Original scientific paper
Accepted 22. 12. 2015.

László Németh

Sectrix Curves on the Sphere

Sectrix Curves on the Sphere

ABSTRACT

In this paper we introduce a class of curves derived from a geometrical construction. These planar curves are the generalization of the less-known sectrix of Ceva. We also present three variations of the sectrix curves on the sphere with using the geometrical construction on the sphere, with the stereographic projection and with a so-called "rolled" transformation.

Key words: sectrix, folium, Chebyshev polynomial, curves on sphere

MSC2010: 51N20

Sektrise na sferi

SAŽETAK

U ovom članku uvodimo klasu krivulja izvedenih geometrijskom konstrukcijom. Takve ravninske krivulje su generalizacija manje poznatih Cevinih sektrisa. Također, prikazujemo tri varijacije sektrisa na sferi, koristeći geometrijsku konstrukciju na sferi, stereografsku projekciju i takozvano "valjano" preslikavanje.

Ključne riječi: sektrisa, folium, Chebyshevjev polinom, krivulje na sferi

1 Sectrix on the plane

The Sectrix of Ceva is a less-known planar curve ([7, p. 314-315]), that is defined with the polar equation

$$\rho = a + 2a\frac{\sin k\varphi \cos(k+1)\varphi}{\sin\varphi}, \quad a>0,\ k\in\mathbb{N},\ \varphi\in[0,2\pi]. \tag{1}$$

Figure 1 shows its shape where $a=1$ and $k=2$. It has two perpendicular axes of symmetry. In this article we use this curve in case $a=1$.

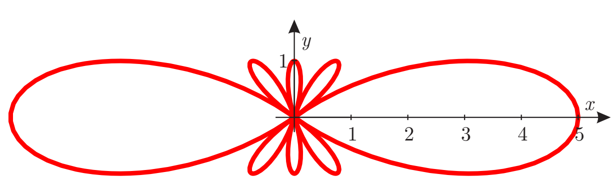

Figure 1: *Sectrix of Ceva ($k=2$).*

If $k=1$ then we get the so-called Ceva Cycloid (Figure 2). It was devised by Ceva, who termed it the cycloidum anomalarum ([2, p. 29], [8]). Its polar equation is

$$\rho = 1 + 2\cos 2\varphi,\ \varphi\in[0,2\pi]. \tag{2}$$

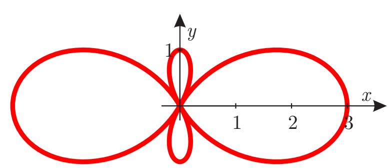

Figure 2: *Ceva Cycloid ($k=1$ or $n=2k+1=3$).*

In [3] a geometrical construction was defined from which a generalization of sectrix of Ceva comes. Let e be a line given by the origin O and angle α between axis x^+ and e, as the angle of polar coordinates of e (Figure 3). Let the point A_0 coincide with O. Let the point A_1 be given on e such that the distance between the points O and A_1 is 1. Let the point A_2 be on axis x such that the distance of A_1 and A_2 is equal also to 1 and $A_2 \neq O$ if it is possible. Then let the new point A_3 be on the line e again such that $A_2A_3 = 1$ and $A_3 \neq A_1$ if it is possible. Recursively, we can define the point A_i ($i \geq 2$) on the line e or on axis x if i is odd or even, respectively, where $A_{i-1}A_i = 1$ and $A_i \neq A_{i-2}$ if it is possible. For all α the point A_i exists. Figure 3 shows the first six points. If α is small enough then A_i is between points O and A_{i+2}. Let angle $OA_{i+1}A_i$ be α_i, then $\alpha_i = i\alpha$ can be proved easily. If A_1 is on the axis x we obtain a similar geometric construction (Figure 4). These constructions gave a new proof for some trigonometric connections [5].

