APPROXIMATION IN SMIRNOV-ORLICZ CLASSES

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Abstract. We use the approximation properties of the Faber polynomials to obtain some direct theorems of the polynomial approximation in Smirnov-Orlicz classes.

1. Introduction and Main Results

Let $\Gamma$ be a rectifiable Jordan curve in the complex plane $\mathbb{C}$ and $G := \text{Int} \Gamma$, $G^- := \text{Ext} \Gamma$. Without loss of generality we may assume $0 \in G$. Let $T := \{w \in \mathbb{C} : |w| = 1\}$, $D := \text{Int} T$ and $D^- := \text{Ext} T$. Let also $w = \varphi(z)$ be the conformal mapping of $G^-$ onto $D^-$, normalized by

$$\varphi(\infty) = \infty, \quad \lim_{z \to \infty} \frac{\varphi(z)}{z} > 0,$$

and let $\psi$ be its inverse.

If $1 \leq p < \infty$, we denote by $L_p(\Gamma)$ and $E_p(G)$ the set of all measurable complex valued functions $f$ on $\Gamma$ such that $|f|^p$ is Lebesgue integrable with respect to arclength, and the Smirnov class of analytic functions in $G$, respectively. Since $\Gamma$ is rectifiable, we have that $\varphi' \in E_1(G^-)$ and $\psi' \in E_1(D^-)$, and hence the functions $\varphi'$ and $\psi'$ admit non-tangential limits almost everywhere (a.e.) on $\Gamma$ and on $T$, and these functions belong to $L_1(\Gamma)$ and $L_1(T)$, respectively (see, for example [10, p. 419]).

Let $h$ be a continuous function on $[0,2\pi]$. Its modulus of continuity is defined by

$$\omega(t,h) := \sup \{|h(t_1) - h(t_2)| : t_1, t_2 \in [0,2\pi], \ |t_1 - t_2| \leq t\}, \quad t \geq 0.$$

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The function $h$ is called Dini-continuous if
\[ \int_0^\pi t^{-1} \omega(t, h) \, dt < \infty. \]

**Definition 1.1** ([17, p. 48]). The curve $\Gamma$ is called Dini-smooth if it has a parametrization $\Gamma : \varphi_0(\tau), \quad 0 \leq \tau \leq 2\pi$ such that $\varphi'_0(\tau)$ is Dini-continuous and $\neq 0$.

If $\Gamma$ is Dini-smooth, then [21]
\begin{equation}
0 < c_1 \leq |\varphi'(z)| \leq c_2 < \infty, \quad z \in \Gamma
\end{equation}
for some constants $c_1$ and $c_2$ independent of $z$.

A function $M(u) : \mathbb{R} \to \mathbb{R}^+$, where $\mathbb{R} := (-\infty, \infty)$ and $\mathbb{R}^+ := (0, \infty)$, is called an $N$-function if it admits of the representation
\[ M(u) = \int_0^{|u|} p(t) \, dt, \]
where the function $p(t)$ is right continuous and nondecreasing for $t \geq 0$ and positive for $t > 0$, which satisfies the conditions
\[ p(0) = 0, \quad p(\infty) := \lim_{t \to \infty} p(t) = \infty. \]

The function
\[ N(v) := \int_0^{|v|} q(s) \, ds, \]
where
\[ q(s) := \sup_{p(t) \leq s} t, \quad (s \geq 0) \]
is defined as complementary function of $M(u)$ [16, p. 11].

Let $M$ be an $N$-function and $N$ be its complementary function. By $L_M(\Gamma)$ we denote the linear space of Lebesgue measurable functions $f : \Gamma \to \mathbb{C}$. By $L_N(\Gamma)$, we denote the linear space of Lebesgue measurable functions $g : \Gamma \to \mathbb{C}$ such that
\[ \int_\Gamma M[\alpha |f(z)||dz| < \infty \]
for some $\alpha > 0$.

The space $L_M(\Gamma)$ becomes a Banach space with the norm
\[ \|f\|_{L_M(\Gamma)} := \sup \left\{ \int_\Gamma |f(z)g(z)||dz| : g \in L_N(\Gamma), \quad \rho(g; N) \leq 1 \right\}. \]
where
\[ \rho (g ; N) := \int_{\Gamma} N [ |g(z)| ] |dz| . \]
The norm \( ||\cdot||_{L_M(\Gamma)} \) is called Orlicz norm and the Banach space \( L_M(\Gamma) \) is called Orlicz space. Every function in \( L_M(\Gamma) \) is integrable on \( \Gamma \) [18, p. 50], i.e.
\[ L_M(\Gamma) \subset L_1(\Gamma) . \]
An \( N - function M \) satisfies the \( \Delta_2 - condition \) if
\[ \limsup_{x \to \infty} M(2x) < M(x) < 1. \]
The Orlicz space \( L_M(\Gamma) \) is reflexive if and only if the \( N - function M \) and its complementary function \( N \) both satisfy the \( \Delta_2 - condition \) [18, p. 113].

