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ON ABSOLUTE NORMAL DOUBLE MATRIX SUMMABILITY METHODS

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Abstract. It is shown that, if T is a row finite nonnegative double summability matrix satisfying certain conditions, then $|T|_k$ summability is stronger than $|\bar{N}, p_n|_k$ summability, for $k \geq 1$. As special summability methods T we consider weighted mean and double Cesáro, $(C, 1, 1)$, methods.

1. Introduction

Let $\sum\sum a_{mn}$ be a doubly infinite series with partial sums s_{mn}. Denote by T the doubly infinite matrix with entries $t_{mnij}, 0 \leq i \leq m, 0 \leq j \leq n$. For any double sequence $\{u_{mn}\}$ we shall define

$$\Delta_{11} u_{mn} = u_{mn} - u_{m+1,n} - u_{m,n+1} + u_{m+1,n+1}.$$

For a four-fold sequence, like t_{mnij}, it will be understood that Δ_{11} operates only on the first two subscripts, and Δ_{ij} operates only on the last two subscripts.

Associated with T are two matrices $\bar{T}$ and $\hat{T}$, where

$$\bar{t}_{mnij} = \sum_{\mu=i}^{m} \sum_{\nu=j}^{n} t_{mn\mu\nu}, m, n, i, j = 0, 1, \ldots,$$

and $\hat{t}_{m-1,n-1,i,j} = \Delta_{11} \bar{t}_{m-1,n-1,i,j}, m, n = 1, 2, \ldots, \hat{t}_{00} = \bar{t}_{00} = t_{00}$.

We shall define

$$T_{mn} = \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} t_{mn\mu\nu} s_{\mu\nu}. \tag{1.1}$$

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Let $\sum a_n$ be a given series with partial sums s_n, (C, α) the Cesáro matrix of order α. If σ_n^α denotes the n-th term of the (C, α)-transform of $\{s_n\}$, then, from Flett [5], $\sum a_n$ is summable $|C, \alpha|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |\sigma_n^\alpha - \sigma_{n-1}^\alpha|^k < \infty. \tag{1.2}$$

An appropriate extension of (1.2) to arbitrary lower triangular matrices would be

$$\sum_{n=1}^{\infty} n^{k-1} |\Delta t_{n-1}|^k < \infty, \tag{1.3}$$

where

$$t_n := \sum_{i=0}^{n} a_{ni} s_i,$$

and Δ is the forward difference operator satisfying $\Delta t_n = t_n - t_{n+1}$.

Such an extension is used in [3]. However, in [4], Bor and Thorpe replace (1.2) with the following definition. A series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$ if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\Delta Z_{n-1}|^k < \infty, \tag{1.4}$$

where $(\bar{N}, p)$ denotes the weighted mean matrix generated by the sequence $\{p_n\}$, with partial sums P_n, and $Z_n := (1/P_n) \sum_{i=0}^{n} p_i s_i$.

Inequality (1.4) is apparently an attempt to extend (1.2) to weighted mean methods by interpreting n to be the reciprocal of the diagonal entries of $(C, 1)$. Unfortunately this is not an appropriate interpretation of (1.2). For, if it were, then, for the matrix methods (C, α), one would have the condition

$$\sum_{n=1}^{\infty} (n^\delta)^{k-1} |\sigma_n^\alpha - \sigma_{n-1}^\alpha|^k < \infty.$$

However, Flett [5, p. 115] continues to use (1.2). In spite of this fact, a number of papers have been published using (1.4). See, e.g., [1, 2, 4, 8].

In [7] the author proved the analog of the theorem in [8], using (1.3) instead of (1.4), and obtained some of the results of [1, 2, 3, 8] as corollaries.

In this paper we obtain a two infinite-dimensional analog of absolute summability as defined in [7].

Adopting the two-dimensional analog of absolute summability as defined in [7], we shall say that the series $\sum \sum a_{mn}$ is absolutely T-summable of order $k \geq 1$ if

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{k-1} |\Delta_{11} T_{m-1,n-1}|^k < \infty. \tag{1.5}$$

We shall define the mn-th term of the double weighted mean transform of a double sequence $\{s_{mn}\}$ by

$$Z_{mn} = \frac{1}{P_{mn}} \sum_{i=0}^{m} \sum_{j=0}^{n} p_{ij} s_{ij}, P_{mn} := \sum_{i=0}^{m} \sum_{j=0}^{n} p_{ij}. \tag{1.6}$$

A doubly infinite weighted mean matrix will be called factorable if there exist sequences $\{p_m\}, \{q_n\}$ such that $p_{mn} = p_m q_n$, and we focus on this case below.

For any sequence $\{u_{ij}\}, \Delta_{0j} := u_{ij} - u_{i,j+1}, \Delta_{i\,0} u_{ij} := u_{ij} - u_{i+1,j}$, and $\Delta_{ij} u_{ij} := u_{ij} - u_{i,j+1} - u_{i+1.j} + u_{i+1.j+1}$.

2. Main result

THEOREM 2.1. *Suppose that a double factorable positive sequence $\{p_{mn}\}$ and a positive matrix T satisfy*

(i) $t_{mnij} \geq t_{m+1,n,i,j}, t_{mnij} \geq t_{m,n+1,i,j}, m \geq i, n \geq j, i, j = 0, 1, \ldots;$

(ii) $P_{mn} t_{mnmn} = O(p_{mn});$

(iii) $\displaystyle\sum_{\mu=0}^{m} t_{mn\mu j} = \sum_{\mu=0}^{m-1} t_{m-1,n,\mu,j} = a(n, j),$

$$\sum_{\nu=0}^{n} t_{mni\nu} = \sum_{\nu=0}^{n-1} t_{m,n-1,i,\nu} = b(m, i),$$

$$0 \leq i \leq m, 0 \leq j \leq n, m, n = 0, 1, \ldots,$$

and

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} t_{mn\mu\nu} = 1, \text{for} \quad m, n = 0, 1, \ldots;$$

(iv) $\Delta_{11} t_{m-1,n-1,i,j} \leq 0$, *for* $0 \leq i < m, 0 \leq j < n, m, n = 1, 2, \ldots;$

(v) $\displaystyle\sum_{i=1}^{m} \sum_{j=1}^{n} t_{ijij} |\hat{t}_{m-1,n-1,i,j}| = O(t_{mnmn});$

(vi) $\displaystyle\sum_{m=i}^{\infty} \sum_{n=j}^{\infty} (mnt_{mnmn})^{k-1} |\hat{t}_{m-1,n-1,i,j}| = O(ijt_{ijij})^{k-1});$