The parametric equation system of the orbits of the points in [3] is determined not only when the point A_1 is on line e, but also when it is on axis x. In case of vertices A_n $(n \geq 1)$ the parametric equation system of the curves is

$$\begin{aligned} x_n(\alpha) = &\quad \cos\alpha\, U_{n-1}(\cos\alpha) \\ y_n(\alpha) = &\quad \sin\alpha\, U_{n-1}(\cos\alpha), \end{aligned} \tag{3}$$

and the polar equation of the curves when $\alpha \in [0, 2\pi]$ is

$$\rho_n(\alpha) = U_{n-1}(\cos\alpha), \tag{4}$$

where $U_{n-1}(x)$ is the Chebyshev polynomial of the second kind. (Some orbits can be seen on Figure 3 and 4.) The recursive definition of the Chebyshev polynomials of the second kind $U_\ell(x)$ is

$$U_0(x) = 1,\ U_1(x) = 2x,\ U_{\ell+1}(x) = 2xU_\ell(x) - U_{\ell-1}(x),\ \ \ell \geq 1. \tag{5}$$

When $|x| \leq 1$ the substitution $x = \cos\varphi$ gives the expressions $\sin\ell\varphi = \sin\varphi\, U_{\ell-1}(\cos\varphi)$ [6].

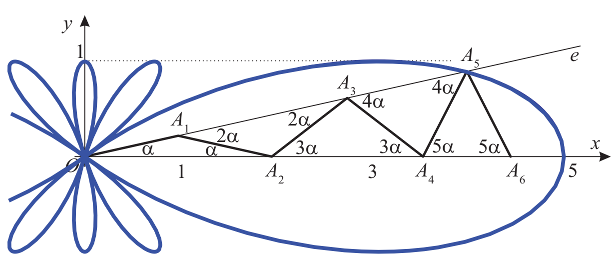

Figure 3: *Generalized sectrix of Ceva in case* $n = 5$.

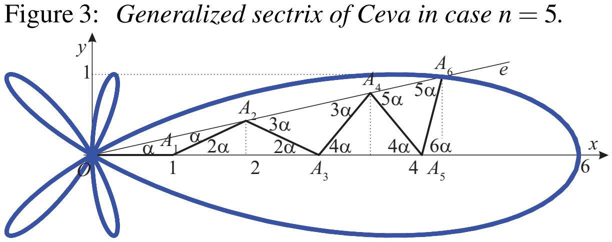

Figure 4: *Generalized sectrix of Ceva in case* $n = 6$.

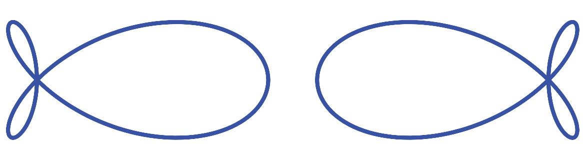

Figure 5: *Folium – curves in case* $n = 4$ and $n = -4$.

Lemma 1 *If* $n = 2k + 1$ *the curves defined by equations* (4) *and* (1) *for* $a = 1$ *are the same.* (Compare the Figures 1 and 3.)

Proof. Since

$$\begin{aligned} \sin(2k+1)\alpha &= \sin(k+k+1)\alpha \\ &= \sin k\alpha \cos(k+1)\alpha + \cos k\alpha \sin(k+1)\alpha \\ &= \sin k\alpha \cos(k+1)\alpha \\ &\quad + \cos k\alpha\, (\sin k\alpha \cos\alpha + \cos k\alpha \sin\alpha) \\ &= \sin k\alpha \cos(k+1)\alpha \\ &\quad + \sin k\alpha \cos k\alpha \cos\alpha + \cos^2 k\alpha \sin\alpha \\ &= \sin k\alpha \cos(k+1)\alpha \\ &\quad + \sin k\alpha\, (\cos(k+1)\alpha + \sin\alpha \sin k\alpha) \\ &\quad + \cos^2 k\alpha \sin\alpha \\ &= 2\sin k\alpha \cos(k+1)\alpha + \sin^2 k\alpha \sin\alpha \\ &\quad + \cos^2 k\alpha \sin\alpha \\ &= 2\sin k\alpha \cos(k+1)\alpha + \sin\alpha, \end{aligned}$$

if $\varphi = \alpha$ then we have

$$\begin{aligned} U_{2k}(\cos\alpha) &= \frac{\sin(2k+1)\alpha}{\sin\alpha} = \frac{2\sin k\alpha \cos(k+1)\alpha + \sin\alpha}{\sin\alpha} \\ &= 1 + 2\frac{\sin k\alpha \cos(k+1)\alpha}{\sin\alpha}. \end{aligned}$$

