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Fine spectra of triangular triple-band matrices on sequence spaces c and ℓ_p, $(0 < p < 1)$

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Abstract. The purpose of this study is to determine the fine spectra of the operator for which the corresponding upper and lower triangular matrices $A(r,s,t)$ and $B(r,s,t)$ are on the sequence spaces c and ℓ_p, where $(0 < p < 1)$, respectively. Further, we obtain the approximate point spectrum, defect spectrum and compression spectrum on these spaces. Furthermore, we give the graphical representations of the spectrum of the triangular triple-band matrix over the sequence spaces c and ℓ_p.

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1. Introduction

It is well known that matrices play an important role in operator theory. The spectrum of an operator generalizes the notion of eigenvalues for matrices. The spectrum of an operator over a Banach space is partitioned into three parts, i.e., the point spectrum, the continuous spectrum and the residual spectrum. Calculation of these three parts of the spectrum of an operator is called determination of the fine spectrum of the operator.

By a sequence space we understand a linear subspace of the space $\omega = \mathbb{C}^{\mathbb{N}_1}$ of all complex sequences containing ϕ, which is the set of all finitely non–zero sequences, where $\mathbb{N}_1$ denotes the set of positive integers. We write ℓ_∞, c, c_0 and bv for spaces of all bounded, convergent, null and bounded variation sequences, which are Banach spaces with the sup-norm $\|x\|_\infty = \sup_{k\in\mathbb{N}} |x_k|$, $x \in \{\ell_\infty, c_0, c\}$ and $\|x\|_{bv} = \sum_{k=0}^{\infty} |x_k - x_{k+1}|$, while ϕ is not a Banach space with respect to any norm, respectively, where $\mathbb{N} = \{0, 1, 2, \dots\}$. Also, by ℓ_p, $(0 < p < \infty)$ we denote the sequence space of all sequences associated with a $p-$absolutely convergent series which is a Banach space with the norm $\|x\|_p = \left(\sum_{k=0}^{\infty} |x_k|^p\right)^{1/p}$, $1 \le p < \infty$ and $\|x\|_p = \sum_{k=0}^{\infty} |x_k|^p$, $0 < p < 1$.

Let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $n, k \in \mathbb{N}$, and write

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$$(Ax)_n = \sum_k a_{nk} x_k \ ; \quad (n \in \mathbb{N},\ x \in D_{00}(A)), \tag{1}$$

where $D_{00}(A)$ denotes the subspace of w consisting of $x \in w$ for which the sum exists as a finite sum. For simplicity in notation, we write the summation without limits running from 0 to ∞ and we use the convention that any term with a negative subscript is equal to naught. More generally, if μ is a normed sequence space, we can write $D_\mu(A)$ for $x \in w$ for which the sum in (1) converges in the norm of μ. We write

$$(\lambda : \mu) = \{A : \lambda \subseteq D_\mu(A)\}$$

for the space of those matrices sending the whole of the sequence space λ into μ in this sense.

We give a short survey concerning the spectrum and the fine spectrum of linear operators defined by some particular triangle matrices over certain sequence spaces. The fine spectrum of the Cesàro operator of order one on the sequence space ℓ_p was studied by Gonzàlez [17], where $1 < p < \infty$. Also, weighted mean matrices of the operator on ℓ_p were investigated by Cartlidge [9]. The spectrum of the Cesàro operator of order one on the sequence spaces bv_0 and bv was investigated by Okutoyi [25, 26]. The spectrum and the fine spectrum of the Rhally operator on the sequence spaces c_0 and c were examined by Yıldırım [30]. The fine spectrum of the difference operator Δ over the sequence spaces c_0 and c were studied by Altay and Başar [4]. The fine spectra of the difference operator Δ over the sequence spaces ℓ_p and bv_p was studied by Akhmedov and Başar [1, 2], where bv_p is the space of p-bounded variation sequences introduced by Başar and Altay [7] with $1 \leq p < \infty$. Also, the fine spectrum of $B(r, s, t)$ over the sequence spaces c_0 and c was studied by Furkan et al. [15]. In 2010, Srivastava and Kumar [27] determined the spectra and the fine spectra of the generalized difference operator Δ_ν on ℓ_1, where Δ_ν is defined by $(\Delta_\nu)_{nn} = \nu_n$ and $(\Delta_\nu)_{n+1,n} = -\nu_n$ for all $n \in \mathbb{N}$, under certain conditions on the sequence $\nu = (\nu_n)$, and they generalized these results by the generalized difference operator Δ_{uv} defined by $\Delta_{uv} x = (u_n x_n + v_{n-1} x_{n-1})_{n \in \mathbb{N}}$ for all $n \in \mathbb{N}$, (see [28]). Karakaya and Altun determined the fine spectra of upper triangular double-band matrices over the sequence spaces c_0 and c [23]. Akhmedov and El-Shabrawy [3], and El-Shabrawy [13, 14] obtained the fine spectrum of the generalized difference operator $\Delta_{a,b}$, defined as a double band matrix with the convergent sequences $\widetilde{a} = (a_k)$ and $\widetilde{b} = (b_k)$ having certain properties, over the sequence spaces c, ℓ_p, $(1 < p < \infty)$ and c_0, respectively. Dutta and Baliarsingh [12, 8] examined the fine spectra of the generalized rth difference operator $\triangle_v^r$ on the sequence spaces ℓ_1 and c.

