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Elementary Constructions for Conics in Hyperbolic and Elliptic Planes

Dedicated to Prof. Dr. Klaus Meirer on the occasion of his 75th birthday.

Elementary Constructions for Conics in Hyperbolic and Elliptic Planes

ABSTRACT

In the Euclidean plane there are well-known constructions of points and tangents of e.g. an ellipse c based on the given axes of c, which make use of the Apollonius definition of c via its focal points or via two perspective affinities (de la Hire's construction). Even there is no parallel relation neither in a hyperbolic plane nor in an elliptic plane it is still possible to modify many of the elementary geometric constructions for conics, such that they also hold for those (regular) non-Euclidean planes. Some of these modifications just replace Euclidean straight lines by non-Euclidean circles. Furthermore we also study properties of Thales conics, which are generated by two pencils of (non-Euclidean) orthogonal lines.

Key words: hyperbolic plane, elliptic plane, conic sections, de la Hire, Apollonius, Thales

MSC2010: 51M04, 51M09

Elementarne konstrukcije konika u hiperboličnoj i eliptičnoj ravnini

SAŽETAK

U Euklidskoj ravnini poznate su konstrukcije točaka i tangenata za npr. elipsu c zadanu osima, pri čemu se koristi Apolonijeva definicija za c preko fokusa ili dva afiniteta (de la Hireova konstrukcija). Iako ne postoje paralelne relacije s hiperboličnom niti eliptičnom ravninom, ipak je moguće modificirati mnoge elementarne konstrukcije vezane za konike tako da one vrijede za (regularne) neeuklidske ravnine. U nekim modifikacijama samo je euklidski pravac zamijenjen s neeuklidskom kružnicom. Također će se proučiti svojstva Talesovih konika koje su generirane s dva pramena (neeuklidskih) okomitih pravaca.

Ključne riječi: hiperbolična ravnina, eliptična ravnina, konika, de la Hire, Apolonije, Tales

1 Introduction: Euclidean focal points

We aim at modifying Euclidean constructions for points and tangents of conics, such that they hold for conics in hyperbolic and elliptic planes. In the Euclidean plane, due to Apollonius, a hyperbola or an ellipse c is defined as geometric locus, where the focal points of c are the essential givens of c. To construct e.g. an ellipse via the classical "gardener's construction" in a hyperbolic plane necessitates the concept of focus in a hyperbolic resp. elliptic geometry. For visualizing the hyperbolic plane we use F. Klein's projective geometric model, elliptic geometry will be visualized on the sphere.

In a Euclidean plane a focus of a conic can be defined in several ways:

Definition 1 (Euclidean) *Given $c = \{P\}$ (ellipse resp. hyperbola), F_1, F_2 are (real) focal points, iff $\overline{PF_1} \pm \overline{PF_2} = const. \; \forall P \in c$, (and similar for a parabola c).*

Definition 2 (Euclidean) *Given $c = \{P\}$ (ellipse resp. hyperbola resp. parabola), F_i are focal points, iff each line through F_i is orthogonal to its conjugate line through F_i, (Figure 1).*

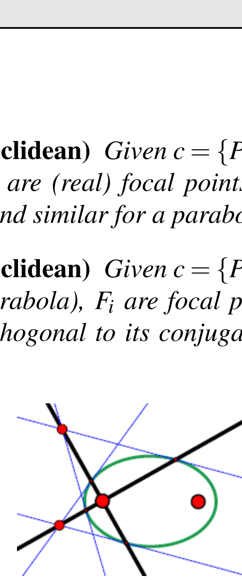

Figure 1: *Conjugate lines through a focus of a conic are orthogonal.*

Definition 3 (Euclidean) *Given $c = \{P\}$ (ellipse resp. hyperbola resp. parabola in the projectively and complex extended Euclidean plane π), F_i are focal points, iff they are intersections of tangents of c passing through the (conjugate imaginary) absolute points I, J of π.*

