Math. Commun. **21**(2016), 23–30

Perfect 1-error-correcting Lipschitz weight codes

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Received May 10, 2015; accepted October 27, 2015

Abstract. Let π be a Lipschitz prime and $p = \pi\pi^\star$. Perfect 1-error-correcting codes in $H(\mathbb{Z})_\pi^n$ are constructed for every prime number $p \equiv 1 (\mathrm{mod}\ 4)$. This completes a result of the authors in an earlier work, *Perfect Mannheim, Lipschitz and Hurwitz weight codes*, (Math. Commun. **19**(2014), 253–276), where a construction is given in the case $p \equiv 3\,(\mathrm{mod}\ 4)$.

AMS subject classifications: 94B60

Key words: perfect codes

1. Introduction

Lipschitz weight codes were introduced by Martinez et al. in [3, 4]. Shortly, consider the ring of quaternions over the integers

$$H(\mathbb{Z}) = \{a_0 + a_1e_1 + a_2e_2 + a_3e_3 \mid a_0, a_1, a_2, a_3 \in \mathbb{Z}\},$$

where

$$e_1^2 = e_2^2 = e_3^2 = -1, \tag{1}$$

and

$$e_1e_2 = -e_2e_1 = e_3, \quad e_2e_3 = -e_3e_2 = e_1, \quad e_3e_1 = -e_1e_3 = e_2. \tag{2}$$

A *Lipschitz prime* is an element $\pi = a_0 + a_1e_1 + a_2e_2 + a_3e_3$ in $H(\mathbb{Z})$ such that

$$p = \pi\pi^\star = (a_0 + a_1e_1 + a_2e_2 + a_3e_3)(a_0 - a_1e_1 - a_2e_2 - a_3e_3) = a_0^2 + a_1^2 + a_2^2 + a_3^2$$

is an odd prime number. The integer $\mathrm{N}(\pi) = \pi\pi^\star$ is called the *norm* of π.

The elements in the left ideal

$$\langle\pi\rangle = \{\lambda\pi \mid \lambda \in H(\mathbb{Z})\}$$

constitute a normal subgroup of the additive group of the ring $H(\mathbb{Z})$. The set of cosets to $\langle\pi\rangle$ in $H(\mathbb{Z})$ constitute an Abelian group denoted as below:

$$H = H(\mathbb{Z})_\pi = H(\mathbb{Z})/\langle\pi\rangle.$$

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In [3], it is proved that the size of $H(\mathbb{Z})_\pi$ is equal to p^2.

Let

$$\mathcal{E} = \{\pm 1, \pm e_1, \pm e_2, \pm e_3\},$$

and let $\mathcal{E}_\pi$ denote the family of cosets to $\langle \pi \rangle$ containing the elements of $\mathcal{E}$. We define the *distance* between the words $(\alpha_1, \ldots, \alpha_n)$ and $(\beta_1, \ldots, \beta_n)$ in the direct product $H(\mathbb{Z})_\pi^n$ of n copies of $H(\mathbb{Z})_\pi$,

$$\mathrm{d}((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n)) = 1,$$

if there is a $j \in [n]$ and an $\epsilon \in \mathcal{E}_\pi$ such that $\beta_j = \alpha_j + \epsilon$ and $\beta_i = \alpha_i$, for $i \neq j$.

A *perfect 1-error-correcting Lipschitz weight code of length* n is a subset C of the direct product H^n of n copies of the group H, such that every element in $C \setminus H^n$ is at distance one from exactly one word of C.

In [1], perfect 1-error-correcting Lipschitz weight codes in $H(\mathbb{Z})_\pi^n$ are constructed for every Lipschitz prime π such that $p = \pi\pi^\star \equiv 3 \pmod 4$, for $p > 3$. The purpose of this paper is to extend this result to the case $p \equiv 1 \pmod 4$.