□

The Cartesian equation of curves (without the point in the origin) defined with (3) or (4) is

$$x^2 + y^2 = U_{n-1}^2\left(\sqrt{\frac{x^2}{x^2+y^2}}\right), \tag{6}$$

where $x^2 + y^2 \neq 0$. We extend the equation (6) to negative values n. The definition of the Chebyshev-polynomials for negative indexes with definition (5) is

$$U_{\ell-1}(x) = 2xU_\ell(x) - U_{\ell+1}(x),\ \ \ell < 1. \tag{7}$$

Now, $U_{-1}(x) = 0$ and $U_m(x) = -U_{-m-2}(x)$, $(m \leq -2)$ and $x^2 + y^2 \neq 0$ implies $n \neq 0$. If $n = 2k + 1$ then equation (6) gives the sectrix of Ceva. Otherwise, if $n = 2k$ we get the union of curves in case n and $-n$ (see Figure 5 and 6).

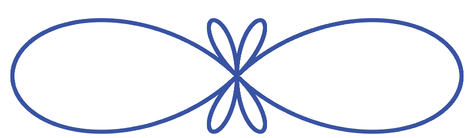

Figure 6: *Union of curves in case* $n = 4$ *and* $n = -4$.

Moreover, the polar equation of folium curve is

$$\rho = \cos\varphi\left(4a\sin^2\varphi - b\right), \quad \varphi \in [0, 2\pi], \tag{8}$$

and the curve defined by equations (4) is the folium curve if $n = -4$ and $a = 2$, $b = 4$, as $U_{-5}(\cos\alpha) = -U_3(\cos\alpha) = -8\cos^3\alpha + 4\cos\alpha = -4\cos\alpha(2\cos^2\alpha - 1) = \cos\alpha(8\sin^2\alpha - 4)$ (see Figure 5).

Here are some Cartesian equations from (6):

- $|n| = 1$: $x^2 + y^2 = 1$ (circle)
- $|n| = 2$: $(x^2 + y^2)^2 = 4x^2$ (two circles)
- $|n| = 3$: $(x^2 + y^2)^3 = (3x^2 - y^2)^2$ (Ceva cycloid, Figure 2)
- $|n| = 4$: $(x^2 + y^2)^4 = 16x^4(x^2 - y^2)^2$ (union of foliums, Figure 6)
- $|n| = 5$: $(x^2 + y^2)^5 = (5x^4 - 10x^2y^2 + y^4)^2$ (sectrix of Ceva, Figure 1).

The two biggest loops are very similar to the loops of the lemniscates. In Figure 7 we can see that the angles between any two lines m_i $(i = 1, \ldots, 2n)$ are the multiples of $\pi/(2n)$, where m_i is a tangent line of the curve in the origin or a line goes through the origin and one of the extrema of the curve ($x = n$ or $y = \pm 1$). For some more details, for more figures and for a generalization see [3, 4]. These curves can also be considered as the generalizations of the well-known rose curves (see in [1]).

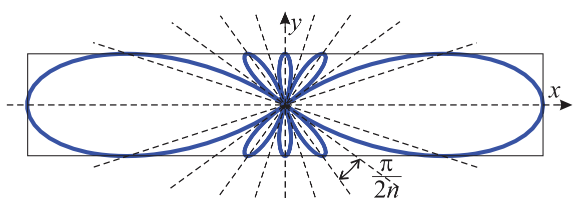

Figure 7: *Curve with some properties in case $n = 5$.*

1.1 Sectrix on the sphere

In this subsection we determine the orbit of the point A_n $(n \geq 1)$ with similar conditions as in Section 1 on a sphere. We consider a sphere with radius 1 with equation $x^2 + y^2 + z^2 = 1$. Let the axes ξ and ψ of the coordinate system on it, with origin $K(1,0,0)$, be main circles according to Figure 8.