Recently, Karaisa [19, 20] and Karaisa and Başar [6] have determined the fine spectrum of matrix operators with the corresponding upper and lower triangular matrices $A(\widetilde{r}, \widetilde{s})$ and $B(\widetilde{r}, \widetilde{s})$ with the convergent sequences $\widetilde{r} = (r_k)$ and $\widetilde{s} = (s_k)$ having certain properties, over the sequence space ℓ_p for $(1 \leq p < \infty)$ and $(1 < p < \infty)$, respectively. Later, Karaisa and Başar [18, 21, 22] have determined the fine spectrum of the upper triangular triple band matrix $A(r, s, t)$ over some sequence spaces. Finally, Dündar and Başar have determined the fine spectrum of the matrix operator $\triangle^+$ defined by an upper triangle double band matrix acting on the sequence space c_0 [10].

This paper is organized as follows: In Section 2, some notations and fundamental definitions are given. In Section 3, the spectrum, the point spectrum, the residual spectrum and the continuous spectrum of the operator $A(r,s,t)$ on the sequence space c have been computed. We also give the approximate point spectrum, the defect spectrum and the compression spectrum of the operator $A(r,s,t)$ over the space c. In Section 4, we have computed the spectrum and the fine spectrum of the lower-triangular triple band matrix $B(r,s,t)$ over the sequence space ℓ_p, $(0<p<1)$ as well. Also, the boundedness and the norm of the operator $B(r,s,t)$ are given. Finally, we give the graphical representation of the spectrum of upper triangular triple-band matrices and conclude the study.

2. Notations and known results

Let X and Y be the Banach spaces, and $T : X \to Y$ a bounded linear operator. By $R(T)$ we denote range of T, i.e.,

$$R(T) = \{y \in Y : y = Tx,\ x \in X\}.$$

By $B(X)$ we also denote the set of all bounded linear operators on X into itself. If $T \in B(X)$, then the adjoint T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*f)(x) = f(Tx)$ for all $f \in X^*$ and $x \in X$.

Let $X \neq \{\theta\}$ be a complex normed space and $T : D(T) \to X$ a linear operator with domain $D(T) \subseteq X$. With T we associate the operator $T_\alpha = T - \alpha I$, where α is a complex number and I is the identity operator on $D(T)$. If T_α has an inverse which is linear, it is denoted by T_α^{-1}, that is, $T_\alpha^{-1} = (T - \alpha I)^{-1}$ and it is called the resolvent operator of T.

Many properties of T_α and T_α^{-1} depend on α, and spectral theory is concerned with those properties. For instance, we are interested in the set of all α's in the complex plane such that T_α^{-1} exists. The boundedness of T_α^{-1} is another property that will be essential. We shall also ask for what α's the domain of T_α^{-1} is dense in X, to name just a few aspects. For our investigation of T, T_α and T_α^{-1}, we need some basic concepts of spectral theory, which are given as follows (see [24, pp. 370-371]):

A regular value α of T is a complex number such that

(R1) T_α^{-1} exists,

(R2) T_α^{-1} is bounded,

(R3) T_α^{-1} is defined on a set which is dense in X.

The resolvent set $\rho(T)$ of T is a set of all regular values α of T. Its complement $\mathbb{C} \setminus \sigma(T)$ in the complex plane $\mathbb{C}$ is called the spectrum of T. Furthermore, the spectrum $\sigma(T)$ is partitioned into three disjoint sets as follows: The point spectrum $\sigma_p(T)$ is a set such that T_α^{-1} does not exist. $\alpha \in \sigma_p(T)$ is called an eigenvalue of T. The continuous spectrum $\sigma_c(T)$ is a set such that T_α^{-1} exists and satisfies (R3) but not (R2). The residual spectrum $\sigma_r(T)$ is a set such that T_α^{-1} exists but does not satisfy (R3).

In this section, before Appell et al. [5], we define the three more subdivisions of the spectrum called the approximate point spectrum, the defect spectrum and the compression spectrum, respectively.

Given a bounded linear operator T in a Banach space X, we call a sequence (x_k) in X a Weyl sequence for T if $\|x_k\| = 1$ and $\|Tx_k\| \to 0$, as $k \to \infty$.