From Definition 3 follows that an ellipse and a hyperbola have four focal points, two of which are real and the other two are conjugate imaginary. For the extension of the concept "focus" to a projective extended hyperbolic or elliptic plane it seems to be an advantage to interpret the pair of Euclidean absolute points I, J at the ideal line u of π as degenerate absolute conic ω. Then the four tangents mentioned in Definition 3 become the common tangents of ω and c, which intersect also in I, J. Therewith one might count the pair of absolute points I, J also as an additional pair of focal points, such that, from an algebraic-geometric point of view, one has three pairs of focal points. A parabola and a circle have one real proper focal point. For the circle, considered as a limit figure of an ellipse, this focal point is the centre and counts for four focal points, while I, J forms the ideal pair of focal points. For a parabola, considered as a limit figure of a hyperbola, the ideal point is a real ideal focal point, while the pair I, J counts twice as imaginary pair of ideal focal points. In Figure 2 the situation is visualized symbolically showing conics "near" the limit cases.

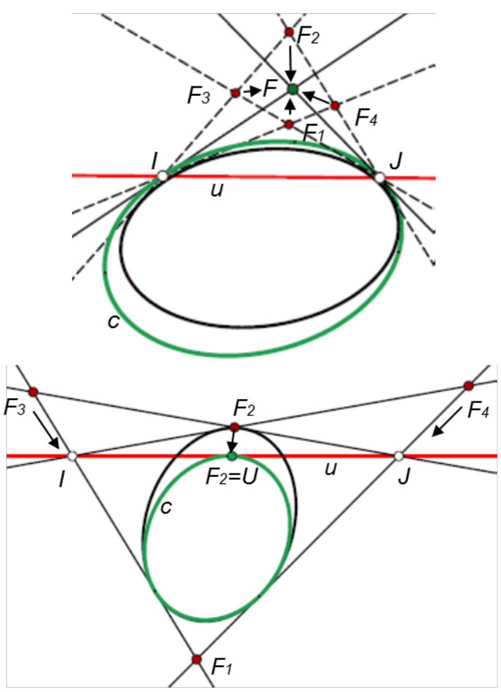

Figure 2: *Symbolic visualization of the 6 focal points of limit cases of a conic c.*

2 Introduction: de la Hire's construction of an ellipse and extensions

In the Euclidean plane π let an ellipse c be given by its axis segments. Then, due to Philippe de la Hire, one constructs points and tangents of an ellipse via two orthogonal perspective affine transformations of c to its two vertex circles, see Figure 3.

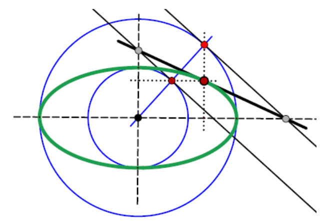

Figure 3: *De la Hire's construction of an ellipse via two orthogonal affinities.*

Remark 1 *It might not be so well-known that de la Hire's construction of an ellipse can easily be modified to construct a hyperbola c given by its axis rectangle (resp. its half axis segments a, b) using two perspective collineations $\kappa_i : c \mapsto c_i$. Thereby c_i is the (twice counted) vertex circle of c and the centres C_i of κ_i are the vertices of c, see Figure 4. Using e.g. the vanishing line v_1 of κ_1 one finds the first image O_1 of the point O of c. Let P_0 be an arbitrary point on c_i and let P_1 be the second intersection point of c_i with line $O_1 \vee P_0$, and P_2 the second intersection point of c_i with line $O \vee P_0$, then the mappings $P_0 \mapsto P_1$, $P_0 \mapsto P_2$ are involutions and $P_1 \mapsto P_2$ is a projectivity on the circle c_i. Therewith the pencils $\{C_1P_1\}$ and $\{C_2P_2\}$ are projective pencils and generate, according to J. Steiner, a conic. If the point O_1 is outside c_i, then this conic is a hyperbola with vertices C_i. If the point $O_1 (\neq C_i)$ is inside c_i, then the conic c becomes an ellipse with vertices C_i.*