Let \( \Gamma_r \) be the image of the circle \( \{ w \in \mathbb{C} : |w| = r, 0 < r < 1 \} \) under some conformal mapping of \( \mathbb{D} \) onto \( G \) and let \( M \) be an \( N - function. \)

**Definition 1.2.** If an analytic function \( f \) in \( G \) satisfies
\[ \int_{\Gamma_r} M [ |f(z)| ] |dz| < \infty \]
uniformly in \( r \), it belongs to Smirnov-Orlicz class \( E_M(G) \).

If \( M(x) = M(x, p) := x^p, 1 < p < \infty \), then the Smirnov-Orlicz class \( E_M(G) \) coincides with the usual Smirnov class \( E_p(G) \).

Every function in the class \( E_M(G) \) has [14] the non-tangential boundary values a.e. on \( \Gamma \) and the boundary function belongs to \( L_M(\Gamma) \), and hence for \( f \in E_M(G) \) we can define the \( E_M(G) \) norm as:
\[ \|f\|_{E_M(G)} := \|f\|_{L_M(\Gamma)}. \]

For \( \zeta \in \Gamma \) we define the point \( \varsigma_h \in \Gamma \) by
\[ \varsigma_h := \psi(\varphi(\zeta)e^{ih}), \quad h \in [0, 2\pi], \]
and also the shift \( T_h f \) for \( f \in L_M(\Gamma) \) as:
\[ T_h f(\zeta) := f(\varsigma_h), \quad \zeta \in \Gamma. \]
Using relation (1), it can be easily verified that, if \( \Gamma \) is Dini-smooth, then \( L_M(\Gamma) \) is invariant under the shift \( T_h f \).

We define the modulus of continuity for \( f \in L_M(\Gamma) \) as:
\[ \omega_M(\delta, f) := \sup_{|h| \leq \delta} \|f - T_h f\|_{L_M(\Gamma)}, \quad \delta \geq 0, \]
which satisfies the conditions
\[ \omega_M(0, f) = 0, \quad \omega_M(\delta, f) \geq 0 \quad \text{for} \ \delta > 0, \]
\[ \lim_{\delta \to 0} \omega_M(\delta, f) = 0, \]
\[ \omega_M(\delta, f + g) \leq \omega_M(\delta, f) + \omega_M(\delta, g) \]
for \( f, g \in E_M(G) \).

For \( f \in E_M(G) \) we put
\[ E_M^p(f, G) := \inf \| f - p_n \|_{L_M(\Gamma)} \]
\[ = \inf \left\{ \sup \left\{ \int_{\Gamma} |(f(\zeta) - p_n(\zeta))g(\zeta)|\, |d\zeta| : \rho(\zeta;N) \leq 1 \right\} \right\}, \tag{5} \]
where \( \inf \) is taken over the polynomials \( p_n \) of degree at most \( n \).

In this work, we considered some problems of the polynomial approximation in Smirnov-Orlicz class \( E_M(G) \). Our new results are the following.

**Theorem 1.3.** Let \( G \) be a finite simply connected domain with the Dini-smooth boundary \( \Gamma \), and let \( E_M(G) \) be a reflexive Smirnov-Orlicz space on \( G \). Then for every \( f \in E_M(G) \) and any natural number \( n \) there exists an algebraic polynomial \( p_n(\cdot, f) \) of degree at most \( n \) such that
\[ \| f - p_n(\cdot, f) \|_{L_M(\Gamma)} \leq c \omega_M(\frac{1}{n}, f) \]
with some constant \( c \) independent of \( n \).

In the more general case, namely when \( \Gamma \) is a Carleson curve, applying the same method of summation, but using different modulus of continuity some direct theorem of approximation theory by polynomials in Smirnov-Orlicz class \( E_M(G) \) is given in [8]. The modulus of continuity \( \omega_M \), used in this work, is simpler than the modulus of continuity considered in [8].