From now on, we call the set

$$\sigma_{ap}(T, X) := \{\alpha \in \mathbb{C} : \text{there exists a Weyl sequence for } T - \alpha I\} \tag{2}$$

the approximate point spectrum of T. Moreover, the subspectrum

$$\sigma_\delta(T, X) := \{\alpha \in \mathbb{C} : T - \alpha I \text{ is not surjective}\} \tag{3}$$

is called the defect spectrum of T.

The two subspectra given by (2) and (3) form (not necessarily disjoint) subdivisions

$$\sigma(T, X) = \sigma_{ap}(T, X) \cup \sigma_\delta(T, X)$$

of the spectrum. There is another subspectrum,i.e.,

$$\sigma_{co}(T, X) = \{\lambda \in \mathbb{C} : \overline{R(T - \alpha I)} \neq X\},$$

which is often called the compression spectrum in the literature. In Goldberg [16], if $T \in B(X)$, with X a Banach space, then there are three possibilities for $R(T)$:

(A) $R(T) = X$,

(B) $R(T) \neq \overline{R(T)} = X$,

(C) $\overline{R(T)} \neq X.$,

and three possibilities for T^{-1}:

(1) T^{-1} exists and is continuous,

(2) T^{-1} exists but is discontinuous,

(3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, and C_3. If an operator is in state C_2, for example, then $\overline{R(T)} \neq X$ and T^{-1} exists but is discontinuous and we can write $\sigma(T, X)C_2$.

By the definitions given above, we can illustrate the subdivisions of the spectrum in the following table:

		1	2	3
		T_α^{-1} exists and is bounded	T_α^{-1} exists and is unbounded	T_α^{-1} does not exist
A	$R(T-\alpha I)=X$	$\alpha\in\rho(T,X)$	–	$\alpha\in\sigma_p(T,X)$ $\alpha\in\sigma_{ap}(T,X)$
B	$\overline{R(T-\alpha I)}=X$	$\alpha\in\rho(T,X)$	$\alpha\in\sigma_c(T,X)$ $\alpha\in\sigma_{ap}(T,X)$ $\alpha\in\sigma_\delta(T,X)$	$\alpha\in\sigma_p(T,X)$ $\alpha\in\sigma_{ap}(T,X)$ $\alpha\in\sigma_\delta(T,X)$
C	$\overline{R(T-\alpha I)}\neq X$	$\alpha\in\sigma_r(T,X)$ $\alpha\in\sigma_\delta(T,X)$ $\alpha\in\sigma_{co}(T,X)$	$\alpha\in\sigma_r(T,X)$ $\alpha\in\sigma_{ap}(T,X)$ $\alpha\in\sigma_\delta(T,X)$ $\alpha\in\sigma_{co}(T,X)$	$\alpha\in\sigma_p(T,X)$ $\alpha\in\sigma_{ap}(T,X)$ $\alpha\in\sigma_\delta(T,X)$ $\alpha\in\sigma_{co}(T,X)$

Table 1: *Subdivisions of the spectrum of a linear operator*

Proposition 1 (see [5], Proposition 1.3, p. 28)**.** *Spectra and subspectra of an operator $T\in B(X)$ and its adjoint $T^*\in B(X^*)$ are related by the following relations:*

(a) $\sigma(T^*,X^*)=\sigma(T,X)$,

(b) $\sigma_c(T^*,X^*)\subseteq\sigma_{ap}(T,X)$,

(c) $\sigma_{ap}(T^*,X^*)=\sigma_\delta(T,X)$,

(d) $\sigma_\delta(T^*,X^*)=\sigma_{ap}(T,X)$,

(e) $\sigma_p(T^*,X^*)=\sigma_{co}(T,X)$,

(f) $\sigma_{co}(T^*,X^*)\supseteq\sigma_p(T,X)$,

(g) $\sigma(T,X)=\sigma_{ap}(T,X)\cup\sigma_p(T^*,X^*)=\sigma_p(T,X)\cup\sigma_{ap}(T^*,X^*)$.

Relations (c)–(f) show that the approximate point spectrum is in a certain sense dual to the defect spectrum, and the point spectrum is dual to the compression spectrum. Equality (g) implies, in particular, that $\sigma(T,X)=\sigma_{ap}(T,X)$ if X is a Hilbert space and T is normal. Roughly speaking, this shows that normal (in particular, self-adjoint) operators on the Hilbert spaces are most similar to matrices in finite dimensional spaces (see [5]).