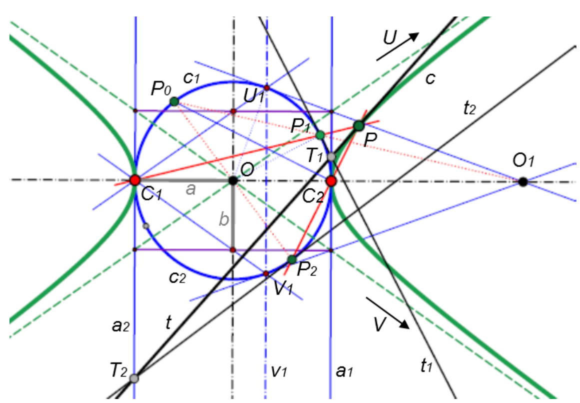

Figure 4: *Modified de la Hire's construction of a hyperbola via two perspective collineations.*

This construction principle is also applicable for constructing a parabola c given by its focal point F and its directrix line l, resp. its vertex C. The construction becomes very simple, if one uses the circle c_1 with centre F and radius $r = |FC|$ as collinear image of c. The diameters d of c_1 different form the axis of the parabola c intersect c in pairs of points with orthogonal tangents intersecting in a point $S \in l$, see Figure 5. The tangents of c_1 in the intersection points $d \cup c_1$ intersect the vertex tangent of c in points of this pair of parabola tangents and $d \perp FS$ according to focus Definition 2.

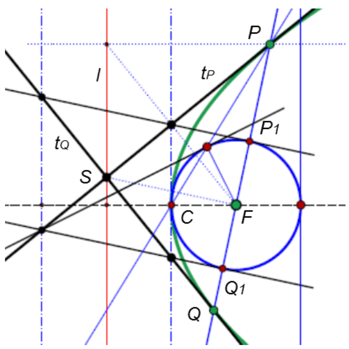

Figure 5: *Modified de la Hire's construction of a parabola via two perspective collineations.*

3 Conics in the hyperbolic and elliptic plane based on focus Definition 1

Starting with two real focus points F_i and the length of the main axis of a conic c it is possible to perform the usual Euclidean construction of points and tangents of c also in a (projectively extended) hyperbolic or an elliptic plane π, see Figure 6.

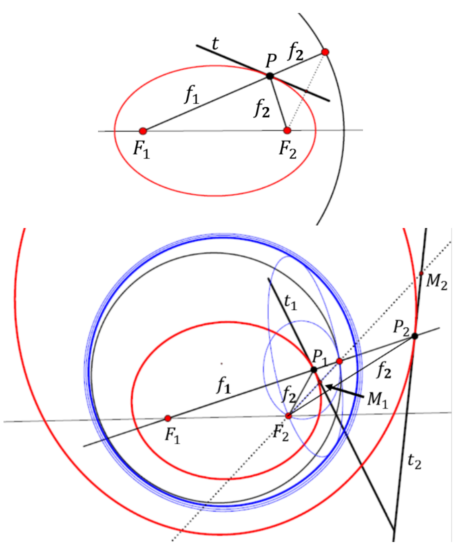

Figure 6: *"Gardener's construction" of an ellipse in the Euclidean and a hyperbolic plane.*

Note that a segment in a regular non-Euclidean plane has two midpoints M_i and therefore one gets two conics to a given proper pair of focal points and main axis length.
It turns out that also focus Definition 2 remains valid in non-Euclidean planes, see Figure 7.

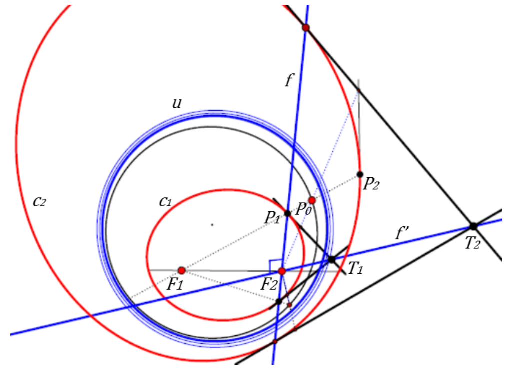

Figure 7: *In a non-Euclidean plane pairs of conjugate lines through a focus of a conic are orthogonal.*

4 Conics in the hyperbolic and elliptic plane based on de la Hire's construction

Obviously it is possible to perform de la Hire's construction (Figure 3) line by line also on the sphere just by avoiding the interpretation of orthogonal perspective affinities, see Figure 8. Central projection of the spherical figure from the midpoint of the sphere onto e.g. the plane of the main vertex circle of the spherical ellipse c gives the Euclidean version of de la Hire's construction.