Similar problems for the spaces \( L_p(\Gamma) \) and \( E_p(G), 1 \leq p < \infty \), have been studied in [1, 2, 4, 5, 10, 11, 12, 15]. All these results were proved under different restrictive conditions on \( \Gamma = \partial G \).

Some inverse problems of approximation theory in Smirnov-Orlicz classes have been investigated by Kokilashvili [14] in the case that \( \Gamma \) is Dini-smooth.

Now let \( K \) be a bounded continuum with the connected complement \( D := \overline{\mathbb{C}} \setminus K \) and let \( f(z) \) be an analytic function on \( K \). It is well known that [20, p. 199] the expansion
\[ f(z) = \sum_{k=0}^{\infty} a_k \Phi_k(z), \quad z \in K, \tag{6} \]
with the Faber coefficients
\[ a_k := \frac{1}{2\pi i} \int_{\partial K} \frac{f(\psi(t))}{t^{k+1}} \, dt, \quad k = 0, 1, 2, \ldots, \]
converges absolutely and uniformly on $K$, where $\Phi_k(z)$, $k = 0, 1, 2, \ldots$, are the Faber polynomials for $K$ that satisfy the relation

$$
\frac{\psi'(w)}{\psi(w)} - z = \sum_{k=0}^{\infty} \frac{\Phi_k(z)}{w^{k+1}}, \quad z \in K, \quad |w| > 1.
$$

The detailed information about the Faber polynomials and their approximation properties can be found in the monographs [8, 19, 20].

Let us introduce the value

$$
R_n(z, f) := f(z) - \sum_{k=0}^{n} a_k\Phi_k(z) = \sum_{k=n+1}^{\infty} a_k\Phi_k(z), \quad z \in K,
$$

and put

$$
\Gamma_R := \{ z \in D : |\varphi(z)| = R \} \quad \text{and} \quad G_R := \text{Int}\Gamma_R, \quad R > 1.
$$

The following theorem characterizes the maximal convergence property of the Faber series (6) in the Smirnov-Orlicz space $E_M(G_R)$.

**Theorem 1.4.** If $f \in E_M(G_R)$, $R > 1$, then

$$
|R_n(z, f)| \leq \frac{c}{R^{n+1}(R-1)} E_M^n(f, G_R) \sqrt{n \ln n}, \quad z \in K
$$

with a constant $c > 0$ independent of $n$ and $z$.

From theorem 1.3 and 1.4, we have the following corollary.

**Corollary 1.5.** Let $K$ be a continuum with connected complement and let $E_M(G_R)$ be a reflexive Smirnov-Orlicz class on $G_R$, $R > 1$. If $f \in E_M(G_R)$, then

$$
|R_n(z, f)| \leq \frac{c}{R^{n+1}(R-1)} \omega_M \left( \frac{1}{n}, f \right) \sqrt{n \ln n}, \quad z \in K,
$$

with $c > 0$.

Theorem 1.4 in the Smirnov spaces $E_p(G)$, $p > 1$, was proved in [20, p. 207].

We use $c, c_1, c_2, \ldots$ to denote constants (which may, in general, differ in different relations) depending only on numbers that are not important for the question of interest.

2. **Auxiliary Results**

Let $\Gamma$ be a rectifiable Jordan curve and $f \in L_1(\Gamma)$. The functions $f^+$ and $f^-$ defined by

$$
f^+(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in G,
$$
and
\[ f^-(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in G^-, \]
are analytic in \( G \) and \( G^- \), respectively and \( f^-(\infty) = 0 \).

Let also
\[ S_{\Gamma} f(z_0) := \lim_{\varepsilon \to 0} \int_{\Gamma \cap \{ \zeta : |\zeta - z_0| \geq \varepsilon \}} \frac{f(\zeta)}{\zeta - z_0} d\zeta, \quad z_0 \in \Gamma \]
be the Cauchy singular integral of \( f \in L_1(\Gamma) \).

If one of the functions \( f^+ \) or \( f^- \) has non-tangential limits a.e. on \( \Gamma \), then \( S_{\Gamma} f(z) \) exists a.e. on \( \Gamma \) and also the other one has non-tangential limits a.e. on \( \Gamma \). Conversely, if \( S_{\Gamma} f(z) \) exist a.e. on \( \Gamma \), then both functions \( f^+ \) and \( f^- \) have non-tangential limits a.e. on \( \Gamma \). In both cases, the formulae
\[ f^+(z) = S_{\Gamma} f(z) + \frac{1}{2} f(z) \]
\[ f^-(z) = S_{\Gamma} f(z) - \frac{1}{2} f(z) \]
hold, and hence
\[ f = f^+ - f^- \]
a.e. on \( \Gamma [9, p. 431] \).