3. Fine spectra of upper triangular triple-band matrices over the space of convergent sequences

In this section, our main focus is on the upper triple-band matrix $A(r,s,t)$, where

$$A(r,s,t)=\begin{bmatrix} r & s & t & 0 & \dots\\ 0 & r & s & t & \dots\\ 0 & 0 & r & s & \dots\\ 0 & 0 & 0 & r & \dots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{bmatrix}.$$

From now on, we assume that s and t are complex parameters which do not vanish simultaneously.

Let us introduce the operator $A(r,s,t)$ from c to itself by

$$A(r,s,t)x = (rx_k + sx_{k+1} + tx_{k+2})_{k=0}^{\infty} \text{ where } x = (x_k) \in c.$$

In this section, we examine the spectrum, the point spectrum, the residual spectrum and the continuous spectrum of the operator $A(r,s,t)$ on the sequence space c. Besides, we give the approximate point spectrum, the defect spectrum and the compression spectrum of the matrix operator $A(r,s,t)$ over the space c.

Lemma 1 (see [29], Theorem 1.3.6, p. 6)**.** *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(c)$ from c to itself if and only if*

(1) the rows of A are in ℓ_1 and their ℓ_1 norms are bounded,

(2) the columns of A are in c,

(3) the sequence of row sums of A is in c.

The operator norm of T is the supremum of the ℓ_1 norms of the rows.

Corollary 1. *$A(r,s,t) : c \to c$ is a bounded linear operator and*

$$\|A(r,s,t)\|_{(c:c)} = |r| + |s| + |t|. \tag{4}$$

Lemma 2 (see [16], p. 59)**.** *T has a dense range if and only if T^* is one to one.*

If $T : c \to c$ is a bounded matrix operator with the matrix A, then $T^* : c^* \to c^*$ acting on $\mathbb{C} \oplus \ell_1$ has a matrix representation of the form $\begin{bmatrix} \chi & 0 \\ b & A^t \end{bmatrix}$, where χ denotes the characteristic of the matrix A and b is the column vector whose k^{th} entry is the limit of the column of A for each $k \in \mathbb{N}$. For $A(r,s,t) : c \to c$, the matrix $A(r,s,t)^* \in B(\ell_1)$ is of the form

$$A(r,s,t)^* = \begin{bmatrix} r+s+t & 0 \\ 0 & A^t(r,s,t) \end{bmatrix}.$$

Theorem 1. *$A(r,s,t) : c \to c$ has a dense range if and only if $\alpha \neq r+s+t$.*

Proof. First, let us show that $\sigma_p[A(r,s,t)^*, \mathbb{C} \oplus \ell_1] = \{r+s+t\}$. Suppose that α is an eigenvalue of the operator $A(r,s,t)^* : \mathbb{C} \oplus \ell_1 \to \mathbb{C} \oplus \ell_1$. Then there exists $f \in \ell_1$ satisfying the system of equations

$$\left.\begin{aligned} (r+s+t)f_0 &= \alpha f_0 \\ rf_1 &= \alpha f_1 \\ sf_1 + rf_2 &= \alpha f_2 \\ tf_1 + sf_2 + rf_3 &= \alpha f_3 \\ &\vdots \end{aligned}\right\} \tag{5}$$

From the above system of linear equation, one can see that $\alpha = r + s + t$ is an eigenvalue corresponding to the eigenvector $(1, 0, 0, 0, \dots)$. Now, suppose that $\alpha \neq r + s + t$. Then, $f_0 = 0$. Let f_k be the first nonzero entry of the sequence f. Then by the k-th equation of (5), $\alpha = r$. But, the $(k+1)$-th equation $f_k = 0$ for $s \neq 0$, which contraditcts our assumption. Hence there is no other eigenvalue. So $\sigma_p[A(r,s,t)^*, \mathbb{C} \oplus \ell_1] = \{r + s + t\}$. $\square$

Lemma 3 (see [16], p. 60). *The adjoint operator T^* of T is onto if and only if T has a bounded inverse.*

Before giving the main theorem of this section, we should note the following remark. In this paper, from now on, if z is a complex number, then by $\sqrt{z}$ we always mean the square root of z with a nonnegative real part. If $Re(\sqrt{z}) = 0$, then $\sqrt{z}$ represents the square root of z with $Im(\sqrt{z}) > 0$. The same results are obtained if $\sqrt{z}$ represents the other square roots.

Theorem 2. *Let s be a complex number such that $\sqrt{s^2} = -s$ and define the set D_1 by*

$$D_1 = \left\{ \alpha \in \mathbb{C} : 2\left|r - \alpha\right| \leq \left|-s + \sqrt{s^2 - 4t(r-\alpha)}\right| \right\}.$$

Then, $\sigma_c[A(r,s,t), c] \subseteq D_1$.