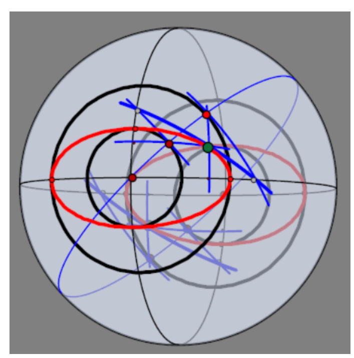

Figure 8: *Spherical version of de la Hire's construction of an ellipse.*

Similarly we can perform de la Hire's construction line by line also in the hyperbolic plane, see Figure 9. Note that in a (regular) non-Euclidean plane a conic c in general has three midpoints, which are the vertices of the polar triangle common to c and the absolute conic (resp. the absolute polarity). In Figure 9 we show a case with three real midpoints, labelled as M_1, M_2, M_3. We call M_1 the proper midpoint and M_2, M_3 ideal midpoints. In Figure 9 conic c has real axes a, b and reality of the axes is a necessary (and sufficient) condition to perform de la Hire's construction. Furthermore, as long as the correspondence between vertex circle and axis is not explicitly given one has two possibilities for de la Hire's construction and gets two conics as results. (Also in the Euclidean case one could permute the two perspective affinities and would get the second solution.)

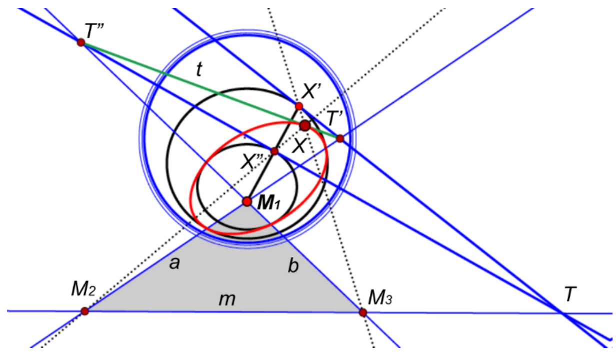

Figure 9: *De la Hire's construction of an ellipse in a hyperbolic plane.*

5 Conics in the hyperbolic and elliptic plane have, in general, six focal points

In the hyperbolic plane it is possible that a conic c has three real pairs of focal points F_i. This is the case, if c and the absolute conic u have four real tangents in common forming a quadrilateral, see Figure 10. All conics c (except u) touching this quadrilateral are hyperbolic confocal. The diagonal triangle of the common tangent quadrilateral is the common midpoint triangle (M_1, M_2, M_3) of all confocal conics. Let c_1 be one of the conics and let P be one of its points. Then it is of interest, whether the "gardener's construction" based on different pairs of focal points will deliver the same conic c_1:

From Projective Geometry (see e.g. [1]) follows that the pairs of lines (PF_1, PF_2), (PF_3, PF_4) and (PF_5, PF_6) are pairs of a hyperbolic involutoric projectivity, which has the tangent t and normal n of c in P as (orthogonal) fixed lines.

The common tangents of a general real conic c and the imaginary absolute conic u of an elliptic plane occur in two conjugate imaginary pairs, such that only one pair of focal points is real.

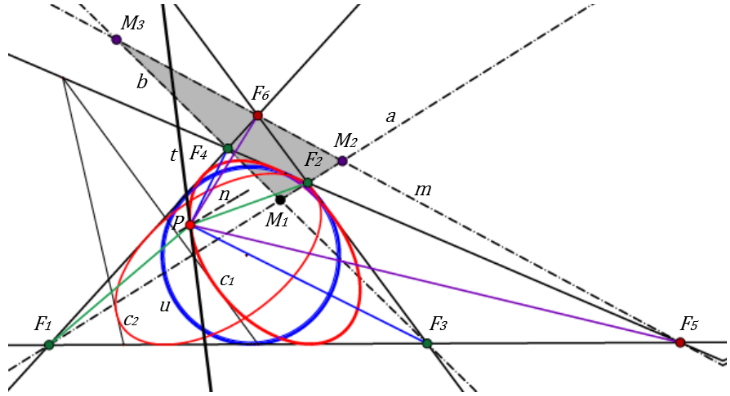

Figure 10: *Confocal pencil of conics with 6 real focal points in the hyperbolic plane.*

Concluding we state that all real pairs of focal points of a conic c in a hyperbolic plane are equivalent with respect to focus definitions 1, 2 and 3.