The linear operator \( S_{\Gamma} : f \to S_{\Gamma} f \) is called the Cauchy singular operator.

For \( z \in \Gamma \) and \( \epsilon > 0 \), let \( \Gamma (z, \epsilon) \) denote the portion of \( \Gamma \) which is inside the open disk of radius \( \epsilon \) centered at \( z \), i.e. \( \Gamma (z, \epsilon) := \{ t \in \Gamma : |t - z| < \epsilon \} \).

Further, let \( |\Gamma (z, \epsilon)| \) denote the length of \( \Gamma (z, \epsilon) \). A rectifiable Jordan curve \( \Gamma \) is called a Carleson curve if
\[ \sup_{\epsilon > 0} \sup_{z \in \Gamma} \frac{1}{\epsilon} |\Gamma (z, \epsilon)| < \infty. \]

**Theorem 2.1** ([13]). Let \( \Gamma \) be a rectifiable Jordan curve and let \( L_M(\Gamma) \) be a reflexive Orlicz space on \( \Gamma \). Then the singular operator \( S_{\Gamma} \) is bounded on \( L_M(\Gamma) \), i.e.
\[ \| S_{\Gamma} f \|_{L_M(\Gamma)} \leq c_3 \| f \|_{L_M(\Gamma)} \text{ for all } f \in L_M(\Gamma) \]
for some constant \( c_3 > 0 \), if and only if \( \Gamma \) is a Carleson curve.

**Theorem 2.2** ([16, p. 67]). For every pair of real valued functions \( u(z) \in L_M(\Gamma), v(z) \in L_N(\Gamma) \) the inequality
\[ \int_{\Gamma} u(z) v(z) dz \leq \rho(u; M) + \rho(v; N) \]
holds.
Theorem 2.3 ([16, p. 74]). For every pair of real valued functions \( u(z) \in L_M(\Gamma), v(z) \in L_N(\Gamma) \) the inequality
\[
\left| \int_{\Gamma} u(z) v(z) \, dz \right| \leq \|u\|_{L_M(\Gamma)} \|v\|_{L_N(\Gamma)}
\]
holds.

The coefficients of the series in (6) are determined by the formulae
\[
a_k := \frac{1}{2\pi i} \int_{\Gamma} \frac{f(s) \varphi'(s)}{\varphi^{k+1}(s)} \, ds = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\psi(t))}{t^{k+1}} \, dt, \quad k = 0, 1, 2, \ldots,
\]
and hence the relation (8) implies that
\[
R_n(z, f) = \frac{1}{2\pi i} \int_{\Gamma} f(\psi(t)) \left[ \sum_{k=n+1}^{\infty} \frac{\Phi_k(z)}{t^{k+1}} \right] \, dt.
\]
If \( p_n(z) \) is a polynomial of degree at most \( n \), then
\[
R_n(z, f) = \frac{1}{2\pi i} \int_{\Gamma} \{ f(\psi(t)) - p_n(\psi(t)) \} \left[ \sum_{k=n+1}^{\infty} \frac{\Phi_k(z)}{t^{k+1}} \right] \, dt.
\]
Since
\[
\Phi_k(z) = [\varphi(z)]^k + E_k(z), \quad \forall z \in K,
\]
where \( E_k(z) \) is analytic on the whole domain \( D \) and \( E_k(\infty) = 0 \), we have
\[
\sum_{k=n+1}^{\infty} \frac{\Phi_k(z)}{t^{k+1}} = \sum_{k=n+1}^{\infty} \frac{[\varphi(z)]^k}{t^{k+1}} + \sum_{k=n+1}^{\infty} \frac{E_k(z)}{t^{k+1}}.
\]
Hence from (15), taking into account (16), we get
\[
|R_n(z, f)| \leq \frac{1}{2\pi} \int_{|t|=R} |f(\psi(t)) - p_n(\psi(t))| \left| \sum_{k=n+1}^{\infty} \frac{w^k}{t^{k+1}} \right| |dt|
\]
\[
+ \frac{1}{2\pi} \int_{|t|=R} |f(\psi(t)) - p_n(\psi(t))| \left| \sum_{k=n+1}^{\infty} E_k(\psi(w)) \frac{1}{t^{k+1}} \right| |dt|.
\]
We shall also use the relations
\[
E_k(\psi(w)) = \frac{1}{2\pi i} \int_{\Gamma} \tau^k F(\tau, w) \, d\tau, \quad |w| \geq r > 1,
\]
and
\[
\frac{1}{2\pi i} \int_{\Gamma} |F(\tau, w)| |d\tau| \leq \sqrt{\frac{r^2}{r^4 - 1} \ln \frac{r^2}{r^2 - 1}}, \quad r > 1, \quad |w| > r > 1,
\]
given in [20, p. 63-205], where
\[
F(\tau, w) := \frac{\psi'(\tau)}{\psi(\tau) - \psi(w)} - \frac{1}{\tau - w}, \quad |\tau| > 1, \quad |w| > 1.
\]

