in A, \alpha'' \geq \alpha$.*

Thus, it sufficient to prove that the sequence (1) does not satisfy condition $ML$. Let us observe that $\pi_1(B)$ is a free group with two generators $a$ and $b$, and $f_*$ is the homomorphism defined by the formulas

$$f_*(a) = aba^{-1}b^{-1}, \quad f_*(b) = a^{-1}b^{-1}ab.$$
$f_*$ is a monomorphism, because $f_*(a) \neq f_*(b)$, but not an epimorphism, because, for example, $f_*(x) \neq a$, for all $x \in \pi_1(B)$. Hence, for any natural $m$ and $n$, $\text{Im} f^m \not\subseteq \text{Im} f^n$ only if $m > n$. It means that the inverse sequence (1) does not satisfy condition $ML$.

6. Movability of the orbit space

**Theorem 6.1.** Let $X$ be a metrizable $G$-space. If $X$ is $G$-movable then for any closed and normal subgroup $H$ of the group $G$, the $H$-orbit space $X|_H$ is also $G$-movable.

**Proof.** Without losing generality one may suppose that $X$ is a closed $G$-invariant subset of some $G - AR(M_G)$-space $Y$ ([18], Theorem 1.1). $X|_H$ is a closed $G$-invariant subset of $Y|_H$ ([5], Theorem 3.1).

Let $\{X_\alpha, \alpha \in A\}$ be the family of all $G$-invariant neighborhoods of $X$ in $Y$. Let us consider the family $\{X_\alpha|_H, \alpha \in A\}$, where each $X_\alpha|_H \in G - ANR(M_G)$ and is a $G$-invariant neighborhood of $X|_H$ in $Y|_H$. Let us prove that the family $\{X_\alpha|_H, \alpha \in A\}$ is cofinal in the family of all neighborhoods of $X|_H$ in $Y|_H$. Let $U$ be an arbitrary neighborhood of $X|_H$ in $Y|_H$. By a theorem of Palais ([17], Proposition 1.1.14), there exists a $G$-invariant neighborhood $V \supset X|_H$ laying in $U$. Let us denote $\hat{V} = (pr)^{-1}(V)$, where $pr : Y \to Y|_H$ is the $H$-orbit projection. It is evident that $\hat{V}$ is a $G$-invariant neighborhood of the space $X$ in $Y$ and $V = \hat{V}|_H$. So in any neighborhood of the space $X|_H$ in $Y|_H$, there is a neighborhood of type $X_\alpha|_H$, where $X_\alpha$ is a $G$-invariant neighborhood of $X$ in $Y$.

Now let us prove the $G$-movability of the space $X|_H$. Let $X$ be $G$-movable. It means that the inverse system $\{X_\alpha, i_{\alpha\alpha'}, A\}$ is $G$-movable. We must prove that the induced inverse system $\{X_\alpha|_H, i_{\alpha\alpha'|_H}, A\}$ is $G$-movable. Let $\alpha \in A$ be any index. By the $G$-movability of the inverse system $\{X_\alpha, i_{\alpha\alpha'}, A\}$, there is $\alpha' \in A, \alpha' > \alpha$, such that for any other index $\alpha'' \in A, \alpha'' > \alpha$, there exists a $G$-mapping $r^{\alpha'\alpha''} : X_{\alpha'} \to X_{\alpha''}$, which makes the following diagram $G$-homotopy commutative.

![Diagram 1](image-url)
It turns out that, for given \( \alpha \in A \), the obtained index \( \alpha' \in A, \alpha' > \alpha \), also satisfies the condition of \( G \)-movability of the inverse system

\[
\{X_\alpha|_H, i_{\alpha}\alpha'|_H, A\}.
\]

This is obvious, because the \( G \)-homotopy commutativity of Diagram 1 implies the \( G \)-homotopy commutativity of the following diagram

\[
\begin{array}{ccc}
X_\alpha|_H & \xrightarrow{i_{\alpha}\alpha'} & X_{\alpha'}|_H \\
\downarrow & & \downarrow r^{\alpha'\alpha''}|_H \\
X_\alpha|_H & \xrightarrow{i_{\alpha}\alpha''} & X_{\alpha''}|_H \\
\end{array}
\]

Diagram 2.

where \( r^{\alpha'\alpha''}|_H : X_{\alpha'}|_H \to X_{\alpha''}|_H \) is induced by the mapping \( r^{\alpha'\alpha''} \). So, the \( G \)-movability of the space \( X|_H \) is proved.