6 The orthoptic of a conic in hyperbolic and elliptic planes

The orthoptic of a given curve c is the set of points, from which one can draw orthogonal tangents to c, see [8]. In a Euclidean plane it is well-known that ellipses c have a real orthoptic circle, hyperbolas have a real or imaginary orthoptic circle, too. (For an equilateral hyperbola c in the projective extended Euclidean plane the pair of asymptotes is the only pair of real orthogonal tangents, such that the orthoptic degenerates into the centerpoint of c alone.) Parabolas have an orthoptic line, the directrix line. So we can say that *conics c have an orthoptic Möbius circle concentric to c*. Figure 11 shows the case of an ellipse in the Euclidean plane. In addition the "pedal curve" for the foci as poles is depicted, too. This pedal curve is the main vertex circle of c. Furthermore we note that this pedal circle is the same for each of the two foci of an ellipse or hyperbola. For a parabola this pedal curve is the vertex tangent, such that also the pedal curve of a conic c for a focus as pole is again a Möbius circle.

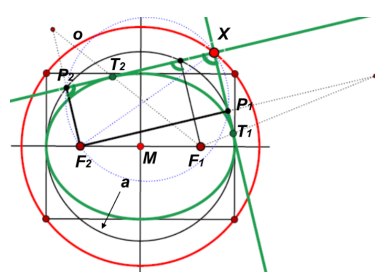

Figure 11: *Orthoptic circle o of an ellipse and its pedal circle for the foci as poles.*

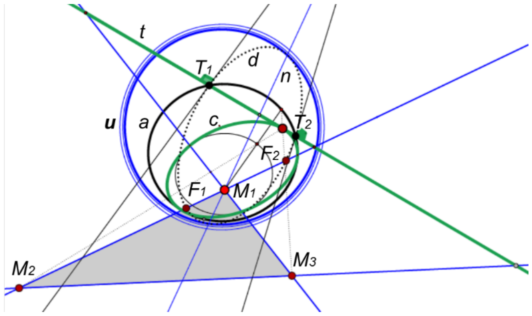

Figure 12: *In a hyperbolic plane π the intersections T_i of a tangent t to an ellipse c with the main vertex circle a and the focal points F_i are concyclic.*

In a hyperbolic plane π it turns out that the main vertex circle a is not the pedal curve of c with respect to a focus as pole, but the intersections T_i of a tangent t to an ellipse c with the main vertex circle a and the focal points F_i are points of a circle, see Figure 12.

Similar to the Euclidean case there is a "Poncelet-circle" p concentric with an ellipse c, containing the vertices of quadrangles subscribed to c. These quadrangles have metric properties with respect to hyperbolic geometry, but the circle p is not the hyperbolic orthoptic curve of the conic c, see Figure 13. This is obvious because of

Poncelet's Theorem: *If a closed $n-$gon is subscribed to a conic p and inscribed to another c, then there exist infinitely many such $n-$gons.*

In the Euclidean case p coincides with the orthoptic o and all Poncelet 4-gons are rectangles.

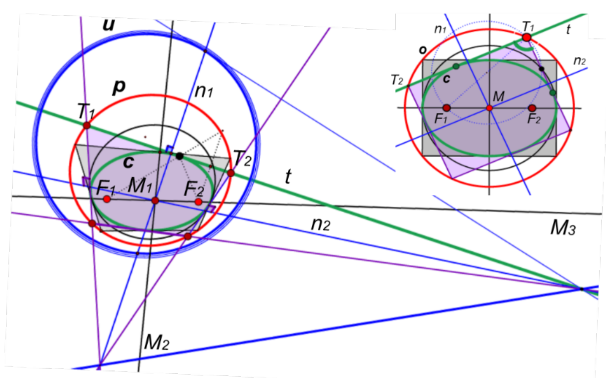

Figure 13: *An ellipse c in a Euclidean or non-Euclidean plane possesses a concentric circle p containing the vertices of Poncelet quadrangles subscribed to c.*

The presented results show that in a (regular) non-Euclidean geometry neither the orthoptic of a conic nor its pedal curve for a focus as pole can be a concentric circle.