(vii) $\displaystyle\sum_{i=1}^{m} \sum_{j=1}^{n} \Big(\frac{p_i}{P_i}\Big) |\Delta_{0j} \hat{t}_{m-1,n-1,i,j}| = O(t_{mnmn});$

(viii) $\displaystyle\sum_{i=1}^{m} \sum_{j=1}^{n} \Big(\frac{q_j}{Q_j}\Big) |\Delta_{i0} \hat{t}_{m-1,n-1,i,j}| = O(t_{mnmn});$

(ix) $\displaystyle\sum_{m=i}^{\infty} \sum_{n=j}^{\infty} (mnt_{mnmn})^{k-1} |\Delta_{0j} \hat{t}_{m-1,n-1,i,j}| = O\Big((ijt_{ijij})^{k-1} \frac{q_j}{Q_j}\Big)$

(x) $\displaystyle\sum_{m=1}^{\infty}\sum_{n=j}^{\infty}(mnt_{mnmn})^{k-1}|\Delta_{i0}\hat{t}_{m-1,n-1,i,j}| = O\Big((ijt_{ijij})^{k-1}\frac{p_i}{P_i}\Big)$

(xi) $\displaystyle\sum_{m=i}^{\infty}\sum_{n=j}^{\infty}(mnt_{mnmn})^{k-1}|\Delta_{ij}\hat{t}_{m-1,n-1,i,j}| = O((ij)^{k-1}t_{ijij}^{k}).$

If $\sum\sum a_{mn}$ *is summable* $|\bar{N}, p_{mn}|_k$*, then it is also summable* $|T|_k, k \geq 1$.

The notation $\Delta_{11}t_{m-1,n-1,r,s} := t_{m-1,n-1,r,s} - t_{m,n-1,r,s} - t_{m-1,n,r,s} + t_{mnrs}$.

Lemma 2.2. *If T satisfies conditions* (i), (iii)*, and* (iv)*, then T satisfies*

(xii) $\hat{t}_{m-1,n-1,i,j} \geq 0$ *for* $0 \leq i \leq m, 0 \leq j \leq n$, $m, n = 1, 2, \ldots$.

Proof. Using (iii), for $0 \leq i < m, 0 \leq j < n$,

$$\begin{aligned}
\hat{t}_{m-1,n-1,i,j} &= \bar{t}_{m-1,n-1,i,j} - \bar{t}_{m,n-1,i,j} - \bar{t}_{m-1,n,i,j} + \bar{t}_{mnij} \\
&= \sum_{r=i}^{m-1}\sum_{s=j}^{n-1} t_{m-1,n-1,r,s} - \sum_{r=i}^{m}\sum_{s=j}^{n-1} t_{m,n-1,r,s} - \sum_{r=i}^{m-1}\sum_{s=j}^{n} t_{m-1,n,r,s} \\
&\quad + \sum_{r=i}^{m}\sum_{s=j}^{n} t_{mnrs} \\
&= 1 - \sum_{r=0}^{m-1}\sum_{s=0}^{j-1} t_{m-1,n-1,r,s} - \sum_{r=0}^{i-1}\sum_{s=j}^{n-1} t_{m-1,n-1,r,s} \\
&\quad - 1 + \sum_{r=0}^{m}\sum_{s=0}^{j-1} t_{m,n-1,r,s} + \sum_{r=0}^{i-1}\sum_{s=j}^{n-1} t_{m,n-1,r,s} \\
&\quad - 1 + \sum_{r=0}^{m-1}\sum_{s=0}^{j-1} t_{m-1,n,r,s} + \sum_{r=0}^{i-1}\sum_{s=j}^{n} t_{m-1,n,r,s} \\
&\quad + 1 - \sum_{r=0}^{m}\sum_{s=0}^{j-1} t_{mnrs} - \sum_{r=0}^{i-1}\sum_{s=j}^{n} t_{mnrs} \\
&= -\sum_{r=0}^{m-1}\sum_{s=0}^{j-1} \Delta_{11}t_{m-1,n-1,r,s} + \sum_{s=0}^{j-1}(t_{m,n-1,m,s} - t_{mnms}) \\
&\quad - \sum_{r=0}^{i-1}\sum_{s=j}^{n-1} \Delta_{11}t_{m-1,n-1,r,s} + \sum_{r=0}^{i-1}(t_{m-1,n,r,n} - t_{mnrn}) \\
&\geq 0,
\end{aligned}$$

since, by (iv), the terms involving the double summation are nonnegative and, from (i), $t_{m,n-1,m,s} \geq t_{mnms}$, and $t_{m-1,n,r,n} \geq t_{mnrn}$.

Suppose that $0 \le i < m, j = n$. Then, since $\bar{t}_{m-1,n-1,i,n} = \bar{t}_{m,n-1,i,n} = 0$, and using (i) and (iii),

$$\begin{aligned}\hat{t}_{m-1,n-1,i,n} &= -\bar{t}_{m-1,n,i,n} + \bar{t}_{mnin} = -\sum_{r=i}^{m-1} t_{m-1,n,r,n} + \sum_{r=i}^{m} t_{mnrn} \\ &= -a(n,n) + \sum_{r=0}^{i-1} t_{m-1,n,r,n} + a(n,n) - \sum_{r=0}^{i-1} t_{mnrn} \\ &= \sum_{r=0}^{i-1}(t_{m-1,n,r,n} - t_{mnrn}) \ge 0.\end{aligned}$$

Similarly, for $i = m, 0 \le j < n, \hat{t}_{m-1,n-1,m,j} \ge 0$.

Finally, $\hat{t}_{m-1,n-1,m,n} = t_{mnmn} \ge 0$. ∎

We remark that condition (xii) can be weakened. All that is required is that the $\hat{t}_{m-1,n-1,i,j}$ be of constant sign for all m and n.

In condition (iii) one needs only that $\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} t_{mn\mu\nu} = c$ for some constant c.

Proof of the Theorem 2.1. To carry out the proof we will solve (1.6) for a_{mn} (formula (2.2)). Then, after using (1.1) to compute $\Delta_{11}T_{m-1,n-1}$ ((2.3)), substitute (2.2) into (2.3) to obtain an expression for $\Delta_{11}T_{m-1,n-1}$ in terms of Z_{mn}. Then apply (1.5).