3. PROOFS OF MAIN RESULTS

PROOF OF THEOREM 1.3. Let \( f \in L_M(\Gamma) \). Then by (2) we have \( f \in L_1(\Gamma) \). Since \( \Gamma \) is Dini-smooth, we have \( f \circ \psi \in L_1(\mathbb{T}) \) and hence we can associate a formal series
\[
\sum_{k=0}^{\infty} a_k w^k + \sum_{k=1}^{\infty} \frac{b_k}{w^k}
\]
with the function \( f \circ \psi \in L_1(\mathbb{T}) \), i.e.
\[
f(\psi(w)) \sim \sum_{k=0}^{\infty} a_k w^k + \sum_{k=1}^{\infty} \frac{b_k}{w^k}.
\]

Let
\[
K_n(\theta) = \sum_{m=-n}^{n} \lambda_m^{(n)} e^{im\theta}
\]
be an even, nonnegative trigonometric polynomial satisfying the conditions
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(\theta) d\theta = 1,
\]
\[
\int_{0}^{\pi} \theta K_n(\theta) d\theta \leq \frac{c_4}{n}
\]
for every natural number \( n \) and with some constant \( c_4 > 0 \) (for example, the Jackson kernel
\[
J_n(\theta) := \frac{3 \left( \sin \frac{n\theta}{2} \right)^4}{n(2n^2 + 1) \left( \sin \frac{\theta}{2} \right)^4}
\]
satisfies the above cited conditions, see [6, p. 203-204]).

Consider the integral
\[
I(\theta, z) := \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\varsigma)}{\varsigma - z} d\varsigma, \quad z \in G.
\]
Using the change of variables $\zeta = \psi (e^{it})$, we obtain

$$I (\theta, z) := \frac{1}{2\pi i} \int_{-\pi}^{\pi} f \left( \psi \left( e^{i(t-\theta)} \right) \right) \frac{\psi' \left( e^{it} \right) e^{it}}{\psi (e^{it}) - z} \, dt,$$

and taking into account the relations (20) and (7), we can write

$$I (\theta, z) \sim \sum_{k=0}^{\infty} a_k \Phi_k (z) e^{-ik\theta}.$$

Since $I (\theta, z) \in L_1 (\mathbb{R})$ and $K_n (\theta)$ is of bounded variation, by the generalized Parseval identity [3, p. 225-228], we obtain

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n (\theta) I (\theta, z) \, d\theta = \sum_{k=0}^{n} \lambda_k^{(n)} a_k \Phi_k (z),$$

which together with (23) implies that

$$\frac{1}{4\pi^2 i} \int_{-\pi}^{\pi} K_n (\theta) \, d\theta \int_{\Gamma} \frac{f (\zeta \theta)}{\zeta - z} \, d\zeta = \sum_{k=0}^{n} \lambda_k^{(n)} a_k \Phi_k (z), \quad z \in G.$$

Hence we see that

$$P_n (z, f) := \frac{1}{4\pi^2 i} \int_{-\pi}^{\pi} K_n (\theta) \, d\theta \int_{\Gamma} \frac{f (\zeta \theta)}{\zeta - z} \, d\zeta, \quad z \in G,$$

is an algebraic polynomial of degree $n$.