Theorem 1 *The orthoptic curve of a conic c in a non-Euclidean plane π is a concentric conic o.*

Proof by applying the absolute polarity to conic c, which maps c to a conic c': The tangent $t' \perp t$ of c maps to an intersection point $T' \in T \cap c'$, see Figure 14.

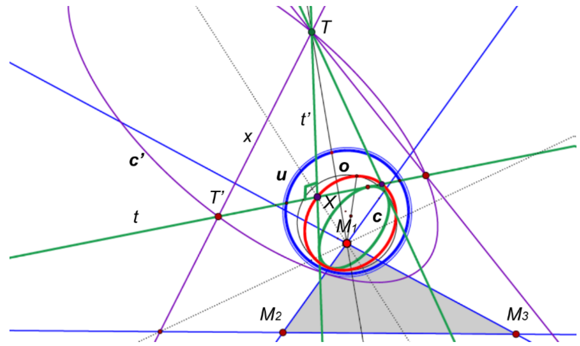

Figure 14: *In a hyperbolic plane π to an ellipse c there exists a concentric orthoptic conic o.*

7 Thales conics as orthoptics of a segment

A segment can be regarded as a singular conic c. Applying Theorem 1 follows:

The non-Euclidean orthoptic curve of a segment c is a concentric conic o, the "Thales-conic" of c.

The orthoptic o of segment c is generated by projective pencils of lines and therefore it is a conic.

Remark 2 *N. Wildberger calls the orthoptic of a segment its "thaloid". As the Euclidean case is called "Thales-circle" we prefer the concept "Thales-conic" comprising also the Euclidean case, c.f. [6]. For generalisations of the classical Thales-circle to Thales-surfaces see also [7].*

The orthoptic o of segment c is generated by projective pencils of lines and therefore it is a conic. It also allows a kinematic generation, see Figure 15: The orthogonal slider cross Σ_3 defines a forced motion keeping the two rods Σ_1, Σ_2 orthogonal, which rotate at the fixed poles P_{01}, P_{02}. Via the three-pole theorem of Aronhold (see [9]) it is possible to construct the instantaneous pole P_{03} and therewith also the tangents of o.

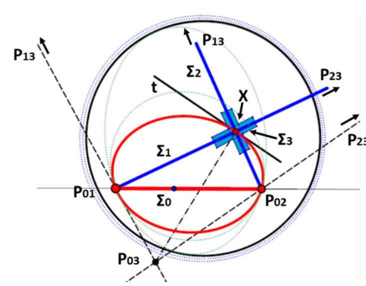

Figure 15: *Kinematic construction of points and tangents of a Thales-conic in a hyperbolic plane.*

8 Bodenmiller's theorem and its non-Euclidean counterparts

As an application of the concepts Thales-conic and orthoptic conic we start with the Euclidean theorem of Bodenmiller and study non-Euclidean versions of it. The theorem reads as follows, see e.g. [2] and Figure 16:

Theorem 2 (Bodenmiller) *Given a general quadrilateral in the Euclidean plane, then its three diagonal segments have Thales-circles belonging to a pencil of circles.*

Usually one calls the (real, coinciding or conjugate imaginary) common points of the Bodenmiller-circles the "Bodenmiller-points" of the quadrilateral. There are several extensions of this theorem, see also [2]:

(1) The midpoints of the Bodenmiller-Thales-circles are collinear with the Gauss-line g. (The Gauss-line of a quadrilateral contains the centres of all regular and singular conics c touching the quadrilateral; such a set of conics is called a 'dual pencil of conics'. The Gauss-line g is an affine geometric concept!)