From (1.6), since the weighted mean method is factorable,

$$\begin{aligned}Z_{mn} &= \frac{1}{P_{mn}} \sum_{\mu=0}^{m}\sum_{\nu=0}^{n} p_{\mu\nu}s_{\mu\nu} = \frac{1}{P_{mn}} \sum_{\mu=0}^{m}\sum_{\nu=0}^{n} p_{\mu\nu} \sum_{i=0}^{\mu}\sum_{j=0}^{\nu} a_{ij} \\ &= \frac{1}{P_{mn}} \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij} \sum_{\mu=i}^{m}\sum_{\nu=j}^{n} p_{\mu\nu} \\ &= \frac{1}{P_m Q_n} \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}(P_m - P_{i-1})(Q_n - Q_{j-1}) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\Big(1 - \frac{P_{i-1}}{P_m}\Big)\Big(1 - \frac{Q_{j-1}}{Q_n}\Big),\end{aligned}$$

and

$$Z_{mn} - Z_{m,n-1} = \frac{q_n}{Q_{n-1}Q_n} \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\Big(1 - \frac{P_{i-1}}{P_m}\Big)Q_{j-1}.$$

It then follows that

$$\Delta_{11}Z_{m-1,n-1} = \frac{p_m q_n}{P_{m-1}P_m Q_{n-1}Q_n} \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}P_{i-1}Q_{j-1}, \tag{2.1}$$

since $P_{-1} = Q_{-1} = 0$.

Solving (2.1) for a_{mn} yields

$$\begin{aligned} a_{mn} =& \frac{P_m Q_n}{p_m q_n} \Delta_{11} Z_{m-1,n-1} - \frac{P_{m-2} Q_n}{p_{m-1} q_n} \Delta_{11} Z_{m-2,n-1} \\ & - \frac{P_m Q_{n-2}}{p_m q_{n-1}} \Delta_{11} Z_{m-1,n-2} + \frac{P_{m-2} Q_{n-2}}{p_{m-1} q_{n-1}} \Delta_{11} Z_{m-2,n-2}. \end{aligned} \tag{2.2}$$

From (1.1),

$$\begin{aligned} T_{mn} &= \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} t_{mn\mu\nu} s_{\mu\nu} = \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} t_{mn\mu\nu} \sum_{i=0}^{\mu} \sum_{j=0}^{\nu} a_{ij} \\ &= \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} \sum_{\mu=i}^{m} \sum_{\nu=j}^{n} t_{mn\mu\nu} = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} \bar{t}_{mnij}. \end{aligned}$$

$$T_{m,n-1} = \sum_{i=0}^{m} \sum_{j=0}^{n-1} a_{ij} \bar{t}_{m,n-1,i,j},$$

and

$$\begin{aligned} T_{mn} - T_{m,n-1} &= \sum_{i=0}^{m} a_{in} \bar{t}_{mnin} + \sum_{i=0}^{m} \sum_{j=0}^{n-1} a_{ij} (\bar{t}_{mnij} - \bar{t}_{m,n-1,i,j}) \\ &= \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} (\bar{t}_{mnij} - \bar{t}_{m,n-1,i,j}), \end{aligned}$$

since $\bar{t}_{m,n-1,i,n} = 0$.

$$\begin{aligned} T_{mn} - T_{m,n-1} =& \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} (\bar{t}_{mnij} - \bar{t}_{m,n-1,i,j}) + \sum_{i=0}^{m} a_{i0} (\bar{t}_{mni0} - \bar{t}_{m,n-1,i,0}) \\ & + \sum_{j=1}^{n} a_{0j} (\bar{t}_{mn0j} - \bar{t}_{m-1,n,0,j}). \end{aligned}$$

By (iii),

$$\begin{aligned} \bar{t}_{mni0} - \bar{t}_{m,n-1,i,0} &= \sum_{r=i}^{m} \sum_{s=0}^{n} t_{mnrs} - \sum_{r=i}^{m} \sum_{s=0}^{n-1} t_{m,n-1,r,s} \\ &= \sum_{r=i}^{m} \left(\sum_{s=0}^{n} t_{mnrs} - \sum_{s=0}^{n-1} t_{m,n-1,r,s} \right) = 0. \end{aligned}$$

Similarly, $\bar{t}_{mn0j} - \bar{t}_{m-1,n,0,j} = 0$.

Thus

$$T_{mn} - T_{m,n-1} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} (\bar{t}_{mnij} - \bar{t}_{m,n-1,i,j})$$

and

$$\Delta_{11} T_{m-1,n-1} = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\Delta_{11}\bar{t}_{m-1,n-1,i,j}. \tag{2.3}$$

Substituting (2.2) into (2.3) yields

$$\begin{aligned}\Delta_{11} T_{m-1,n-1} = \sum_{i=1}^{m}\sum_{j=1}^{n}\hat{t}_{m-1,n-1,i,j}\Big[&\frac{P_iQ_j\Delta_{11}Z_{i-1,j-1}}{p_iq_j} - \frac{P_{i-2}Q_j\Delta_{11}Z_{i-2,j-1}}{p_{i-1}q_j}\\ &- \frac{P_iQ_{j-2}\Delta_{11}Z_{i-1,j-2}}{p_iq_{j-1}} + \frac{P_{i-2}Q_{j-2}\Delta_{11}Z_{i-2,j-2}}{p_{i-1}q_{j-1}}\Big]\end{aligned}$$

Using the substitutions $r = i-1$ in the second sum, $s = j-1$ in the third sum, and both substitutions in the fourth sum, we have

$$\begin{aligned}\Delta_{11} T_{m-1,n-1} =& \sum_{i=1}^{m}\sum_{j=1}^{n}\hat{t}_{m-1,n-1,i,j}\frac{P_iQ_j\Delta_{11}Z_{i-1,j-1}}{p_iq_j}\\ &- \sum_{r=1}^{m-1}\sum_{j=1}^{n}\hat{t}_{m-1,n-1,r+1,j}\frac{P_{r-1}Q_j\Delta_{11}Z_{r-1,j-1}}{p_rq_j}\\ &- \sum_{i=1}^{m}\sum_{s=1}^{n-1}\hat{t}_{m-1,n-1,i,s+1}\frac{P_iQ_{s-1}\Delta_{11}Z_{i-1,s-1}}{p_iq_s}\\ &+ \sum_{r=1}^{m-1}\sum_{s=1}^{n-1}\hat{t}_{m-1,n-1,r+1,s+1}\frac{P_{r-1}Q_{s-1}\Delta_{11}Z_{r-1,s-1}}{p_rq_s}\\ =& \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{p_iq_j}\Delta_{11}Z_{i-1,j-1}[P_iQ_j\hat{t}_{m-1,n-1,i,j}\\ &- P_{i-1}Q_j\hat{t}_{m-1,n-1,i+1,j} - P_iQ_{j-1}\hat{t}_{m-1,n-1,i,j+1}\\ &+ P_{i-1}Q_{j-1}\hat{t}_{m-1,n-1,i+1,j+1}],\end{aligned}$$

since

$$\begin{aligned}\hat{t}_{m-1,n-1,m+1,j} = \hat{t}_{m-1,n-1,i,n+1} &= \hat{t}_{m-1,n-1,m+1,j+1}\\ &= \hat{t}_{m-1,n-1,i+1,n+1} = 0,\end{aligned}$$

and $P_{-1} = Q_{-1} = 0$.