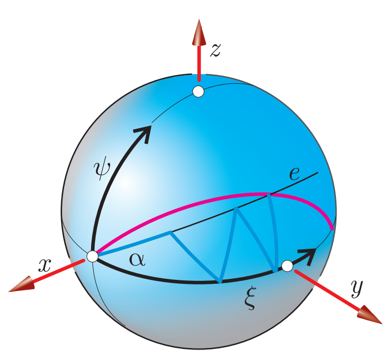

Figure 8: *Construction on the sphere.*

Let e also be a main circle through point K and let the rotation angle between the axis ξ and the "line" e be α, where $0 \leq \alpha \leq 2\pi$. Moreover, let the distance between two consecutive points be d, where $0 < d < \pi/2$.

Figure 9 demonstrates the construction in a plane with coordinate axes ξ and ψ. (Compare Figures 8 and 9.) Moreover, Figure 9 shows an odd case when A_1 lies on e and $n = 5$. Let α_i $(i \geq 1)$ be the angles $A_iA_{i-1}A_{i+1}$ and $A_{i-1}A_{i+1}A_i$ and let $2\beta_i$ be the angle $A_{i-1}A_iA_{i+1}$. (If $\alpha_i < \pi/2$ then the point A_i $(i \geq 2)$ is further from the origin then A_{i-2}.)

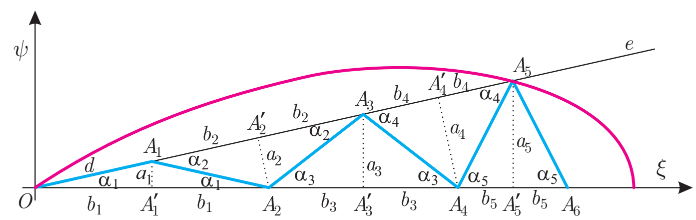

Figure 9: *Construction in the plane with coordinate axes ξ and ψ.*

Let the orthogonal projection of A_i to ξ or e in case if i is odd or even, respectively, be A'_i. We denote by b_i and a_i the spherical segments $A_{i-1}A'_i = A_{i+1}A'_i$ and $A_iA'_i$, respectively. Now we determine the angles α_i recursively.

Lemma 2 *If $i \geq 2$ then*

$$\alpha_i = \pi - 2\beta_{i-1} - \alpha_{i-2},$$

where $\alpha_0 = 0$, $\alpha_1 = \alpha$ *and* $\beta_{i-1} = \operatorname{arccot}(\cos d \tan \alpha_{i-1})$.

Proof: From the triangle $A_0A_1A_2$ we obtain at point A_1 that $\alpha_2 = \pi - 2\beta_1$ (see Figure 9). We suppose the lemma holds for any j from 2 up to $i - 1$. From the triangle $A_{i-2}A_{i-1}A_i$ $(i \geq 3)$ we obtain at point A_{i-1} that $\alpha_{i-2} + \alpha_i = \pi - 2\beta_{i-1}$ and by the use of the spherical trigonometric identity

$$\cot \alpha_{i-1} \cot \beta_{i-1} = \cos d$$

in the right angled triangle $A_{i-2}A_{i-1}A'_{i-1}$ we get the lemma. □

From the triangle $A_{i-1}A'_iA_i$ using the spherical trigonometry the next lemma holds.

Lemma 3

$$\begin{aligned} \sin a_i &= \sin d \cdot \sin \alpha_i, \\ \tan b_i &= \tan d \cdot \cos \alpha_i. \end{aligned}$$

Theorem 1 follows from the summation of the lemmas.