**Corollary 6.2.** Let \( X \) be a metrizable \( G \)-space. If \( X \) is \( G \)-movable, then the orbit space \( X|_G \) is movable.

**Proof.** In the case \( H = G \) from the last theorem we obtain that the orbit space \( X|_G \) with the trivial action of the group \( G \) is \( G \)-movable. Therefore, it will be movable by Corollary 3.5.

Corollary 6.2 in general is not invertible:

**Example 6.3.** Let \( \Sigma \) be a solenoid. It is known ([4], Theorem 13.5) that \( \Sigma \) is a non-movable compact metrizable Abelian group. By Corollary 3.5, the solenoid \( \Sigma \) with the natural group action is not \( \Sigma \)-movable although the orbit space \( \Sigma|_G \) as a one-point set is movable.

The converse of Corollary 6.2 is true if the group \( G \) is a Lie group and the action is free (see Theorem 7.2).

### 7. Equivariant movability of a free \( G \)-space

**Theorem 7.1.** Let \( G \) be a compact Lie group and let \( Y \) be a metrizable \( G - AR(M_G) \)-space. Suppose that a closed invariant subset \( X \) of \( Y \) has an invariant neighborhood whose orbits have the same type. If the orbit space \( X|_G \) is movable, then \( X \) is equivariantly movable.

**Proof.** The orbit space \( X|_G \) is closed in \( Y|_G \), which is a \( G - AR(M) \)-space. Let \( U \) be an arbitrary invariant neighborhood of \( X \) in \( Y \). By the assumption of the theorem, it follows that there exists a cofinal family of neighborhoods of \( X \) in \( Y \), whose orbits have the same type. Therefore, one
may suppose that all orbits of the neighborhood $U$ have the same type. The orbit set $U|_G$ will be a neighborhood of $X|_G$ in $Y|_G$. From the movability of $X|_G$ it follows that, for the neighborhood $U|_G$, there is a neighborhood $\tilde{V}$ of the space $X|_G$ in $Y|_G$, which lies in the neighborhood $U|_G$ and contracts to any preassigned neighborhood of the space $X|_G$.

Let us denote $V = (pr)^{-1}(\tilde{V})$, where $pr : Y \to Y|_G$ is the orbit projection. It is evident that $V$ is an invariant neighborhood of the space $X$ lying in $U$. Let us prove that $V$ contracts in $U$ to any preassigned invariant neighborhood of $X$. Let $W$ be any invariant neighborhood of $X$ in $Y$. We must prove the existence of an equivariant homotopy $F : V \times I \to U$, which satisfies the condition

$$F(x,0) = x, \quad F(x,1) \in W,$$

for any $x \in V$. Since $W|_G$ is a neighborhood of the space $X|_G$ in $Y|_G$, there is a homotopy $\tilde{F} : V|_G \times I \to U|_G$ such that

$$F(\tilde{x},0) = \tilde{x}, \quad \tilde{F}(\tilde{x},1) \in W|_G,$$

for any $\tilde{x} \in V|_G$. The homotopy $\tilde{F} : V|_G \times I \to U|_G$ preserves the $G$-orbit structure, because $V \subset U$ and all orbits of $U$ have the same types (see Diagram 3).

![Diagram 3](image-url)

By the covering homotopy theorem of Palais ([17], Theorem 2.4.1), there is an equivariant homotopy $F : V \times I \to U$, which covers the homotopy $\tilde{F}$ and satisfies $F(x,0) = i(x) = x$. That is, the following diagram is commutative (Diagram 4).

![Diagram 4](image-url)
F : V x I → U is the designed equivariant homotopy. It only remains
to verify that F(x, 1) ∈ W. But this immediately follows from (2) and the
commutativity of Diagram 4.