2. Notation

The coset $a + \langle \pi \rangle$ to the left ideal $\langle \pi \rangle$ in the ring $H(\mathbb{Z})$ is denoted by $\overline{a}$.[‡]

It is important for our results that the Abelian group $H(\mathbb{Z})_\pi$, consisting of the cosets to the left ideal $\langle \pi \rangle$, is a left module over the ring $H(\mathbb{Z})$, see [1]. We remind that, as a left module over $H(\mathbb{Z})$, the left distributive rule holds in $H(\mathbb{Z})_\pi$, that is,

$$\lambda(\overline{a} + \overline{b}) = \lambda\overline{a} + \lambda\overline{b},$$

is true for every $\lambda \in H(\mathbb{Z})$ and $\overline{a}, \overline{b} \in H(\mathbb{Z})_\pi$.

For example, with $\pi = 2 - 3e_2$, we get that $13e_i = e_i(2 + 3e_2)(2 - 3e_2) \in \langle \pi \rangle$, for $i = 1, 2, 3$, and that $\overline{e_3} = \overline{5e_1}$, as

$$4e_1(2 - 3e_2) \in \langle \pi \rangle \Longrightarrow 8e_1 - 12e_3 \in \langle \pi \rangle \Longrightarrow -12e_3 \in -8e_1 + \langle \pi \rangle.$$

Also used in an example below is the fact that

$$(2 + 2e_3)\overline{-e_3} = \overline{-2e_3 + 2} = \overline{2} - \overline{2e_3} = \overline{2} - \overline{10e_1} = \overline{2} + \overline{3e_1} = \overline{2 + 3e_1}, \tag{3}$$

and that

$$\overline{0} = e_1\overline{0} = e_1(\overline{8e_1} - \overline{12e_3}) = \overline{-8} + \overline{12e_2} = \overline{5} - \overline{e_2},$$

that is, $\overline{e_2} = \overline{5}$.

Finally, we let $\mathcal{E}_0$ and $\mathcal{E}_1$ denote the following sets

$$\mathcal{E}_0 = \{\pm\overline{1}, \pm\overline{e_1}\}, \quad \mathcal{E}_1 = \{\pm\overline{e_2}, \pm\overline{e_3}\}.$$

[‡]To simplify reading of the text, one of the reviewers suggested a change of the notation used in "Part One" [1] of this study.

3. Preliminaries

Throughout this paper we are only concerned with the case when $p = \pi^\star \pi \equiv 1 (\bmod 4)$ is a prime number. It then follows from the Christmas theorem of Fermat that p is the sum of two squares. Henceforth, we consider the case when $\pi = a_0 + a_2 e_2$, whereby $a_0^2 + a_2^2$ is equal to a prime number p.

We note that $p \in \langle \pi \rangle$, as $p = \pi^\star \pi$. Thus $\overline{p} = \overline{0}$ in $H(\mathbb{Z})_\pi$, and elements in $H(\mathbb{Z})_\pi$ can be described as 4-tuples $x_0 + x_1\overline{e_1} + x_2\overline{e_2} + x_3\overline{e_3}$, where we may assume that $x_i \in \mathbb{Z}_p$ for $i = 0, 1, 2, 3$. The element $\overline{a_0 + a_2 e_2}$ is equal to $\overline{0}$ in $H(\mathbb{Z})_\pi$. Hence, if we let $\mathbf{i}$ denote the element $-a_0/a_2$ in the finite field $\mathbb{Z}_p$, then $-\mathbf{i}\overline{1} + \overline{e_2} = \overline{0}$ in $H(\mathbb{Z})_\pi$, and furthermore, as $\overline{0} = e_1(\overline{a_0} + \overline{a_2 e_2}) = \overline{a_0 e_1} + \overline{a_2 e_3}$, we get that

$$\mathbf{i}\overline{e_1} - \overline{e_3} = \overline{0}. \tag{4}$$

Hence, with this notation,

$$\overline{x + y e_3} = \overline{x + \mathbf{i} y e_1}, \quad \overline{x e_1 + y e_2} = \overline{\mathbf{i} y + x e_1}, \tag{5}$$

and

$$\overline{x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3} = \overline{x_0 + \mathbf{i} x_2 + (x_1 + \mathbf{i} x_3) e_1},$$