Theorem 1 *The equation system of the sectrix on the sphere is*

$$\begin{aligned} x_n(\alpha) &= \cos\psi_n(\alpha)\cos\xi_n(\alpha) \\ y_n(\alpha) &= \cos\psi_n(\alpha)\sin\xi_n(\alpha) \\ z_n(\alpha) &= \sin\psi_n(\alpha), \end{aligned}$$

where

$$\begin{aligned} \xi_n(\alpha) &= \begin{cases} 2(b_1+b_3+\cdots+b_{n-2})+b_n & \text{if } n=2k+1, \\ 1+2(b_2+b_4+\cdots+b_{n-2})+b_n & \text{if } n=2k, \end{cases} \\ \psi_n(\alpha) &= a_n. \end{aligned}$$

We mention that from the triangle $OA_n'A_n$ the equation

$$\sin\xi_n(\alpha) = \cot\alpha\cdot\tan\psi_n(\alpha)$$

gives the implicit connection between the coordinates.

By using the parametric equations from Theorem 1, we obtain the visualizations of the curves in the software *Maple 17* (from *Maplesoft*). In this article, we improve the quality of Maple-graphics by re-rendering in the software *POV-Ray*.

Figures 10–12 show some curves on the sphere in case $n = 3$, 4 and 5.

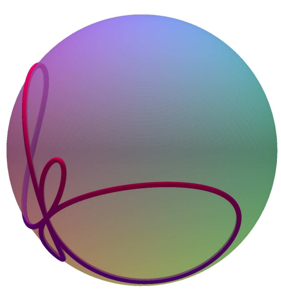

Figure 10: *Sectrix curve on sphere in case $n = 3$ and $d = \pi/6$.*

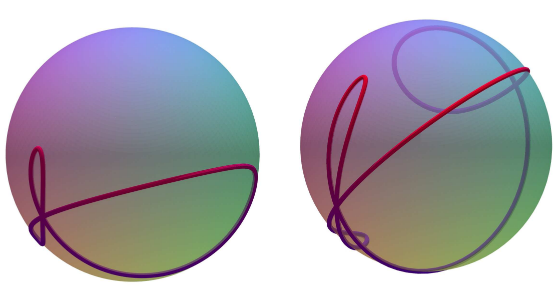

Figure 11: *Sectrix curves on sphere in case $n = 4$, $d = \pi/6$ and $d = \pi/3$.*

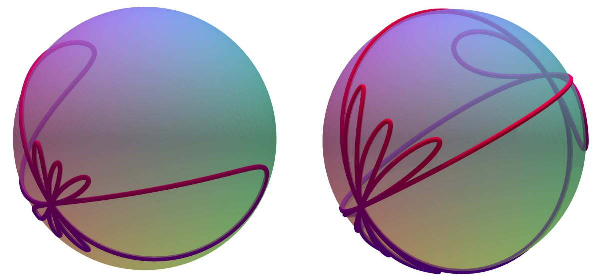

Figure 12: *Sectrix curves on sphere in case $n = 5$, $d = \pi/8$ and $d = \pi/4$.*

1.2 Sectrix curves with stereographic projection

We obtain similar curves on the surface of a sphere with the stereographic projection of the sectrix. Let the curve be in plane $z = -R$ and project it from point $N(0,0,R)$ into the sphere with centre $(0,0,0)$ and radius R (Figure 13).

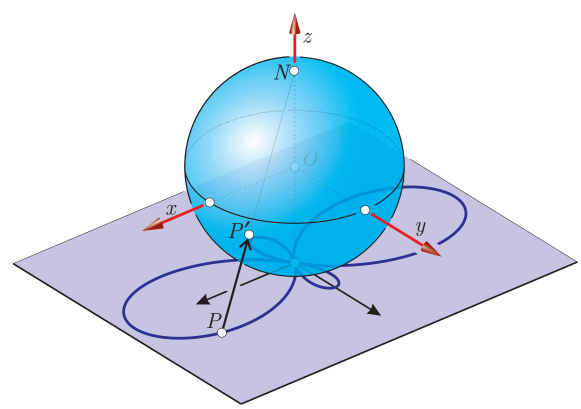

Figure 13: *Stereographic projection.*

In this case one can easily gain that the stereographic projection of a general point $P(x,y,-R)$ from the plane is $P'(cx,cy,1-2c)$, where $c = 4R^2/(x^2+y^2+4R^2)$.