Proof. Let $y = (y_k) \in \ell_1$. Then, by solving the equation $A_\alpha(r,s,t)^* x = y$ for $x = (x_k)$ in terms of y, we obtain

$$\begin{aligned}
x_0 &= \frac{y_0}{r+s+t-\alpha}, \\
x_1 &= \frac{y_1}{r-\alpha}, \\
x_2 &= \frac{y_2}{r-\alpha} + \frac{-sy_1}{(r-\alpha)^2}, \\
x_3 &= \frac{y_3}{r-\alpha} + \frac{-sy_2}{(r-\alpha)^2} + \frac{(s^2 - t(r-\alpha))y_1}{(r-\alpha)^3}, \\
&\vdots
\end{aligned}$$

and if we denote $a_1 = 1/(r-\alpha)$, $a_2 = -s/(r-\alpha)^2$, $a_3 = [s^2 - t(r-\alpha)]/(r-\alpha)^3$, we have

$$\begin{aligned}
x_0 &= \frac{y_0}{r+s+t-\alpha}, \\
x_1 &= a_1 y_1, \\
x_2 &= a_1 y_2 + a_2 y_1, \\
x_3 &= a_1 y_3 + a_2 y_2 + a_3 y_1, \\
&\vdots \\
x_n &= a_1 y_n + a_2 y_{n-1} + \cdots + a_n y_1 = \sum_{k=1}^{n} a_{n+1-k} y_k \quad \text{for } n \geq 1.
\end{aligned} \tag{6}$$

Now, we must find a_n. We have $y_n = tx_{n-2} + sx_{n-1} + (r-\alpha)x_n$, for $n \geq 3$ and if we use relation (6), we have

$$\begin{aligned} y_n &= t\sum_{k=1}^{n-2} a_{n-1-k}y_k + s\sum_{k=1}^{n-1} a_{n-k}y_k + (r-\alpha)\sum_{k=1}^{n} a_{n+1-k}y_k \\ &= y_1[ta_{n-2} + sa_{n-1} + (r-\alpha)a_n] + y_2[ta_{n-3} + sa_{n-2} + (r-\alpha)a_{n-1}] \\ &\quad + \cdots + y_{n-1}[sa_1 + (r-\alpha)a_2] + y_n a_1(r-\alpha) \quad \text{for all}\, n \geq 3. \end{aligned}$$

This implies that $ta_{n-2} + sa_{n-1} + (r-\alpha)a_n = 0$, $ta_{n-3} + sa_{n-2} + (r-\alpha)a_{n-1} = 0, \ldots, sa_1 + (r-\alpha)a_2 = 0$, $(r-\alpha)a_1 = 1$. In fact, this sequence is obtained recursively by letting

$$a_1 = \frac{1}{r-\alpha}, \quad a_2 = \frac{-s}{(r-\alpha)^2} \quad \text{and} \quad ta_{n-2} + sa_{n-1} + (r-\alpha)a_n = 0 \quad \text{for all} \quad n \geq 3.$$

The characteristic equation of the recurrence relation is $(r-\alpha)\lambda^2 + s\lambda + t = 0$. If $\Delta = s^2 - 4t(r-\alpha) \neq 0$, then the straightforward calculation gives that

$$a_n = \frac{\lambda_1^n - \lambda_2^n}{\sqrt{s^2 - 4t(r-\alpha)}} \quad \text{for all} \quad n \geq 1, \; \lambda_1 = \frac{-s+\sqrt{\Delta}}{2(r-\alpha)}, \quad \lambda_2 = \frac{-s-\sqrt{\Delta}}{2(r-\alpha)}. \tag{7}$$

By (6), one can see that

$$|x_n| \leq \sum_{k=1}^{n} |a_{n+1-k}||y_k|, \quad \text{for all} \quad n \in \mathbb{N}_1,$$

and we have

$$\begin{aligned} |x_0| + |x_1| + \cdots + |x_n| &\leq \frac{|y_0|}{|r+s+t-\alpha|} + \sum_{k=1}^{1} |a_{2-k}||y_k| + \sum_{k=1}^{2} |a_{3-k}||y_k| \\ &\quad + \cdots + \sum_{k=1}^{n} |a_{n+1-k}||y_k| \\ &= \frac{|y_0|}{|r+s+t-\alpha|} + \sum_{j=1}^{n} |a_j||y_1| + \sum_{j=1}^{n-1} |a_j||y_2| + \cdots + \sum_{j=1}^{1} |a_j||y_n| \\ &\leq \frac{|y_0|}{|r+s+t-\alpha|} + \sum_{j=1}^{n} |a_j|(|y_1| + |y_2| + \cdots + |y_n|) \end{aligned}$$

for all $n \in \mathbb{N}$. By letting $n \to \infty$, we get

$$\|x\|_1 \leq |\|y\|_1 \sum_{j=1}^{\infty} |a_j| + \frac{|y_0|}{|r+s+t-\alpha|}.$$