The quantity in brackets can be written in the form

$$
\begin{aligned}
&Q_j[p_i\hat{t}_{m-1,n-1,i,j} + P_{i-1}(\hat{t}_{m-1,n-1,i,j} - \hat{t}_{m-1,n-1,i+1,j})]\\
&\qquad - p_iQ_{j-1}\hat{t}_{m-1,n-1,i,j+1}\\
&\qquad - P_{i-1}Q_{j-1}(\hat{t}_{m-1,n-1,i,j+1} - \hat{t}_{m-1,n-1,i+1,j+1})\\
&= p_iq_j\hat{t}_{m-1,n-1,i,j} + p_iQ_{j-1}(\hat{t}_{m-1,n-1,i,j} - \hat{t}_{m-1,n-1,i,j+1})\\
&\qquad + q_jP_{i-1}(\hat{t}_{m-1,n-1,i,j} - \hat{t}_{m-1,n-1,i+1,j})\\
&\qquad + P_{i-1}Q_{j-1}(\hat{t}_{m-1,n-1,i,j} - \hat{t}_{m-1,n-1,i+1,j}\\
&\qquad\qquad - \hat{t}_{m-1,n-1,i,j+1} + \hat{t}_{m-1,n-1,i+1,j+1}).
\end{aligned}
$$

Therefore

$$\Delta_{11}T_{m-1,n-1} = I_1 + I_2 + I_3 + I_4, \quad \text{say.}$$

Substituting into (1.5) and using Hölder's inequality, we have

$$
\begin{aligned}
J_1 &:= \sum_{m=1}^{M}\sum_{n=1}^{N}(mn)^{k-1}|I_1|^k\\
&= \sum_{m=1}^{M}\sum_{n=1}^{N}(mn)^{k-1}\Big|\sum_{i=1}^{m}\sum_{j=1}^{n}\hat{t}_{m-1,n-1,i,j}\Delta_{11}Z_{i-1,j-1}\Big|^k\\
&\le \sum_{m=1}^{M}\sum_{n=1}^{N}(mn)^{k-1}\sum_{i=1}^{m}\sum_{j=1}^{n}t_{ijij}^{1-k}|\hat{t}_{m-1,n-1,i,j}||\Delta_{11}Z_{i-1,j-1}|^k\times\\
&\qquad\times\Big|\sum_{i=1}^{m}\sum_{j=1}^{n}t_{ijij}|\hat{t}_{m-1,n-1,i,j}|\Big|^{k-1}.
\end{aligned}
$$

Using (v), (vi), and (1.5),

$$
\begin{aligned}
J_1 &= O(1)\sum_{m=1}^{M}\sum_{n=1}^{N}(mnt_{mnmn})^{k-1}\sum_{i=1}^{m}\sum_{j=1}^{n}t_{ijij}^{1-k}|\hat{t}_{m-1,n-1,i,j}||\Delta_{11}Z_{i-1,j-1}|^k\\
&= O(1)\sum_{i=1}^{M}\sum_{j=1}^{N}t_{ijij}^{1-k}|\Delta_{11}Z_{i-1,j-1}|^k\sum_{m=i}^{M}\sum_{n=j}^{N}(mnt_{mnmn})^{k-1}|\hat{t}_{m-1,n-1,i,j}|\\
&= O(1)\sum_{i=1}^{M}\sum_{j=1}^{N}t_{ijij}^{1-k}|\Delta_{11}Z_{i-1,j-1}|^k(ijt_{ijij})^{k-1}\\
&= O(1)\sum_{i=1}^{M}\sum_{j=1}^{N}(ij)^{k-1}|\Delta_{11}Z_{i-1,j-1}|^k = O(1).
\end{aligned}
$$

Using (vii), (ix) and (ii),

$$
\begin{aligned}
J_2 &:= \sum_{m=1}^{M}\sum_{n=1}^{N}(mn)^{k-1}|I_2|^k \\
&= \sum_{m=1}^{M}\sum_{n=1}^{N}(mn)^{k-1}\Big|\sum_{i=1}^{m}\sum_{j=1}^{n}\Big(\frac{Q_{j-1}}{q_j}\Big)\Delta_{11}Z_{i-1,j-1}(\Delta_{0j}\hat{t}_{m-1,n-1,i,j})\Big|^k \\
&\leq \sum_{m=1}^{M}\sum_{n=1}^{N}(mn)^{k-1}\sum_{i=1}^{m}\sum_{j=1}^{n}\Big(\frac{Q_j}{q_j}\Big)^k\Big(\frac{P_i}{p_i}\Big)^{k-1}|\Delta_{oj}\hat{t}_{m-1,n-1,i,j}|\times \\
&\qquad\quad \times|\Delta_{11}Z_{i-1,j-1}|^k\times\Big|\sum_{i=1}^{m}\sum_{j=1}^{n}\Big(\frac{p_i}{P_i}\Big)||\Delta_{oj}\hat{t}_{m-1,n-1,i,j}|\Big|^{k-1} \\
&= O(1)\sum_{m=1}^{M}\sum_{n=1}^{N}(mnt_{mnmn})^{k-1}\sum_{i=1}^{m}\sum_{j=1}^{n}\Big(\frac{Q_j}{q_j}\Big)^k\Big(\frac{P_i}{p_i}\Big)^{k-1}\times \\
&\qquad\quad \times|\Delta_{oj}\hat{t}_{m-1,n-1,i,j}|\Delta_{11}Z_{i-1,j-1}|^k \\
&= O(1)\sum_{i=1}^{M}\sum_{j=1}^{N}\Big(\frac{Q_j}{q_j}\Big)^k\Big(\frac{P_i}{p_i}\Big)^{k-1}|\Delta_{11}Z_{i-1,j-1}|^k\sum_{m=i}^{M}\sum_{n=j}^{N}(mnt_{mnmn})^{k-1}\times \\
&\qquad\quad \times|\Delta_{0j}\hat{t}_{m-1,n-1,i,j}| \\
&= O(1)\sum_{i=1}^{M}\sum_{j=1}^{N}\Big(\frac{Q_j}{q_j}\Big)^k\Big(\frac{P_i}{p_i}\Big)^{k-1}(ijt_{ijij})^{k-1}\frac{q_j}{Q_j}|\Delta_{11}Z_{i-1,j-1}|^k \\
&= O(1)\sum_{i=1}^{M}\sum_{j=1}^{N}(ij)^{k-1}|\Delta_{11}Z_{i-1,j-1}|^k = O(1).
\end{aligned}
$$