Since the kernel $K_n (\theta)$ is an even function, we have

$$P_n (z, f) = \frac{1}{4\pi^2 i} \int_{0}^{\pi} K_n (\theta) \, d\theta \int_{\Gamma} \left[ f (\zeta \theta) + f (\zeta (-\theta)) \right] \frac{d\zeta}{\zeta - z},$$

and by (3) and (9), we conclude that

$$P_n (z, f) = \frac{1}{2\pi} \int_{0}^{\pi} K_n (\theta) \left[ (T_\theta f)^+ (z) + (T_{-\theta} f)^+ (z) \right] \, d\theta, \quad z \in G.$$

Now let $f \in E_M (G)$ and $z' \in G$. Multiplying both side of the equality

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n (\theta) \, d\theta = 1$$
by \( f^+ (z') \) we get
\[
f (z') = f^+ (z') = \frac{1}{2\pi} \int_{-\pi}^{\pi} f^+ (z') K_n (\theta) \, d\theta = \frac{1}{2\pi} \int_{0}^{\pi} 2f^+ (z') K_n (\theta) \, d\theta,
\]
and hence
\[
f (z') = P_n (z', f) = \frac{1}{2\pi} \int_{0}^{\pi} K_n (\theta) \left\{ 2f^+ (z') - \left[ (T_{\theta} f)^+ (z') + (T_{-\theta} f)^+ (z') \right] \right\} d\theta.
\]

Taking the limit \( z' \to z \in \Gamma \) along all nontangential paths inside \( \Gamma \) and using (10), we obtain
\[
f (z) - P_n (z, f) = \frac{1}{2\pi} \int_{0}^{\pi} K_n (\theta) \left[ 2S_{\Gamma} f (z) + f (z) - S_{\Gamma} (T_{\theta} f) (z) - \frac{1}{2} T_{\theta} f (z) \right.
\]
\[\quad \quad - S_{\Gamma} (T_{-\theta} f) (z) - \frac{1}{2} (T_{-\theta} f) (z)] d\theta
\]
\[\quad = \frac{1}{2\pi} \int_{0}^{\pi} K_n (\theta) \left[ S_{\Gamma} (f - T_{\theta} f) (z) + S_{\Gamma} (f - T_{-\theta} f) (z) \right] d\theta
\]
\[\quad + \frac{1}{4\pi} \int_{0}^{\pi} K_n (\theta) \left[ (f - T_{\theta} f) (z) + (f - T_{-\theta} f) (z) \right] d\theta
\]
for almost all \( z \in \Gamma \).

Taking the supremum over all functions \( g \in L_N (\Gamma) \) with \( \rho (g; N) \leq 1 \) in the last relation, we have
\[
\|f - P_n (\cdot, f)\|_{L_N (\Gamma)} = \sup_{\Gamma} \int_{\Gamma} |f (z) - P_n (z, f)| |g (z)| |dz|
\]
\[
\leq \sup_{\Gamma} \left| \frac{1}{2\pi} \int_{0}^{\pi} K_n (\theta) \left[ S_{\Gamma} (f - T_{\theta} f) (z) + S_{\Gamma} (f - T_{-\theta} f) (z) \right] d\theta \right| |g (z)| |dz|
\]
\[\quad + \sup_{\Gamma} \int_{\Gamma} \left| \frac{1}{4\pi} \int_{0}^{\pi} K_n (\theta) \left[ (f - T_{\theta} f) (z) + (f - T_{-\theta} f) (z) \right] d\theta \right| |g (z)| |dz|\]
\[
\begin{align*}
\leq & \sup_\Gamma \left\{ \frac{1}{2\pi} \int_0^\pi K_n(\theta) \left( |S_{\Gamma} (f - T_\theta f) (z)| \\
& \quad + |S_{\Gamma} (f - T_{(-\theta)} f) (z)| \right) d\theta \right\} |g(z)||dz|
+ \sup_\Gamma \left\{ \frac{1}{4\pi} \int_0^\pi K_n(\theta) \left| |(f - T_\theta f) (z)| \right| d\theta \right\} |g(z)||dz|
\end{align*}
\]
and by Fubini’s theorem
\[
\|f - P_n (\cdot, f)\|_{L_M(\Gamma)}
\leq \frac{1}{2\pi} \int_0^\pi K_n(\theta) \left\{ \sup_\Gamma \left[ |S_{\Gamma} (f - T_\theta f) (z)| \right] \\
+ |S_{\Gamma} (f - T_{(-\theta)} f) (z)| |g(z)||dz| \right\} d\theta
+ \frac{1}{4\pi} \int_0^\pi K_n(\theta) \left\{ \sup_\Gamma \left[ |(f - T_\theta f) (z)| \right] \\
+ |(f - T_{(-\theta)} f) (z)| |g(z)||dz| \right\} d\theta
\leq \frac{1}{2\pi} \int_0^\pi K_n(\theta) \left[ \|S_{\Gamma} (f - T_\theta f)\|_{L_M(\Gamma)} + \|S_{\Gamma} (f - T_{(-\theta)} f)\|_{L_M(\Gamma)} \right] d\theta
+ \frac{1}{4\pi} \int_0^\pi K_n(\theta) \left[ \|f - T_\theta f\|_{L_M(\Gamma)} + \|f - T_{(-\theta)} f\|_{L_M(\Gamma)} \right] d\theta.
\]
Now applying (11), we get
\[
\|f - P_n (\cdot, f)\|_{L_M(\Gamma)} \leq c_5 \int_0^\pi K_n(\theta) \left[ \|f - T_\theta f\|_{L_M(\Gamma)} + \|f - T_{(-\theta)} f\|_{L_M(\Gamma)} \right] d\theta,
\]
and recalling the definition (4) of \(\omega_M (\delta, f)\), we obtain
\[
\|f - P_n (\cdot, f)\|_{L_M(\Gamma)} \leq c_6 \int_0^\pi K_n(\theta) \omega_M (\theta, f) d\theta \leq c_7 \omega_M (\frac{1}{n}, f) \int_0^\pi K_n(\theta) (n\theta + 1) d\theta.
\]
Consequently from (21) and (22), we have