(2) The common cord b of the three Bodenmiller-Thales-circles of a quadrilateral Q contains four orthocentres O_i of the four partial triangles of Q. (Obviously b is orthogonal to the Gauss-line g and it contains, in algebraic sense, 6 remarkable points of Q.)

(3) The pencil of Bodenmiller-Thales-circles of a quadrilateral Q consists of the orthoptics to the conics of the dual pencil of conics to Q.

Even though statement (3) is obvious and easy to prove, it seems not to be mentioned explicitly in relevant references:

For the proof we use a linear combination of two diagonal segments of Q, interpreted as singular conics of the dual pencil to Q and get a certain conic c of this pencil. The same linear combination of the corresponding Bodenmiller-Thales-circles then gives the orthoptic circle of that c.

Now the question arises, what about the Bodenmiller statements in non-Euclidean geometries?

Theorem 3 (Non-Euclidean Bodenmiller)

Given a quadrilateral Q, then the Thales-conics over the 3 diagonal segments belong to a pencil of conics.

With a special linear combination of two singular conics (i.e. two diagonal segments) of the dual pencil of conics to Q one might get the third singular conic. Applying the same linear combination to the corresponding "Bodenmiller-Thales-conics" will give the Bodenmiller-conic of the third diagonal segment, see Figure 17. We call the common points of the Bodenmiller-Thales-circles again "Bodenmiller-points".

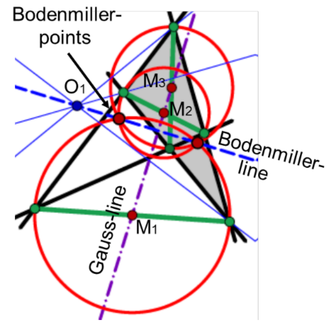

Figure 16: *Bodenmiller-circles, Bodenmiller-points and Gauss-line of a quadrilateral.*

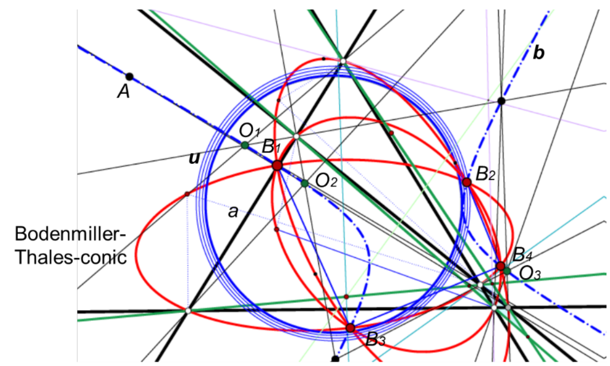

Figure 17: *Quadrilateral Q in a hyperbolic plane and its 3 Bodenmiller-Thales-conics. Its Bodenmiller-conic b contains 4 Bodenmiller-points B_i, the absolute poles of the sides of Q and 4 orthocentres O_i of the partial triangles of Q.*

Furthermore we conclude with the same arguments that the pencil of Bodenmiller-Thales-conics consists of the orthoptic conics to the dual pencil of conics touching quadrilateral Q, thus extending statement (3) also to non-Euclidean cases.

As a counterpart to the extension (2) of Bodenmiller's theorem we find the following:

Theorem 4 (non-Euclidean version of Bodenmiller property (2)) *Given a general quadrilateral Q in a hyperbolic or elliptic plane π, then (in algebraic sense) the four Bodenmiller-points B_i, the four absolute poles of the sides of Q, and the four orthocentres O_i of the partial triangles of Q are points of a single conic b, the "Bodenmiller-conic" of Q, (see Figure 17).*

Remark 3 *We mentioned that in the Euclidean case the Bodenmiller-line b contains 6 points, the two Bodenmiller-points and the four orthocentres of the partial triangles of quadrilateral Q. If we add the ideal line u to b, thus receiving a "singular Bodenmiller-conic", then we find the*

Euclidean absolute points as well as the ideal points of the directions orthogonal to the four sides of Q as additional points on the singular Bodenmiller-conic $(b \cup u)$. Therewith we find 12 remarkable points of a quadrilateral on its Bodenmiller-conic in all cases, the Euclidean and non-Euclidean planes.