and, trivially, in $\mathbb{Z}_p$ we have

$$\mathbf{i}^2 = -1. \tag{6}$$

We say that a selection of coset representatives $\overline{H} = \overline{H(\mathbb{Z})_\pi}$ to $\langle \pi \rangle$ in $H(\mathbb{Z})$ is a *complete selection of coset representatives* if no two elements of $\overline{H(\mathbb{Z})_\pi}$ are congruent modulo π, and if all cosets to $\langle \pi \rangle$ are represented in $\overline{H(\mathbb{Z})_\pi}$, that is,

$$|\overline{H(\mathbb{Z})_\pi}| = |H(\mathbb{Z})_\pi|.$$

As in [1], we say that the set $\overline{H}$ is $\mathcal{E}$*-homogeneous* if

$$\bar{h}\epsilon = \bar{h}'\epsilon \Longrightarrow \bar{h} = \bar{h}'$$

for every $\epsilon \in \mathcal{E}_\pi$ and $\bar{h}, \bar{h}' \in \overline{H}$. In [1], the following proposition is proved:

Proposition 1. *Let $\pi = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3$ be a Lipschitz prime with $p = \pi\pi^\star$. Then, for any two distinct elements e_i and e_j in $\{e_0 = 1, e_1, e_2, e_3\}$ such that p does not divide $a_i^2 + a_j^2$, it is true that*

$$C_{i,j} = \{x_i e_i + x_j e_j : x_i, x_j \in \mathbb{Z}_p\}$$

is a complete selection of coset representatives to $\langle \pi \rangle$ in $H(\mathbb{Z})$. Furthermore, $C_{i,j}$ is $\mathcal{E}$-homogeneous.

A code C is a *group code* if it is a subgroup of H^n, or equivalently, as H^n is a finite group,

$$c, c' \in C \Longrightarrow c - c' \in C.$$

We say that a group code C in H^n is an (n, k)*-code* if the size of C is equal to $|H|^k$.

A more general version of the next theorem is proved in [1].

Theorem 1. *Let H and $\mathcal{E}_\pi$ be constituted as above, and let $\overline{H}$ be a complete selection of coset representatives to $\langle\pi\rangle$. Assume that the norm of π is an odd prime number. Let $n = (|H| - 1)/(|\mathcal{E}_\pi|)$. If $g_1 = 1$, g_2, ..., g_n are elements in $\overline{H}$, satisfying the following three conditions:*

(i) $|g_i\mathcal{E}_\pi| = |\mathcal{E}_\pi|$, *for* $i = 2, 3, \ldots, n$;

(ii) $g_i\mathcal{E}_\pi \cap g_j\mathcal{E}_\pi = \emptyset$, *for* $i \neq j$;

(iii) $H \setminus \{0\} = \mathcal{E}_\pi \cup g_2\mathcal{E}_\pi \cup \ldots \cup g_n\mathcal{E}_\pi$;

then the null-space C of the matrix

$$\mathbf{H} = \begin{pmatrix} 1 & g_2 & \ldots & g_n \end{pmatrix}$$

is a perfect 1-error-correcting group $(n, n-1)$-code in H^n.

Indeed, the code C is defined by the elements g_i, for $i \in [n]$, and may not be altered by a change of these elements. It follows from the results in [2] that it suffices that these elements have the above properties and belong to some ring $\mathcal{R}$ such that H is a left module over $\mathcal{R}$. For the sake of convenience in relation to the presentation of our result,instead of $H(\mathbb{Z})$ we consider the ring

$$H(\mathbb{Z}_p) = \{a_0 + a_1e_1 + a_2e_2 + a_3e_3 \mid a_0, a_1, a_2, a_3 \in \mathbb{Z}_p\},$$

where e_1, e_2 and e_3 have the properties described in Eq. (1) and Eq. (2), and where $p = \pi\pi^\star$. It follows from Proposition 1 that the Abelian group $H(\mathbb{Z})_\pi$ is isomorphic to the Abelian group formed by the cosets to the left ideal $\langle\pi\rangle$ in the ring $H(\mathbb{Z}_p)$.

Thus, in order to prove the existence of a perfect 1-error-correcting Lipschitz weight code of length n, it suffices to prove the existence of a partition of the space as indicated in the theorem, where g_i, for $i \in [n]$, belongs to $H(\mathbb{Z}_p)$. In fact, such partitions are constructed in Section 5 for the cases considered in this paper.