We must show that $\sum_{j=1}^{\infty} |a_j| < \infty$. There are two cases here:

Case 1. If $\Delta = s^2 - 4t(r-\alpha) \neq 0$, relation (7) holds for all $k \in \mathbb{N}_1$. Now, we show that if $|\lambda_1| < 1, |\lambda_2| < 1$. Assume that $|\lambda_1| < 1$. So we have

$$\left|-s + \sqrt{s^2 - 4t(r-\alpha)}\right| < |2(r-\alpha)|\,.$$

Since $\sqrt{s^2} = -s$, one can see that

$$\left|1 + \sqrt{1 - \frac{4t(r-\alpha)}{s^2}}\right| < \left|\frac{2(r-\alpha)}{-s}\right|.$$

Since $|1-\sqrt{z}| \leq |1+\sqrt{z}|$ for any $z \in \mathbb{C}$, we get

$$\left|1 - \sqrt{1 - \frac{4t(r-\alpha)}{s^2}}\right| < \left|\frac{2(r-\alpha)}{-s}\right|.$$

It follows that $|\lambda_2| < 1$. Now, for $|\lambda_1| < 1$ we can see that

$$\sum_{j=1}^{\infty} |a_j| \leq \frac{1}{|\sqrt{\Delta}|}\left(\sum_{j=1}^{\infty} |\lambda_1|^j + \sum_{j=1}^{\infty} |\lambda_2|^j\right).$$

$x = (x_k) \in \ell_1$ since $|\lambda_1| < 1$. Hence, $A_\alpha(r,s,t)^*$ is onto. By Lemma 3, $A_\alpha(r,s,t)$ has a bounded inverse. This means that

$$\sigma_c[A(r,s,t),c] \subseteq \left\{\alpha \in \mathbb{C} : 2\big|r-\alpha\big| \leq \big|-s + \sqrt{s^2 - 4t(r-\alpha)}\big|\right\} = D_1.$$

Case 2. If $\Delta = s^2 - 4t(r-\alpha) = 0$, calculation on the recurrence sequence gives

$$a_n = \left(\frac{2n}{-s}\right)\left[\frac{-s}{2(r-\alpha)}\right]^n \qquad \text{for all } \; n \geq 1. \tag{8}$$

Now, for $|-s| < 2|r-\alpha|$ we can see that

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n-1}}\right| = \left|\frac{-s}{2(r-\alpha)}\right| < 1.$$

Since $\sum_{k=1}^{\infty} |a_k|$ is convergent, $x = (x_k) \in \ell_1$. Hence, $A_\alpha(r,s,t)^*$ is onto. By Lemma 3, $A_\alpha(r,s,t)$ has a bounded inverse. This means that

$$\sigma_c[A(r,s,t),c] \subseteq \left\{\alpha \in \mathbb{C} : 2\big|r-\alpha\big| \leq \big|-s\big|\right\} \subseteq D_1.$$

$\square$

Theorem 3. *Define D_2 by*

$$D_2 = \left\{\alpha \in \mathbb{C} : 2\big|r-\alpha\big| < \big|-s + \sqrt{s^2 - 4t(r-\alpha)}\big|\right\}.$$

Then $\sigma_p[A(r,s,t),c] = D_2 \cup \{r+s+t\}$.

Proof. Let $A(r,s,t)x = \alpha x$ for $\theta \neq x \in c$. Then, by solving the system of linear equations

$$\left.\begin{aligned} rx_0 + sx_1 + tx_2 &= \alpha x_0 \\ rx_1 + sx_2 + tx_3 &= \alpha x_1 \\ rx_2 + sx_3 + tx_4 &= \alpha x_2 \\ &\vdots \\ rx_{k-2} + sx_{k-1} + tx_k &= \alpha x_{k-2} \\ &\vdots \end{aligned}\right\}$$

we have

$$\left.\begin{aligned} x_2 &= \frac{-s}{t}x_1 - \frac{r-\alpha}{t}x_0 \\ x_3 &= \frac{s^2 - t(r-\alpha)}{t^2}x_1 + \frac{s(r-\alpha)}{t^2}x_0 \\ &\vdots \\ x_n &= \frac{a_n(r-\alpha)^n}{t^{n-1}}x_1 - \frac{a_{n-1}(r-\alpha)^n}{t^{n-1}}x_0 \end{aligned}\right\} \tag{9}$$

for all $n \geq 2$. Assume that $\alpha \in D_2$. Then we choose $x_0 = 1$ and $x_1 = 2(r-\alpha)/[-s+\sqrt{s^2 - 4t(r-\alpha)}]$. We will show that $x_n = x_1^n$ for all $n \geq 2$. Since λ_1 and λ_2 are roots of the characteristic equation $(r-\alpha)\lambda^2 + s\lambda + t = 0$, we must have