\[ \| f - P_n (\cdot, f) \|_{L_M (\Gamma)} \leq c_8 \omega_M \left( \frac{1}{n}, f \right), \]

which proves Theorem 1.3. \( \square \)

**Proof of Theorem 1.4.** Let \( z \in \Gamma, 1 < r < R \) and \( p_n (z) \) be the best approximating polynomial of degree at most \( n \) to the function \( f \in E_M (G_R) \).

Denoting

\[ I_1 := \frac{1}{2\pi} \int_{|t| = R} |f(\psi(t)) - p_n(\psi(t))| \left| \sum_{k=n+1}^{\infty} \frac{w^k}{k+1} \right| |dt|, \]
\[ I_2 := \frac{1}{2\pi} \int_{|t| = R} |f(\psi(t)) - p_n(\psi(t))| \left| \sum_{k=n+1}^{\infty} E_k(\psi(w)) \frac{1}{t^{k+1}} \right| |dt|, \]

by virtue of (17), we see that

\[ (24) \quad |R_n(z,f)| \leq I_1 + I_2. \]

Using relations (1) and (13), we have

\[ I_1 = \frac{1}{2\pi} \int_{\Gamma_R} |f(\zeta) - p_n(\zeta)| \left| \sum_{k=n+1}^{\infty} \frac{[\varphi(\zeta)]^k}{[\varphi(\zeta)]^{k+1}} \right| \left| \varphi'(\zeta) \right| |d\zeta| \]
\[ \leq \frac{c_9}{2\pi} \int_{\Gamma_R} |f(\zeta) - p_n(\zeta)| \left| \sum_{k=n+1}^{\infty} \frac{[\varphi(\zeta)]^k}{[\varphi(\zeta)]^{k+1}} \right| |d\zeta| \]
\[ \leq \frac{c_9}{2\pi} \left\{ \sup_{\Gamma_R} \int_{\Gamma_R} |f(\zeta) - p_n(\zeta)| |g(\zeta)| |d\zeta| \right\} \cdot \left\{ \sup_{\Gamma_R} \left| \sum_{k=n+1}^{\infty} \frac{[\varphi(\zeta)]^k}{[\varphi(\zeta)]^{k+1}} \right| |h(\zeta)| |d\zeta| \right\}, \]

where the suprema are taken over all functions \( g \in L_N (\Gamma) \) with \( \rho(g; N) \leq 1 \) and \( h \in L_M (\Gamma) \) with \( \rho(h; M) \leq 1 \), respectively. By virtue of (5)

\[ I_1 \leq \frac{c_{10} E_n^M (f, G_R)}{2\pi} \sup \left\{ \int_{\Gamma_R} \left| \sum_{k=n+1}^{\infty} \frac{[\varphi(\zeta)]^k}{[\varphi(\zeta)]^{k+1}} \right| |h(\zeta)| |d\zeta| : \rho(h; M) \leq 1 \right\} \]
\[ \leq \frac{c_{10} E_n^M (f, G_R)}{2\pi} \sup \left\{ \int_{\Gamma_R} \frac{|\varphi(\zeta)|^{n+1}}{|\varphi(\zeta) - \varphi(z)|} |h(\zeta)| |d\zeta| : \rho(h; M) \leq 1 \right\}, \]
\[
\leq \frac{c_{11}E_n^M(f, G_R)}{2\pi} \cdot \frac{r^{n+1}}{R^{n+1}(R - r)} \cdot \sup_{F_R} \left\{ \int_{F_R} |h(\zeta)| |d\zeta| : \rho(h; M) \leq 1 \right\},
\]