**Theorem 7.2.** Let G be a compact Lie group. A metrizable free G-space
X is equivariantly movable if and only if the orbit space X\_|\_G is movable.

**Proof.** The necessity in a more general case was proved in Corollary 6.2.
Let us prove the sufficiency. Let the orbit space X\_|\_G be movable. One can
consider the G-space X as a closed and invariant subset of some G-ANR(MG)-
space Y. Let P ⊂ X be any orbit. From the existence of slices it follows that
around P there is such an invariant neighborhood U(P) in Y that typeQ ≥
typeP, for any orbit Q from U(P) ([5], Corollary 5.5). Since the action of the
group G on X is free, typeQ = typeP = typeG, for any orbit Q lying in U(P).
Let us denote V = ∪{U(P); P ∈ X\_|\_G}. It is evident that V is an invariant
neighborhood of the space X in Y and that all of its orbits have the same
type. Then, by Theorem 7.1, X is equivariantly movable.

Example 6.3 shows that the assumption that G is a Lie group is essential
in the above theorem. The Example 8.1 which follows shows that the condition
of freeness of the action of the group G is also essential in the above theorem.

8. Example of a non-free not Z₂-movable space with a movable
orbit space

**Example 8.1.** Let us consider the space P = lim\_B\_f constructed in
Example 5.1. Let us define an action of the group Z₂ = \{e, g\} on the space
B by the formulas

\begin{align*}
e(z, 1) &= (z, 1) \\
e(1, t) &= (1, t) \\
g(z, 1) &= (1, z^{-1}) \\
g(1, t) &= (t^{-1}, 1),
\end{align*}

for any z and t from S. B is a Z₂-ANR(MZ₂) space with the fixed-point
b₀ = (1, 1).

**Proposition 8.2.** The mapping f : B → B, defined by formulas (3), is
equivariant.

**Proof.** It is necessary to prove the following two equalities:

\begin{align*}
f(g(z, 1)) &= g(f(z, 1)) \\
f(g(1, t)) &= g(f(1, t)),
\end{align*}

for any z and t from S. Let us prove the first one. Consider the following
cases:
Theorem 3.1.13). Let ~ another choice of the representatives of the classes \([x_n, y_n] \in X\), where \((x_1, x_2, \ldots)\) is a sequence in \(X\), and \(x_1, x_2, \ldots\) are selected from the classes \([x_1, x_2, \ldots] \in X\) in such way that \((x_1, x_2, \ldots) \in P\) or what is the same \(f(x_{n+1}) = x_n\), for any \(n = 1, 2, \ldots\). Let us prove that the mapping \(h\) is defined correctly. Let \(\bar{x}_1, \bar{x}_2, \ldots\) be some other representatives of the classes \([x_1, x_2, \ldots] \in X\), respectively, satisfying the conditions \(f(\bar{x}_{n+1}) = \bar{x}_n\) for any \(n \in N\). Since each class \([x_n] \in X\) has two representatives: \(x_n\) and \(g x_n\), where \(g \in Z_2 = \{e, g\}\), either \(\bar{x}_n = g x_n\) or \(\bar{x}_n = x_n\). But it is obvious that, if for some \(n_0 \in N\), \(\bar{x}_{n_0} = g x_{n_0}\), then, for any \(n \in N\), \(\bar{x}_n = g x_n\), because \(f\) is equivariant. Thus, in the case of another choice of the representatives of the classes \([x_1, x_2, \ldots] \in X\), we have

\[
 h(([x_1], [x_2], \ldots)) = ([\bar{x}_1, \bar{x}_2, \ldots]) = [(g x_1, g x_2, \ldots)] = [g(x_1, x_2, \ldots)] = [(x_1, x_2, \ldots)].
\]

However, \(h\) is a continuous bijection and thus, it is a homeomorphism ([8], Theorem 3.1.13).

Consequently,

\[
P|Z_2 = \lim \{B|Z_2, f|Z_2]\]

The second equality of (4) is proved in a similar way. \(\square\)}

**Proposition 8.3.** \(P\) is a connected, compact, metrizable and equivariantly non-movable \(Z_2\)-space which is free at all points except at the only fixed point \((b_0, b_0, \ldots)\) and \(sh(P|Z_2) = 0\).