Thus the stereographic projection of the curve with equation (3) is

$$\begin{aligned} x_n(\alpha) &= c(\alpha)\cos\alpha\, U_{n-1}(\cos\alpha), \\ y_n(\alpha) &= c(\alpha)\sin\alpha\, U_{n-1}(\cos\alpha), \\ z_n(\alpha) &= 1-2c(\alpha), \end{aligned} \tag{9}$$

where

$$c(\alpha) = \frac{4R^2}{U_{n-1}^2(\cos\alpha)+4R^2}, \qquad \alpha\in[0,2\pi].$$

Figures 14 and 15 give some examples in case $n = 3$, 4, 5, 6, 7 and 10 where $R = 1$ and in the figures we rotated the curves around axis z for better visualization.

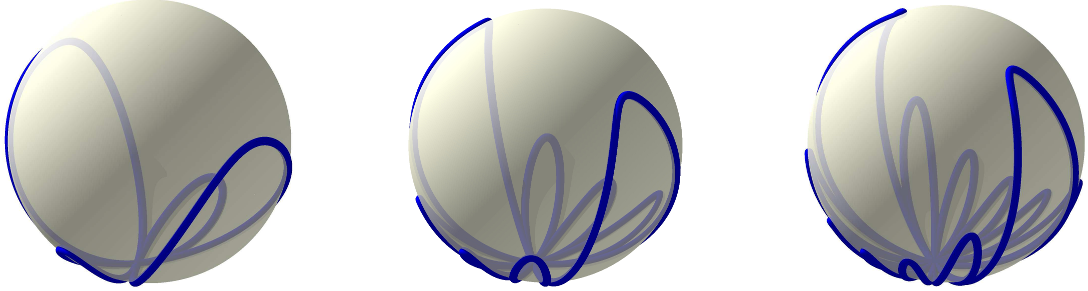

Figure 14: *Stereographic projection of sectrix in case $n = 3$, 5 and 7.*

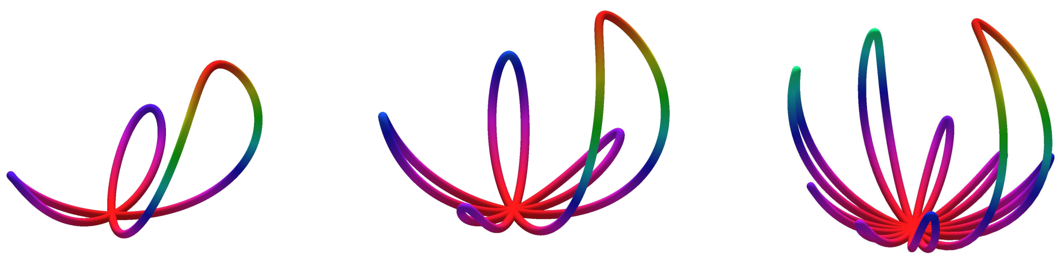

Figure 15: *Stereographic projection of sectrix in case $n = 4$, 6 and 10.*

1.3 Rolled sectrix on the sphere

In this subsection we give curves which are "rolled" to sphere. Let the radius of the sphere with centre $O(0,0,0)$ be R and let the plane of sectrix be the plane $z = -R$ (with point $S(0,0,-R)$) according to Figure 16.

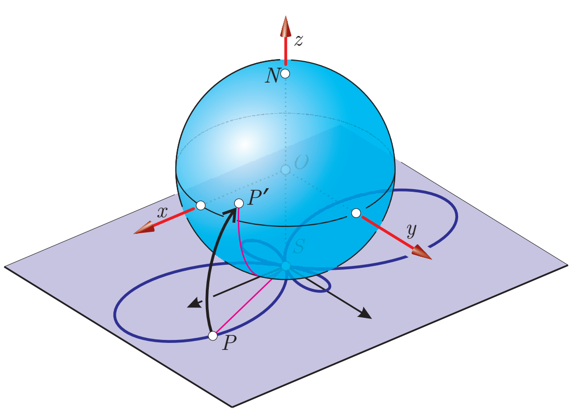

Figure 16: *Rolling of the sectrix onto sphere.*

Let P with parameter α be one of the points of the sectrix defined by (3), take the plane Π incident to the axis z and parallel to direction α (thus P is on Π) and let $P' \in \Pi$ be a point on the sphere so that the length of arc SP' be equal to $\rho_n(\alpha)$ from polar equation (4). In that way we project the point of the sectrix onto the sphere and the equation system of the curves ($\alpha \in [0,2\pi]$) is