4. Some lemmas

Throughout this section, when not stated otherwise, $\pi = a_0 + a_2e_2$, where $a_0^2 + a_2^2$ is equal to a prime number $p \equiv 1 (\mathrm{mod}\ 4)$, although some of the lemmas are true for every Lipschitz prime π.

Let $\mathcal{D}(\overline{a + be_1})$ denote the set

$$\mathcal{D}(\overline{a + be_1}) = \{\overline{\pm(a \pm be_1)}\} \cup \{\overline{\pm(b \pm ae_1)}\} \cup \{\overline{\pm(\mathbf{i}a \pm \mathbf{i}be_1)}\} \cup \{\overline{\pm(\mathbf{i}b \pm \mathbf{i}ae_1)}\}. \quad (7)$$

Let $\mathcal{Q}$ denote the following subgroup of the multiplicative group $\mathbb{Z}_p^\star$ of the finite field $\mathbb{Z}_p$:

$$\mathcal{Q} = \{1, -1, \mathbf{i}, -\mathbf{i}\}.$$

The first lemma is an immediate consequence of the fact that $\mathcal{Q}$ is a subgroup of $\mathbb{Z}_p^\star$.

Lemma 1. *If $\overline{x + ye_1} \in \mathcal{D}(\overline{a + be_1})$, then $\mathcal{D}(\overline{x + ye_1}) = \mathcal{D}(\overline{a + be_1})$.*

Corollary 1. *There is a sequence* $\overline{a_i + b_i e_1}$, $i = 1, 2, \ldots, s$, *of elements in* $H(\mathbb{Z})_\pi$ *such that the sets* $\mathcal{D}(\overline{a_i + b_i e_1})$ *partitions* $H(\mathbb{Z})_\pi$, *that is,*

$$H(\mathbb{Z})_\pi \setminus \{0\} = \bigcup_{i=1}^{s} \mathcal{D}(\overline{a_i + b_i e_1}),$$

and

$$i \neq j \Longrightarrow \mathcal{D}(\overline{a_i + b_i e_1}) \cap \mathcal{D}(\overline{a_j + b_j e_1}) = \emptyset.$$

Proof. The corollary follows from the fact that from Lemma 1 we may deduce that every non-zero element $\overline{x + y e_1}$ of $H(\mathbb{Z})_\pi$ belongs to exactly one of the sets $\mathcal{D}(\overline{a + b e_1})$. □

Lemma 2. *For any element* $a, b \in \mathbb{Z}_p$,

$$|\mathcal{D}(\overline{a + b e_1})| = \begin{cases} 8, & \text{if } ab(a^2 + b^2)(a^2 - b^2) = 0, \\ 16, & \text{otherwise.} \end{cases}$$

Proof. We consider the case $a^2 + b^2 = 0$, that is, when $b = \mathbf{i}a$ or $b = -\mathbf{i}a$. The other cases are treated similarly. From the definition in Eq. (7), we get that

$$\mathcal{D}(\overline{a + \mathbf{i}a e_1}) = \mathcal{D}(\overline{a - \mathbf{i}a e_1}) = \{\overline{\pm(a \pm \mathbf{i}a e_1)}\} \cup \{\overline{\pm(\mathbf{i}a \pm a e_1)}\}.$$

As $\mathbf{i} \neq \pm 1$ and $a \neq 0$, this is a set consisting of eight distinct elements. □

Lemma 3. *For any element* $a, b \in \mathbb{Z}_p$,

$$\mathcal{D}(\overline{a + b e_1}) = \begin{cases} a\mathcal{E}_\pi, & \text{if } b = 0, \\ b\mathcal{E}_\pi, & \text{if } a = 0, \\ (a + a e_3)\mathcal{E}_\pi, & \text{if } a^2 + b^2 = 0, \\ (a + a e_1)\mathcal{E}_\pi, & \text{if } a^2 - b^2 = 0, \\ (a + b e_1)\mathcal{E}_\pi \cup (b + a e_1)\mathcal{E}_\pi, & \text{if } ab(a^2 + b^2)(a^2 - b^2) \neq 0. \end{cases}$$