and by (12)

\[
(25) \quad \sup_{F_R} \left\{ \int_{F_R} |h(\zeta)| |d\zeta| : \rho(h; M) \leq 1 \right\} \leq 1 + N(1) \text{mes}\Gamma_R \leq c_{12},
\]

and therefore

\[
(26) \quad I_1 \leq \frac{c_{13}E_n^M(f, G_R) r^{n+1}}{2\pi R^{n+1}(R - r)}.
\]

Now, we estimate the integral \(I_2\). By (18) we have

\[
I_2 = \frac{1}{2\pi} \int_{|t|=R} |f(\psi(t)) - p_n(\psi(t))| \left| \sum_{k=n+1}^{\infty} \frac{1}{2\pi} \int_{|\tau|=r} \frac{r^k}{l^{k+1}} F(\tau, w) \, d\tau \right| |dt|
\]

\[
\leq \frac{1}{2\pi} \int_{|t|=R} |f(\psi(t)) - p_n(\psi(t))| \left\{ \frac{1}{2\pi} \int_{|\tau|=r} \left| \sum_{k=n+1}^{\infty} \frac{r^k}{l^{k+1}} |F(\tau, w)| |d\tau| \right| \right\} |dt|
\]

\[
\leq \frac{1}{2\pi} \int_{|t|=R} |f(\psi(t)) - p_n(\psi(t))| \left\{ \frac{1}{2\pi} \int_{|\tau|=r} \frac{\tau^{n+1}}{l^{n+1}(l - \tau)} |F(\tau, w)| |d\tau| \right\} |dt|.
\]

Applying Fubini’s theorem

\[
I_2 \leq \frac{r^{n+1}}{2\pi R^{n+1}} \int_{|\tau|=r} |F(\tau, w)| \left\{ \frac{1}{2\pi} \int_{|t|=R} \frac{|f(\psi(t)) - p_n(\psi(t))|}{|t - \tau|} \right\} |d\tau|.
\]
and changing the variables in the last integral and using (13), we have

\[
I_2 \leq \frac{r^{n+1}}{2\pi R^{n+1}} \int_{|\tau|=r} |F(\tau, w)| \left\{ \frac{1}{2\pi} \int_{\Gamma_H} |f(\zeta) - p_n(\zeta)| \frac{\left| \varphi'(\zeta) \right|}{\left| \varphi(\zeta) - \varphi'(\zeta) \right|} |d\zeta| \right\} |d\tau| \\
\leq \frac{c_{14} r^{n+1}}{4\pi^2 R^{n+1} (R - r)} \int_{|\tau|=r} |F(\tau, w)| \left\{ \| f(\zeta) - p_n(\zeta) \|_{L_M(\Gamma_H)} \times \right. \\
\left. \frac{\left\| \varphi'(\cdot) \right\|}{\left\| \varphi(\cdot) - \varphi'(\cdot) \right\|_{L_N(\Gamma_H)}} \right\} |d\tau| \\
\leq \frac{c_{15} r^{n+1}}{2\pi R^{n+1} (R - r)} E_n^M(f, G_R) \sqrt{\frac{r^2}{r^4 - 1} \ln \frac{r^2}{r^2 - 1}}.
\]

From this, by repeating the arguments given in (25) and using (19), we conclude that

\[
(27) \quad I_2 \leq \frac{c_{15} r^{n+1}}{2\pi R^{n+1} (R - r)} E_n^M(f, G_R) \sqrt{\frac{r^2}{r^4 - 1} \ln \frac{r^2}{r^2 - 1}}.
\]

Now, the inequalities (26), (27) and (24) imply that

\[
|R_n(z, f)| \leq \frac{c_{16} r^{n+1} E_n^M(f, G_R)}{2\pi R^{n+1} (R - r)} \sqrt{\frac{r^2}{r^4 - 1} \ln \frac{r^2}{r^2 - 1}}.
\]

Consequently, setting \( z \in K \) and \( r := 1 + \frac{1}{n} \) in this estimate, we obtain the inequality

\[
|R_n(z, f)| \leq \frac{c_{17}}{R^{n+1} (R - 1)} E_n^M(f, G_R) \sqrt{n \ln n},
\]

with \( c_{17} > 0 \).

\[\square\]

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**References**