Using the same argument for J_2, and using the estimates (viii), (x), and (ii),

$$
\begin{aligned}
J_3 &:= \sum_{m=1}^{M}\sum_{n=1}^{N}(mn)^{k-1}|I_3|^k \\
&= \sum_{m=1}^{M}\sum_{n=1}^{N}(mn)^{k-1}\Big|\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{P_{i-1}}{p_i}(\Delta_{i0}\hat{t}_{m-1,n-1,i,j})\Delta_{11}Z_{i-1,j-1}\Big|^k \\
&= O(1).
\end{aligned}
$$

$$\begin{aligned}\hat{t}_{m-1,n-1,i,j} =&\bar{t}_{m-1,n-1,i,j} - \bar{t}_{m,n-1,i,j} - \bar{t}_{m-1,n,i,j} + \bar{t}_{mnij} \\ =&\sum_{r=i}^{m-1}\sum_{s=j}^{n-1} t_{m-1,n-1,r,s} - \sum_{r=i}^{m}\sum_{s=j}^{n-1} t_{m-1,n-1,r,s} - \sum_{r=i}^{m-1}\sum_{s=j}^{n} t_{m-1,n,r,s} \\ &+ \sum_{r=i}^{m}\sum_{s=j}^{n} t_{mnrs} \\ =&\sum_{r=i}^{m}\sum_{s=j}^{n} \Delta_{11} t_{m-1,n-1,r,s}.\end{aligned}$$

Therefore

$$\hat{t}_{m-1,n-1,i,j} - \hat{t}_{m-1,n-1,i,j+1} = \sum_{r=i}^{m} \Delta_{11} t_{m-1,n-1,r,s},$$

and hence

$$\begin{aligned}\Delta_{ij}\hat{t}_{m-1,n-1,i,j} :=&\hat{t}_{m-1,n-1,i,j} - \hat{t}_{m-1,n-1,i+1,j} \\ &- \hat{t}_{m-1,n-1,i,j+1} + \hat{t}_{m-1,n-1,i+1,j+1} \\ =&\Delta_{11} t_{m-1,n-1,i,j}.\end{aligned} \tag{2.4}$$

Using (2.4) and Hölder's inequality,

$$\begin{aligned}J_4 :=& \sum_{m=1}^{M}\sum_{n=1}^{N}(mn)^{k-1}|I_4|^k \\ =& \sum_{m=1}^{M}\sum_{n=1}^{N}(mn)^{k-1}\Big|\sum_{i=1}^{m}\sum_{j=1}^{n}\big(\frac{P_{i-1}Q_{j-1}}{p_iq_j}\big)\Delta_{ij}\hat{t}_{m-1,n-1,i,j}\Delta_{11}Z_{i-1,j-1}\Big|^k \\ \le& \sum_{m=1}^{M}\sum_{n=1}^{N}(mn)^{k-1}\Big[\sum_{i=1}^{m}\sum_{j=1}^{n}\Big(\frac{P_iQ_j}{p_iq_j}\Big)^k|\Delta_{ij}t_{m-1,n-1,i,j}||\Delta_{11}Z_{i-1,j-1}|^k\Big]\times \\ &\times\Big[\sum_{i=1}^{m}\sum_{j=1}^{n}|\Delta_{11}t_{m-1,n-1,i,j}|\Big]^{k-1}.\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j=1}^{m}|\Delta_{11}t_{m-1,n-1,i,j}| =& |\Delta_{11}t_{m-1,n-1,m,n}| + \sum_{i=1}^{m-1}|\Delta_{11}t_{m-1,n-1,i,n}| \\ &+ \sum_{j=1}^{n-1}|\Delta_{11}t_{m-1,n-1,m,j}| + \sum_{i=1}^{m}\sum_{j=1}^{n}|\Delta_{11}t_{m-1,n-1,i,j}| \\ =& M_1 + M_2 + M_3 + M_4, \quad \text{say.}\end{aligned}$$

$$M_1 = t_{mnmn}.$$

Using (i) and (iii),

$$\begin{aligned}M_2 &= \sum_{i=1}^{m-1} |-t_{m-1,n,i,n} + t_{mnin}| = \sum_{i=1}^{m-1}(t_{m-1,n,i,n} - t_{mnin})\\ &= a(n,n) - t_{m-1,n,0,n} - a(n,n) + t_{mn0n} + t_{mnmn}\\ &\le t_{mnmn},\end{aligned}$$

and

$$\begin{aligned}M_3 &= \sum_{j=1}^{n-1} |-t_{m,n-1,m,j} + t_{mnmj}| = \sum_{j=1}^{n-1}(t_{m,n-1,m,j} - t_{mnmj})\\ &= b(m,m) - t_{m,n-1,m,0} - b(m,m) + t_{mnmn} + t_{mnm0}\\ &\le t_{mnmn},\end{aligned}$$

Using (iv), (iii), and (1),

$$\begin{aligned}M_4 &= \sum_{i=1}^{m-1}\sum_{j=1}^{n-1} |\Delta_{11} t_{m-1,n-1,i,j}| \le \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |\Delta_{11} t_{m-1,n-1,i,j}|\\ &= -\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}(t_{m-1,n-1,i,j} - t_{m,n-1,i,j} - t_{m-1,n,i,j} + t_{mnij})\\ &= -\sum_{i=0}^{m-1}[b(m-1,i) - b(m,i) - b(m-1,i) + t_{m-1,n,i,n} + b(m,i) - t_{mnin}]\\ &= -a(n,n) + a(n,n) - t_{mnmn}.\end{aligned}$$