Proof. As in the proof of the previous lemma, we just treat the case $a^2 + b^2 = 0$, and here just when $b = \mathbf{i}a$. We consider products to the left of the elements in $\mathcal{E}_\pi$ with elements $a + a e_3$, where $a \in \mathbb{Z}_p \setminus \{0\}$. For instance, we get

$$(a + a e_3)\overline{e_2} = \overline{\mathbf{i}a - a e_1} = \overline{b - a e_1} \in \mathcal{D}(\overline{a + b e_1}),$$

etc. □

We close this section with the following observation: Let d_1, d_2, ..., $d_{(p-1)/4}$ be a family of coset representatives to $\mathcal{Q}$ in $\mathbb{Z}_p^\star$. The following easily verified relations are useful when deriving partitions of the set $H(\mathbb{Z})_\pi$ into cosets of $\mathcal{E}_\pi$:

$$\bigcup_{a \in \mathbb{Z}_p \setminus \{0\}} \mathcal{D}(\overline{a + \mathbf{i}a e_1}) = \bigcup_{i=1}^{(p-1)/4} (d_i + d_i e_3)\mathcal{E}_\pi. \tag{8}$$

$$\bigcup_{a\in\mathbb{Z}_p\setminus\{0\}} \mathcal{D}(\overline{a+ae_1}) = \bigcup_{i=1}^{(p-1)/4} (d_i+d_ie_1)\mathcal{E}_\pi. \tag{9}$$

$$\bigcup_{a\in\mathbb{Z}_p\setminus\{0\}} \mathcal{D}(\overline{a}) = \bigcup_{i=1}^{(p-1)/4} d_i\mathcal{E}_\pi. \tag{10}$$

5. Construction of necessary partitions

Theorem 2. *For every prime number p with $p \equiv 1 (\mathrm{mod}\ 4)$ there is a Lipschitz prime π, with $\pi^\star\pi = p$, and a sequence of $t = (p^2-1)/8$ elements $g_1, \ldots, g_t$ of $H(\mathbb{Z}_p)$ such that*

$$H(\mathbb{Z})_\pi \setminus \{0\} \ = \ g_1\mathcal{E}_\pi \cup \ldots \cup g_t\mathcal{E}_\pi \ ,$$
$$i \neq j \Longrightarrow g_i\mathcal{E}_\pi \cap g_j\mathcal{E}_\pi \ ,$$

and $|g_i\mathcal{E}_\pi| = |\mathcal{E}_\pi|$, for $i = 1, \ldots, t$ and $j \neq i$.

Proof. The theorem follows immediately from Corollary 1, Lemma 2 and Lemma 3, as for every element $g \in H(\mathbb{Z}_p)$,

$$|g\mathcal{E}_\pi| \leq |\mathcal{E}_\pi| = 8 \ .$$

□

In order to be able to apply Theorem 1 we note that the set

$$\{a+be_1 \in C_{0,1} \mid p \not| a^2+b^2\} \cup \{a \pm ae_3 \in C_{0,3} \mid p \not| a \in \mathbb{Z}_p\},$$

forms a complete set of coset representatives to $\langle\pi\rangle$ in $H(\mathbb{Z}_p)$.

We illustrate the construction described in the proof above in the next example.

Example 1. *Let $\pi = 2-3e_2$. Then $p = 13$, $\mathbf{i} = 2/3 = 5$, $\overline{e_2} = \overline{5}$ and $\overline{e_3} = \overline{5e_1}$, see Section 2 and Section 3.*

The group $\mathcal{Q} = \{\pm 1, \pm 5\}$ has for example the coset representatives $d_1 = 1$, $d_2 = 2$ and $d_3 = 4$ in the multiplicative group of the field $\mathbb{Z}_{13}$. To form the partition of the set $H(\mathbb{Z})_\pi$ into left cosets of $\mathcal{E}_\pi$ we begin by using equations (8), (9) and (10) letting

$$g_i = d_i + d_ie_3, \quad g_{3+i} = d_i + d_ie_1, \quad g_{6+i} = d_i,$$

for $i = 1, 2, 3$. This provides nine of the $(p^2-1)/|\mathcal{E}_\pi| = (13^2-1)/8 = 21$ requested left cosets $g_i\mathcal{E}_\pi$ to $\mathcal{E}_\pi$.