**Proof.** \(P\) is a \(Z_2\)-space because it is an inverse limit of \(Z_2-ANR(MZ_2)\)-spaces \(B\) and \(f\) is an equivariant mapping. The uniqueness of the fixed point is evident. The connectedness, compactness and metrizability follows from the properties of inverse systems ([8], Theorem 6.1.20, Corollary 4.2.5). The non \(Z_2\)-movability follows from Proposition 5.3 and Corollary 3.5.

Let us prove that \(sh(P|Z_2) = 0\) and thus the orbit space \(P|Z_2\) is movable.

Let \(X = \lim \{B|Z_2, f|Z_2\}\). \(X\) is equivimorphic to the orbit space \(P|Z_2\).

Indeed, let us define a mapping \(h : X \to P|Z_2\) in the following way:

\[
h(([x_1], [x_2], \ldots)) = ([x_1, x_2, \ldots])
\]

where \(([x_1], [x_2], \ldots) \in X\), and \(x_1, x_2, \ldots\) are selected from the classes \([x_1], [x_2], \ldots\) in such way that \((x_1, x_2, \ldots) \in P\) or what is the same \(f(x_{n+1}) = x_n\), for any \(n = 1, 2, \ldots\). Let us prove that the mapping \(h\) is defined correctly. Let \(\bar{x}_1, \bar{x}_2, \ldots\) be some other representatives of the classes \([x_1], [x_2], \ldots\), respectively, satisfying the conditions \(f(\bar{x}_{n+1}) = \bar{x}_n\) for any \(n \in N\). Since each class \([x_n] \in X\) has two representatives: \(x_n\) and \(g x_n\), where \(g \in Z_2 = \{e, g\}\), either \(\bar{x}_n = g x_n\) or \(\bar{x}_n = x_n\). But it is obvious that, if for some \(n_0 \in N\), \(\bar{x}_{n_0} = g x_{n_0}\), then, for any \(n \in N\), \(\bar{x}_n = g x_n\), because \(f\) is equivariant. Thus, in the case of another choice of the representatives of the classes \([x_1], [x_2], \ldots\), we have

\[
h(([x_1], [x_2], \ldots)) = ([\bar{x}_1, \bar{x}_2, \ldots]) = [(g x_1, g x_2, \ldots)] = [g(x_1, x_2, \ldots)] = [(x_1, x_2, \ldots)].
\]

However, \(h\) is a continuous bijection and thus, it is a homeomorphism ([8], Theorem 3.1.13).

Consequently,

\[
P|Z_2 = \lim \{B|Z_2, f|Z_2\},
\]
where $B|_{Z_2} \cong S$ and the mapping $\bar{f} = f|_{Z_2} : S \to S$ is defined by the formulas:

$$
\bar{f}(z) = \begin{cases} 
z^4, & 0 \leq \arg(z) \leq \frac{\pi}{2} \\
z^{-4}, & \frac{\pi}{2} \leq \arg(z) \leq \frac{3\pi}{2} \\
z^4, & \frac{3\pi}{2} \leq \arg(z) \leq 2\pi 
\end{cases}
$$

for any $z \in S$. Thus, we conclude that the orbit space $P|_{Z_2}$ is a limit of the inverse sequence

$$S \leftarrow S \leftarrow S \leftarrow \cdots$$

By formula (5), the mapping $\bar{f}$ induces a homomorphism $\bar{f}_* : \pi_1(S) \to \pi_1(S)$, which acts as follows:

$$\bar{f}_*(a) = aa^{-1}a^{-1}a,$$

where $a \in \pi_1(S) \cong Z$ is the generator of the group $Z$. From the above formula, it follows that $\bar{f}_*$ is the null-homomorphism and thus, $\deg \bar{f} = 0$. For any $k = 1, 2, \cdots, \bar{f}_k$ is also a null-homomorphism and thus, $\deg \bar{f}_k = 0$. Therefore, by the classical Hopf theorem ([10], Section 2.8, Theorem $H^n$) all $\bar{f}_k : S \to S$ are null-homotopic and $sh(P|_{Z_2}) = 0$.

References


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