Therefore, using (xi) and (ii),

$$\begin{aligned}J_4 &= O(1)\sum_{m=1}^{M}\sum_{n=1}^{N}(mnt_{mnmn})^{k-1}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(\frac{P_iQ_j}{p_iq_j}\right)^k \times\\ &\qquad \times |\Delta_{ij}\hat{t}_{m-1,n-1,i,j}||\Delta_{11} Z_{i-1,j-1}|^k\\ &= O(1)\sum_{i=1}^{M}\sum_{j=1}^{N}\left(\frac{P_iQ_j}{p_iq_j}\right)^k |\Delta_{11} Z_{i-1,j-1}|^k \times\\ &\qquad \times \sum_{m=i}^{M}\sum_{n=j}^{N}(mnt_{mnmn})^{k-1}|\Delta_{ij}\hat{t}_{m-1,n-1,i,j}|\\ &= O(1)\sum_{i=1}^{M}\sum_{j=1}^{N}\left(\frac{P_iQ_j}{p_iq_j}\right)^k |\Delta_{11} Z_{i-1,j-1}|^k (ij)^{k-1}(t_{ijij})^k\\ &= O(1)\sum_{i=1}^{M}\sum_{j=1}^{N}(ij)^{k-1}|\Delta_{11} Z_{i-1,j-1}|^k = O(1).\end{aligned}$$

□

3. Applications

We shall now use the Theorem to obtain some inclusion relations between weighted mean methods and double Cesáro matrices.

Lemma 3.1. *If T is a positive factorable weighted mean matrix, then conditions* (i) *and* (iii) *of Theorem 2.1 are automatically satisfied.*
Also $\Delta_{11} t_{m-1,n-1,i,j} \geq 0$ *for* $0 \leq i < m, 0 \leq j < n, m, n = 1, 2, \ldots$.

Proof. If T is a weighted mean matrix, then

$$t_{mnij} = \frac{p_{ij}}{P_{mn}} \geq \frac{p_{ij}}{P_{m+1,n}} = t_{m+1,n,i,j}$$

and

$$t_{mnij} \geq \frac{p_{ij}}{P_{m,n+1}} = t_{m,n+1,i,j}.$$

Since

$$P_{mn} = \sum_{i=0}^{m} \sum_{j=0}^{n} p_{ij},$$

condition (iii) is clearly satisfied, since T is factorable.
For $0 \leq i < m, 0 \leq j < n$,

$$\begin{aligned}
\Delta_{11} t_{m-1,n-1,i,j} &= t_{m-1,n-1,i,j} - t_{m,n-1,i,j} - t_{m-1,n,i,j} + t_{mnij} \\
&= \frac{p_i q_j}{P_{m-1} Q_{n-1}} - \frac{p_i q_j}{P_m Q_{n-1}} - \frac{p_i q_j}{P_{m-1} Q_m} + \frac{p_i q_j}{P_m Q_n} \\
&= p_i q_j \Big(\frac{1}{Q_{n-1}} \Big(\frac{1}{P_{m-1}} - \frac{1}{P_m} \Big) - \frac{1}{Q_n} \Big(\frac{1}{P_{m-1}} - \frac{1}{P_m} \Big) \Big) \\
&= \frac{p_i q_j p_m q_n}{P_{m-1} P_m Q_{n-1} Q_n} \geq 0. \qquad (3.1)
\end{aligned}$$

□

Note that (3.1) is the opposite of condition (iv) of Theorem 2.1.

Corollary 3.2. *Suppose that $\{p_m\}, \{p\prime_m\}, \{q_m\}, \{q\prime_m\}$ are positive sequences such that*

(i) $q_n/Q_n = O(q'_n/Q'_n), q'_n/Q'_n = O(q_n/Q_n)$,
(ii) $p_m/P_m = O(p'_m/P'_m), p'_m/P'_m = O(p_m/P_m)$,
(iii) $\displaystyle\sum_{m=i}^{\infty} \sum_{n=j}^{\infty} \frac{(mn)^{k-1}}{P_{m-1} Q_{n-1}} \Big(\frac{p_m q_n}{P_m Q_n} \Big)^k = O\Big(\frac{(ijp_i q_j)^{k-1}}{(P_i Q_j)^k} \Big)$.

Then, if $\sum \sum a_{mn}$ is summable $|\bar{N}, p'_m, q'_n|_k$, it is summable $|\bar{N}, p_m, q_n|_k$.

PROOF. Suppose that $\sum\sum a_{mn}$ is summable $|\bar{N}, p'_m q'_n|_k$. Then, if we set $T = (\bar{N}, p_m q_n)$ in the Theorem, from Lemma 3.1, conditions (i) and (iii) are automatically satisfied, and condition (ii) of the Theorem is implied by (i) and (ii) above.

Condition (iv) of the Theorem is used to prove that

$$\hat{t}_{m-1,n-1,i,j} \geq 0. \tag{3.2}$$

But, from (2.3) and (3.4) below, (3.2) is automatically satisfied.

It remains to show that the remaining conditions of the Theorem are implied by the conditions of Corollary 3.2.

We may write condition (v) of Theorem 2.1 as

$$\begin{aligned} J_5 &:= t_{mnmn}|\hat{t}_{m-1,n-1,m,n}| + \sum_{i=1}^{m-1} t_{inin}|\hat{t}_{m-1,n-1,i,n}| \\ &\quad + \sum_{j=1}^{n-1} t_{mjmj}|\hat{t}_{m-1,n-1,m,j}| + \sum_{i=1}^{m-1}\sum_{j=1}^{m-1} t_{ijij}|\hat{t}_{m-1,n-1,i,j}| \\ &= J_{51} + J_{52} + J_{53} + J_{54}, \quad \text{say.} \end{aligned}$$

By property (iii)

$$J_{51} = t_{mnmn}^2 \leq t_{mnmn}.$$

$$\begin{aligned} \hat{t}_{m-1,n-1,i,n} &= -\bar{t}_{m-1,n-1,i,n} + \bar{t}_{mnin} \\ &= -\sum_{r=i}^{m-1} t_{m-1,n,r,n} + \sum_{r=i}^{m} t_{mnin} \\ &= -a(n,n) + \sum_{r=0}^{i-1} t_{m-1,n-1,r,n} + a(n,n) - \sum_{r=0}^{i-1} t_{mnin} \\ &= \sum_{r=0}^{i-1}\Big(\frac{p_r q_n}{P_{m-1}Q_n} - \frac{p_r q_n}{P_m Q_n}\Big) = \frac{p_m P_{i-1} q_n}{P_m P_{m-1} Q_n}. \end{aligned} \tag{3.3}$$

Therefore

$$J_{52} = \sum_{i=1}^{m-1} \frac{p_i q_n}{P_i Q_n}\frac{p_m P_{i-1} q_n}{P_m P_{m-1} Q_n} \leq \frac{p_m q_n}{PmQ_n}\frac{1}{P_{m-1}}\sum_{i=1}^{m-1} p_i \leq \frac{p_m q_n}{P_m Q_n} = t_{mnmn}.$$