$$\lambda_1\lambda_2 = \frac{t}{r-\alpha} \quad \text{and} \quad \lambda_1 - \lambda_2 = \frac{\sqrt{\Delta}}{r-\alpha}$$

combining $x_1 = 1/\lambda_1$ with relation (9) one can see that

$$\begin{aligned} x_n &= \frac{a_n(r-\alpha)^n}{t^{n-1}}x_1 - \frac{a_{n-1}(r-\alpha)^n}{t^{n-1}}x_0 && (10) \\ &= \left(\frac{r-\alpha}{t}\right)^{n-1}(r-\alpha)(-a_{n-1}x_0 + a_n x_1) && (11) \\ &= \frac{1}{(\lambda_1\lambda_2)^{n-1}}\frac{r-\alpha}{\sqrt{\Delta}}(-\lambda_1^{n-1} + \lambda_2^{n-1} + \lambda_1^{n-1} - \lambda_2^n\lambda_1^{-1}) \\ &= \frac{1}{\lambda_1^{n-1}\lambda_2^{n-1}}\left(\frac{1}{\lambda_1 - \lambda_2}\right)\lambda_2^{n-1}\left(\frac{\lambda_1-\lambda_2}{\lambda_1}\right) \\ &= \frac{1}{\lambda_1^n} \\ &= x_1^n. && (12) \end{aligned}$$

The same result is obtained in the case $\triangle = 0$ as follows; $\Delta = 0$ implies that $s^2 - 4t(r-\alpha) = 0$. Thus, $r - \alpha = \frac{s^2}{4t}$ and $\lambda_1 = \lambda_2 = -\frac{2t}{s}$. By (8), we get

$$a_n = \left(\frac{2n}{-s}\right)\left(\frac{-2t}{s}\right)^n. \tag{13}$$

Then, substituting (11) and choosing $x_0 = 1$ and $x_1 = -\frac{s}{2t}$, we can write

$$\begin{aligned} x_n &= \left(\frac{s}{-2t}\right)^{2n-2} \frac{s^2}{4t}\left[\frac{2n-2}{s}\left(\frac{-2t}{s}\right)^{n-1} - \frac{2n}{s}\left(\frac{-2t}{s}\right)^{n}\left(\frac{s}{-2t}\right)\right] \\ &= \left(\frac{s}{-2t}\right)^{2n-2}\left(\frac{-2t}{s}\right)^{n-1}\frac{s^2}{4t}\left(\frac{2n-2}{s} - \frac{2n}{s}\right) \\ &= \left(\frac{s}{-2t}\right)^{n}\left(\frac{-2t}{s}\right)\frac{s^2}{4t}\left(\frac{-2}{s}\right) \\ &= \left(\frac{s}{-2t}\right)^{n} \\ &= x_1^n. \end{aligned}$$

Since $|x_1| < 1$ and $x_1 = 1$, i.e., $\alpha = r + s + t$, $x = (x_n) \in c$. This shows that $D_2 \cup \{r + s + t\} \subseteq \sigma_p[A(r, s, t), c]$.

Now, we assume that $\alpha \notin D_2$, i.e, $|\lambda_1| \leq 1$. We must show that $\alpha \notin \sigma_p[A(r, s, t), c]$. In this situation, we examine the following three cases.

Case 1. $|\lambda_2| < |\lambda_1| < 1$. In this case, we have $s^2 \neq 4t(r - \alpha)$ and from relation (9) we obtain that

$$\begin{aligned} x_n &= \frac{a_n(r-\alpha)^n}{t^{n-1}}x_1 - \frac{a_{n-1}(r-\alpha)^n}{t^{n-1}}x_0 \\ &= \left(\frac{r-\alpha}{t}\right)^{n-1}(r-\alpha)(-a_{n-1}x_0 + a_n x_1) \\ &= \frac{r-\alpha}{\sqrt{\Delta}(\lambda_1\lambda_2)^{n-1}}(-\lambda_1^{n-1}x_0 + \lambda_2^{n-1}x_0 + \lambda_1^{n-1}x_1 - \lambda_2^{n}x_1) \\ &= \frac{r-\alpha}{\sqrt{\Delta}}\left[\left(\frac{1}{\lambda_1^{n-1}} - \frac{1}{\lambda_2^{n-1}}\right)x_0 + \left(\frac{\lambda_1}{\lambda_2^{n-1}} - \frac{\lambda_2}{\lambda_1^{n-1}}\right)x_1\right] \\ &= \frac{r-\alpha}{\sqrt{\Delta}}\left[\frac{1}{\lambda_1^{n-1}}(x_0 - \lambda_2 x_1) + \frac{1}{\lambda_2^{n-1}}(-x_0 + \lambda_1 x_1)\right]. \end{aligned}$$

Now, if $-x_0 + \lambda_1 x_1 = 0$ and $x_0 - \lambda_2 x_1 = 0$, then we have $\lambda_1 = \lambda_2$, which is a contradiction. Otherwise, $x = (x_k) \notin c$.