It still remains to ask for the non-Euclidean counterpart of the Gauss-line g of a quadrilateral Q. Each diagonal segment has two non-Euclidean midpoints and it is not surprising that these all together 6 points belong to one conic, which we might call "Gauss-conic" g of Q. But note that this Gauss-conic does not contain the midpoint triplets of the regular conics c of the dual pencil of conics to Q! Therefore this conic cannot act as a suitable counterpart for the Euclidean Gauss-line. Let us add the absolute pole of each diagonal d_i to its midpoint pair M_1, M_2 as the third midpoint M_3 of a singular conic, see Figure 18. So we have set of 9 midpoints, which define a one parameter set of cubic curves.

This gives rise to following:

Conjecture (non-Euclidean version of Gauss-line property (1)) *Given a general quadrilateral Q in a hyperbolic or elliptic plane π, then the midpoint triplets of all conics touching Q belong to a cubic curve, the "Gauss-cubic" g_3.*

To prove this conjecture one might again use the idea of linear combining two singular conics: In the dual pencil of conics touching Q we use a pencil parameter t such that the singular conics correspond to parameter values 0, ∞, 1. To each of these special parameter values there will be a point triplet (M_1, M_2, M_3) $(t = 0, 1, \infty)$ of midpoints forming the basis points of a pencil of cubics. By a continuity argument with $t \in \mathbb{R}$ it will be possible to find an ordering within the triplets. Then $\{M_1(t)\} =: g_1$, $\{M_2(t)\} =: g_2$, $\{M_3(t)\} =: g_3$ are three parameterized arcs of curves which are conjectured to belong to one cubic of the pencil.

9 Concluding remarks

Even though there exist already many results on "elementary non-Euclidean geometry", see for example recent KoG-issues, there are still many open questions concerning circles and conics connected with triangles, quadrangles and quadrilaterals. In this article we focussed on the concept orthoptic of a conic and, as a special case, the Thales-conic to a segment and applied the results generalizing theorems for quadrilaterals, which are well-known in Euclidean geometry, namely the theorems of Bodenmiller and Gauss.

Still open questions are e.g. the focus loci of (dual) pencils of conics, Wallace-Simson's theorem, its extension to quadrilaterals as well as its connections with Miquel's theorem. One also could look for conics which have one or more properties of an equilateral hyperbola. Especially in higher dimensional non-Euclidean spaces (see [3]) there is little known about classical elementary geometry and one will find a large field of activity there.

Maybe this topics seem not to be "mathematical mainstream", but as C. Kimberling's incredibly growing list of now more than 7000 remarkable points of a triangle shows, see [5], there is a worldwide community dealing with what can be called "advanced elementary geometry". This might justify that paper, too.

After finishing this paper the author found an announcement of a book concerning conics, see [4], which will be available in Spring 2016 and also contains a chapter about conics in non-Euclidean geometries.

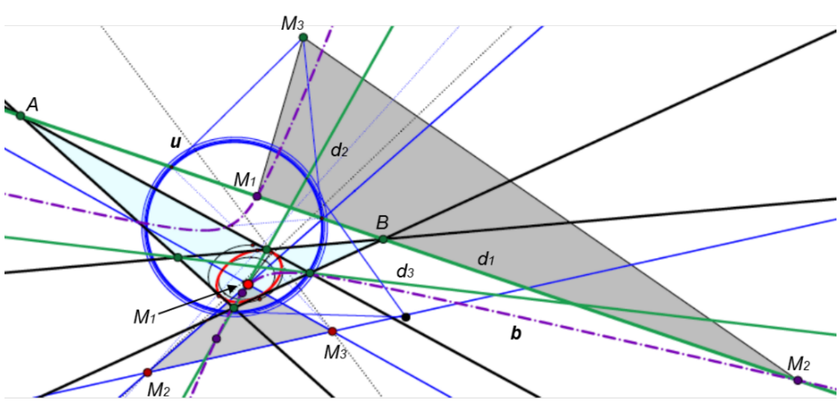

Figure 18: *Quadrilateral Q in a hyperbolic plane and its Gauss-conic g, which does not contain the midpoint triplets of the conics touching Q.*

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