Similarly, $J_{53} = O(t_{mnmn})$.

$$\begin{aligned} \bar{t}_{m-1,n-1,i,j} &= \sum_{r=i}^{m-1}\sum_{s=j}^{n-1} t_{m-1,n-1,i,j} = \sum_{r=i}^{m-1}\sum_{s=j}^{n-1} \frac{p_i q_j}{P_{m-1}Q_{n-1}} \\ &= \Big(\frac{P_{m-1} - P_{i-1}}{P_{m-1}}\Big)\Big(\frac{Q_{n-1} - Q_{j-1}}{Q_{n-1}}\Big). \end{aligned}$$

$$
\begin{aligned}
\hat{t}_{m-1,n-1,i,j} &= \bar{t}_{m-1,n-1,i,j} - \bar{t}_{m,n-1,i,j} - \bar{t}_{m-1,n,i,j} + \bar{t}_{mnij} \\
&= \Big(1 - \frac{P_{i-1}}{P_{m-1}}\Big)\Big(1 - \frac{Q_{i-1}}{Q_{n-1}}\Big) - \Big(1 - \frac{P_{i-1}}{P_m}\Big)\Big(1 - \frac{Q_{j-1}}{Q_{n-1}}\Big) \\
&\quad - \Big(1 - \frac{P_{i-1}}{P_{m-1}}\Big)\Big(1 - \frac{Q_{i-1}}{Q_n}\Big) + \Big(1 - \frac{P_{i-1}}{P_m}\Big)\Big(1 - \frac{Q_{i-1}}{Q_n}\Big) \\
&= \Big(1 - \frac{Q_{j-1}}{Q_{n-1}}\Big)\Big(1 - \frac{P_{i-1}}{P_{m-1}} - 1 + \frac{P_{i-1}}{P_m}\Big) \\
&\quad - \Big(1 - \frac{Q_{j-1}}{Q_n}\Big)\Big(1 - \frac{P_{i-1}}{P_{m-1}} - 1 + \frac{P_{i-1}}{P_m}\Big) \\
&= \Big(1 - \frac{Q_{j-1}}{Q_{n-1}}\Big)\Big(-\frac{P_{i-1}p_m}{P_m P_{m-1}}\Big) - \Big(1 - \frac{Q_{j-1}}{Q_{n-1}}\Big)\Big(-\frac{P_{i-1}p_m}{P_m P_{m-1}}\Big) \\
&= -\frac{p_m P_{i-1}}{P_m P_{m-1}}\Big(1 - \frac{Q_{j-1}}{Q_{n-1}} - 1 + \frac{Q_{j-1}}{Q_n}\Big) \\
&= \frac{p_m q_n P_{i-1} Q_{j-1}}{P_m P_{m-1} Q_n Q_{n-1}}. \qquad (3.4)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
J_{54} &= \sum_{i=1}^{m-1}\sum_{j=1}^{n-1} \frac{p_i q_j}{P_i Q_j} \frac{p_m q_n P_{i-1} Q_{j-1}}{P_m P_{m-1} Q_n Q_{n-1}} \\
&\le \frac{p_m q_n}{P_m P_{m-1} Q_m Q_{m-1}} P_{m-1} Q_{n-1} = \frac{p_m q_n}{P_m Q_n} = t_{mnmn},
\end{aligned}
$$

and condition (v) is satisfied.

Using (3.4) and condition (ii) of Corollary 3.2,

$$
\begin{aligned}
&\sum_{m=i}^{\infty}\sum_{n=j}^{\infty} \Big(\frac{mnp_m q_n}{P_m Q_n}\Big)^{k-1} \frac{p_m q_n P_{i-1} Q_{j-1}}{P_m P_{m-1} Q_n Q_{n-1}} \\
&\qquad = P_{i-1} Q_{j-1} \sum_{m=i}^{\infty}\sum_{n=j}^{\infty} \frac{(mn)^{k-1}}{P_{m-1} Q_{n-1}} \Big(\frac{p_m q_n}{P_m Q_n}\Big)^{k} \\
&\qquad = P_{i-1} Q_{j-1} O\Big(\frac{(ijp_i q_j)^{k-1}}{(P_i Q_j)^k}\Big) = O((ijt_{ijij})^{k-1}),
\end{aligned}
$$

and condition (vi) is satisfied.

Using condition (ii) of Corollary 3.2 we may write condition (vii) of the Theorem as

$$\begin{aligned}
&O(1)\Big[\frac{p_m}{P_m}|\Delta_{0n}\hat{t}_{m-1,n-1,m,n}| + \sum_{i=1}^{m-1}\Big(\frac{p_i}{P_i}\Big)|\Delta_{0n}\hat{t}_{m-1,n-1,i,n}| \\
&\qquad + \sum_{j=1}^{n-1}\Big(\frac{p_m}{P_m}\Big)|\Delta_{0j}\hat{t}_{m-1,n-1,m,j}| + \sum_{i=1}^{m-1}\sum_{j=1}^{n-1}\Big(\frac{p_i}{P_i}\Big)|\Delta_{0j}\hat{t}_{m-1,n-1,i,j}|\Big] \\
&\quad = O(1)[J_{71} + J_{72} + J_{73} + J_{74}], \quad \text{say.}
\end{aligned}$$

$$\begin{aligned}
\Delta_{0n}\hat{t}_{m-1,n-1,i,n} &= \hat{t}_{m-1,n-1,i,n} - \hat{t}_{m-1,n-1,i,n+1} \\
&= \hat{t}_{m-1,n-1,i,n}.
\end{aligned}$$

Using (3.3),

$$J_{71} = \frac{p_m}{P_m}t_{mnmn} \le t_{mnmn},$$

and

$$J_{72} = \sum_{i=1}^{m-1}\Big(\frac{p_i}{P_i}\Big)\frac{p_m P_{i-1} q_n}{P_m P_{m-1} Q_n} \le \frac{p_m q_n}{P_m Q_n} = t_{mnmn}.$$

Similarly, $J_{73} \le t_{mnmn}$. Using (3.4),

$$\begin{aligned}
\Delta_{0j}\hat{t}_{m-1,n-1,i,j} &= \hat{t}_{m-1,n-1,i,j} - \hat{t}_{m-1,n-1,i,j+1} \\
&= \frac{p_m q_n P_{i-1} Q_{j-1}}{P_m P_{m-1} Q_n Q_{n-1}} - \frac{p_m q_n P_{i-1} Q_j}{P_m P_{m-1} Q_n Q_{n-1}} \\
&= -\frac{p_m q_n P_{i-1} q_j}{P_m P_{m-1} Q_n Q_{n-1}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
J_{74} &= \sum_{i=1}^{m-1}\sum_{j=1}^{n-1}\Big(\frac{p_i}{P_i}\Big)\frac{p_m q_n}{P_m P_{m-1} Q_n Q_{n-1}} \\
&\le \frac{p_m q_n}{P_m P_{m-1} Q_n Q_{n-1}} P_{i-1} Q_{n-1} \\
&= \frac{p_m q_n}{P_m Q_n} = t_{mnmn},
\end{aligned}$$

and condition (vii) is satisfied.