$$\begin{aligned} x_n(\alpha) &= R\cos(r(\alpha))\cos\alpha, \\ y_n(\alpha) &= R\cos(r(\alpha))\sin\alpha, \\ z_n(\alpha) &= R\sin(r(\alpha)), \end{aligned} \tag{10}$$

where

$$r(\alpha) = \frac{U_{n-1}(\cos\alpha)}{R} - \frac{\pi}{2}.$$

Figures 17 and 18 show some examples of the rolled sectrix curves, where the curves are rotated for better visualization.

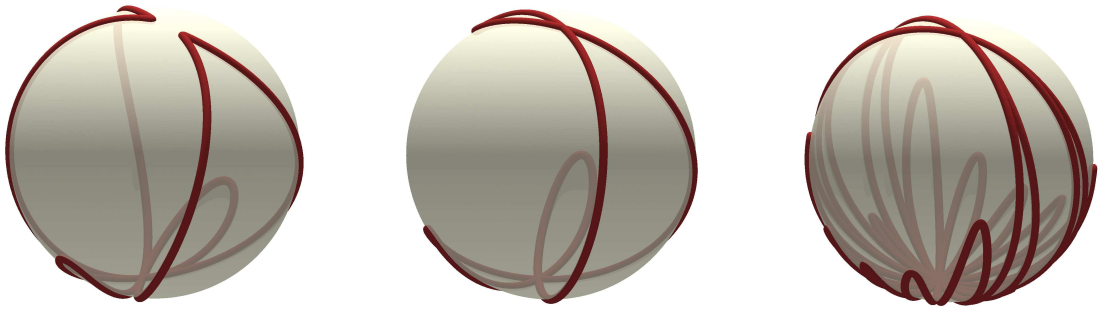

Figure 17: *Sectrix rolled onto sphere* ($n = 3,\ 4,\ 7$ *and* $R = 1$).

Figure 18: *Sectrix rolled onto sphere* ($n = 5,\ 6,\ 9$ *and* $R = 1,\ 2,\ 2$).

References

[1] S. GORJANC, Rose surfaces and their visualizations, *Journal for Geometry and Graphics*, **13**(1) (2010), 1–9.

[2] E. S. LOOMIS, The Cycloid of Ceva. §2.7 in *The Pythagorean Proposition: Its Demonstrations Analyzed and Classified and Bibliography of Sources for Data of the Four Kinds of Proofs*, 2nd ed. Reston, VA: National Council of Teachers of Mathematics, (1968), 29–30.

[3] L. NÉMETH, A new type of lemniscate, *NymE SEK Tudományos Közlemények XX.* Természettudományok 15. Szombathely, (2014), 9–16.

[4] L. NÉMETH, Rose curves with Chebyshev polynomials, *Journal for Geometry and Graphics*, **19**(2), (2015), 219–226.

[5] L. NÉMETH, A geometrical proof of sum of $\cos n\varphi$, *Studies of the University of Žilina, Mathematical Series*, **27** (2015), 63–66.

[6] T. J. RIVLIN, *Chebyshev polynomials*, New York Wiley (1990).

[7] E. V. SHINKIN, *Handbook and Atlas of Curves*, CRC Press (1995).

[8] E. W. WEISSTEIN, "Cycloid of Ceva". From MathWorld–A Wolfram Web Resource. `http://mathworld.wolfram.com/CycloidofCeva.html`

László Németh
e-mail: nemeth.laszlo@emk.nyme.hu

Institute of Mathematics
University of West Hungary
9400 Sopron, Ady E. u. 5, Hungary