The remaining cosets to $\mathcal{E}_\pi$ are formed recursively using Lemma 1 and the case $ab(a^2+b^2)(a^2-b^2) \neq 0$ of Lemma 3. We obtain "the next two cosets $g_{2(k+1)}\mathcal{E}_\pi$ and $g_{2(k+1)+1}\mathcal{E}_\pi$" by first choosing an element $\overline{a+be_1}$ such that

$$\overline{a+be_1} \notin g_1\mathcal{E}_\pi \cup g_2\mathcal{E}_\pi \cup \ldots \cup g_{2k+1}\mathcal{E}_\pi,$$

(where we assume that $k \geq 4$). We then let $g_{2k+2} = a+be_1$ and $g_{2k+3} = b+ae_1$. When this procedure terminates, we have obtained the requested partition of $H(\mathbb{Z})_\pi$.

For example, we may let

$$g_{10}=1+2e_1, g_{11}=2+e_1, g_{12}=1+3e_1, g_{13}=3+e_1, g_{14}=1+4e_1, g_{15}=4+e_1,$$
$$g_{16}=1+6e_1, g_{17}=6+e_1, g_{18}=2+4e_1, g_{19}=4+2e_1, g_{20}=2+6e_1, g_{21}=6+2e_1.$$

The element $\overline{1}$ of $H(\mathbb{Z})_\pi$ belongs to the coset $d_1\mathcal{E}_\pi$. For the error-correcting procedure we may thus form the matrix

$$\mathbf{H} = \begin{bmatrix} 1 & 1+e_3 & 2+2e_3 & 4+4e_3 & 1+2e_1 & 2+e_1 & 1+e_1 & 1+3e_1 & 3+e_1 & \cdots \end{bmatrix}.$$

The perfect 1-error-correcting code C is the null space in $H(\mathbb{Z}_p)^{21}$ of $\mathbf{H}$, the number of words of C is $|C| = 169^{20}$. If, after a transmission, the word $\overline{x_1}\,\overline{x_2}\ldots\overline{x_{21}}$ is received by giving the syndrome

$$\mathbf{H}\begin{bmatrix}\overline{x_1}\\ \vdots \\ \overline{x_{21}}\end{bmatrix} = \begin{bmatrix}\overline{2+3e_1}\end{bmatrix},$$

then the error $\epsilon = \overline{-e_3}$ has appeared in the third coordinate position as from Section 2

$$\overline{2+3e_1} = (2+2e_3)\overline{(-e_3)}.$$

From the theorem above and Theorem 1, we immediately get the following corollary:

Corollary 2. *To every prime number p such that $p \equiv 1(\bmod\ 4)$ there is a Lipschitz prime π of norm $\mathrm{N}(\pi) = p$ such that there exists a perfect 1-error-correcting Lipschitz weight code in $H(\mathbb{Z})^n_\pi$, where $n = (p^2-1)/8$.*

Thus, combining with the results of [1], we get

Theorem 3. *To every prime number $p > 3^{\S}$ there is a Lipschitz prime π of norm $\mathrm{N}(\pi) = p$ such that there exists a perfect 1-error-correcting Lipschitz weight code in $H(\mathbb{Z})^n_\pi$, where $n = (p^2-1)/8$.*

A final remark is that the crucial part in our construction of perfect 1-error-correcting Lipschitz weight codes is the derivation of a partition of $H(\mathbb{Z})_\pi$ into **left cosets** of $\mathcal{E}_\pi$. As soon as this problem is solved, we can easily extend the construction with parity-check matrices, like in Theorem 4 in "Part One" of this study [1], in order to obtain codes of other lengths and sizes. Thus we now know, cf. Section 5.2 of [1], that for every prime number $p > 3$ there is a perfect 1-error-correcting Lipschitz weight code C of length $n = (p^{2l}-1)/8$ and size $|C| = p^{2k}$, where $k = (p^{2l}-1)/8 - l$, and l is any non-negative integer.

Acknowledgement

We are grateful to one of the reviewers who encouraged us to write a more clear presentation.

§The case $p = 3$ is out of any general interest.

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