The validity of condition (viii) of Theorem 2.1 is proved similarly.

Using (iii), (i), and (ii) of Corollary 3.2, condition (ix) becomes

$$\begin{aligned}
&\sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\Big(\frac{mnp_mq_n}{P_mQ_n}\Big)^{k-1}\frac{p_mq_nP_{i-1}q_j}{P_mP_{m-1}Q_nQ_{n-1}}\\
&\qquad = P_{i-1}q_j\sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\frac{(mn)^{k-1}}{P_{m-1}Q_{n-1}}\Big(\frac{p_mq_n}{P_mQ_n}\Big)^k\\
&\qquad = P_{i-1}q_jO\Big(\frac{(ijp_iq_j)^{k-1}}{(P_iQ_j)^k}\Big) = O\Big(\Big(\frac{ijp_i}{P_i}\Big)^{k-1}\Big(\frac{q_j}{Q_j}\Big)^k\Big)\\
&\qquad = O\Big((ijt_{ijij})^{k-1}\frac{q_j}{Q_j}\Big) = O\Big((ijt_{ijij})^{k-1}\frac{q'_j}{Q'_j}\Big).
\end{aligned}$$

Condition (x) is proved in a similar manner.

Using (3.4),

$$\begin{aligned}
\Delta_{ij}\hat{t}_{m-1,n-1,i,j} &= \Delta_{11}t_{m-1,n-1,i,j}\\
&= t_{m-1,n-1,i,j} - t_{m,n-1,i,j} - t_{m-1,n,i,j} + t_{mnij}\\
&= \frac{p_iq_j}{P_{m-1}Q_{n-1}} - \frac{p_iq_j}{P_mQ_{n-1}} - \frac{p_iq_j}{P_{m-1}Q_n} + \frac{p_iq_j}{P_mQ_n}\\
&= p_iq_j\Big[\frac{1}{Q_{n-1}}\Big(\frac{1}{P_{m-1}} - \frac{1}{P_m}\Big) - \frac{1}{Q_n}\Big(\frac{1}{P_{m-1}} - \frac{1}{P_m}\Big)\Big]\\
&= \frac{p_iq_jp_mq_n}{P_mP_{m-1}Q_nQ_{n-1}}.
\end{aligned}$$

Therefore, using (iii) of Corollary 3.2,

$$\begin{aligned}
&\sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\Big(\frac{mnp_mq_n}{P_mQ_n}\Big)^{k-1}\frac{p_iq_jp_mq_n}{P_mP_{m-1}Q_nQ_{n-1}}\\
&\qquad = p_iq_j\sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\frac{(mn)^{k-1}}{P_{m-1}Q_{n-1}}\Big(\frac{p_mq_n}{P_mQ_n}\Big)^k\\
&\qquad = p_iq_jO\Big(\frac{(ijp_iq_j)^{k-1}}{(P_iQ_j)^k}\Big) = O\Big((ij)^{k-1}\Big(\frac{p_iq_j}{P_iQ_j}\Big)^k\Big)\\
&\qquad = O((ij)^{k-1}t_{ijij}^k),
\end{aligned}$$

and condition (xi) is satisfied. ∎

Corollary 3.3. *Let $\{p_m\},\{q_n\}$ be positive sequences satisfying*

(i) $mp_m = O(P_m), P_m = O(mp_m)$,

(ii) $nq_n = O(Q_n), Q_n = O(nq_n)$.

Then $\sum\sum a_{mn}$ is summable $|C,1,1|_k$ if and only if it is also summable $|\bar{N},p_mq_n|_k, k\geq 1$.

PROOF. First apply Corollary 3.2 with $p'_m = q'_n = 1$ for all n.

Conditions (i) and (ii) of Corollary 3.3 imply conditions (i) and (ii) of Corollary 3.2.

Using conditions (i) and (ii) of Corollary 3.3,

$$\sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\frac{(mn)^{k-1}}{P_{m-1}Q_{n-1}}\Big(\frac{p_mq_n}{P_mQ_n}\Big)^k$$
$$= O(1)\sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\frac{p_mq_n}{P_mP_{m-1}Q_nQ_{n-1}} = \frac{O(1)}{P_iQ_j} = O\left(\frac{(ijp_iq_j)^{k-1}}{(P_iQ_j)^k}\right),$$

and condition (iii) of Corollary 3.2 is satisfied.

With $T = (C, 1, 1)$, i.e., $p_m = q_n = 1$ for each m and n, condition (iii) of Corollary 3.2 becomes

$$\sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\frac{1}{m(m+1)n(n+1)} = \frac{1}{ij} = O\left(\frac{(ijp_iq_j)^{k-1}}{(P_iQ_j)^k}\right).$$

□

References

[1] H. Bor, *On two summability methods*, Math. Proc. Cambridge Phil. Soc. **97** (1985), 147–149.

[2] H. Bor, *A note on two summability methods*, Proc. Amer. Math. Soc. **98** (1986), 81–84.

[3] H. Bor, *On the relative strength of two absolute summability methods*, Proc. Amer. Math. Soc. **113** (1991), 1009–1014.

[4] H. Bor and B. Thorpe, *A note on two absolute summability methods*, Analysis **12** (1992), 1–3.

[5] T.M. Flett, *On an extension of absolute summability and theorems of Littlewood and Paley*, Proc. London Math. Soc. **7** (1957), 113–141.

[6] B.E. Rhoades, *Absolute comparison theorems for double weighted mean and double Cesáro means*, Math. Slovaca **48** (1998), 285–291.

[7] B.E. Rhoades, *Inclusion theorems for absolute matrix summability methods*, J. Math. Anal. Appl. **238** (1999), 82–90.

[8] M. Sarigöl, *On absolute normal matrix summability methods*, Glasnik Mat. **28** (1993), 53–60.

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