Case 2. $|\lambda_2| = |\lambda_1| < 1$. In this case, we have $s^2 = 4t(r - \alpha)$ and using the formula

$$a_n = \left(\frac{2n}{-s}\right)\left[\frac{-s}{2(r-\alpha)}\right]^n \quad \text{for all } n \geq 1. \tag{14}$$

Substituting (14) into (11), we get the following

$$x_n = \frac{2(r-\alpha)}{s\lambda_1^{n-1}}\left[x_0(n-1) - nx_1\lambda_1\right].$$

If $x_0 = x_1 = 0$, then $x = \theta$, which is a contradiction. Otherwise, $x = (x_k) \notin c$ since $1/|\lambda_1| > 1$.

Case 3. $|\lambda_2| = |\lambda_1| = 1$. In this case, we have $s^2 = 4t(r-\alpha)$ and so we have $|-s/2t| = 1$. Substituting (13) into (11), we obtain the following

$$x_n = \left(\frac{-s}{2t}\right)^{n-1}\left[-(n-1)\frac{-s}{2t}x_0 + nx_1\right].$$

If $x_0 = x_1 = 0$, then $x = \theta$, which is a contradiction. Otherwise, $x = (x_k) \notin c$, which means $\alpha \notin \sigma_p[A(r,s,t),c]$. Thus $\sigma_p[A(r,s,t),c] \subseteq D_2 \cup \{r+s+t\}$. This completes the proof. □

Theorem 4. $\sigma_r[A(r,s,t),c] = \sigma_p[A^*(r,s,t),c^*] \setminus \sigma_p[A(r,s,t),c] = \emptyset$.

Proof. For $\alpha \in \sigma_p[A^*(r,s,t),c^*] \setminus \sigma_p[A(r,s,t),c]$, the operator $A(r,s,t) - \alpha I$ is one to one. Thus, $[A(r,s,t) - \alpha I]^{-1}$ exists. On the other hand, $A(r,s,t)^* - \alpha I$ is not one to one. Hence, $A(r,s,t) - \alpha I$ does not have a dense range in c by Lemma 2. From Theorem 3 and Theorem 1, we have $\sigma_r[A(r,s,t),c] = \emptyset$. This completes the proof. □

Theorem 5. *Let s be a complex number such that $\sqrt{s^2} = -s$. Then,*

$$\sigma[A(r,s,t),c] = D_1.$$

Proof. The inclusion

$$\left\{\alpha \in \mathbb{C} : 2\left|r-\alpha\right| < \left|-s + \sqrt{s^2 - 4t(r-\alpha)}\right|\right\} \subseteq \sigma[A(r,s,t),c]$$

holds by Theorem 3. Since the spectrum of any bounded operator is closed [24] , we have

$$\left\{\alpha \in \mathbb{C} : 2\left|r-\alpha\right| \leq \left|-s + \sqrt{s^2 - 4t(r-\alpha)}\right|\right\} \subseteq \sigma[A(r,s,t),c]. \tag{15}$$

Again, Theorems 2-4 give that

$$\sigma[A(r,s,t),c] \subseteq \left\{\alpha \in \mathbb{C} : 2\left|r-\alpha\right| \leq \left|-s + \sqrt{s^2 - 4t(r-\alpha)}\right|\right\}. \tag{16}$$

By combining (15) and (16), one can observe that $\sigma[A(r,s,t),c] = D_1$, as desired. □

Theorem 6. $\sigma_c[A(r,s,t),c] = D_3 \setminus \{r+s+t\}$, *where*

$$D_3 = \left\{\alpha \in \mathbb{C} : 2\left|r-\alpha\right| = \left|-s + \sqrt{s^2 - 4t(r-\alpha)}\right|\right\}.$$

Proof. Since the sets $\sigma_c[A(r,s,t),c]$, $\sigma_r[A(r,s,t),c]$ and $\sigma_p[A(r,s,t),c]$ are pairwise disjoint and their union is $\sigma[A(r,s,t),c]$. Thus the proof is immediate from Theorems 3-5. □

Theorem 7. *If $|t| > |-s|$, $r \in \sigma[A(r,s,t),c]A_3$.*

Proof. From Theorem 3, $r \in \sigma_p[A(r,s,t),c]$. Thus, $[A(r,s,t)-\alpha I]^{-1}$ does not exist. It is sufficient to show that $A(r,s,t)-Ir$ is onto and for given $y=(y_k)\in c$, we have to find $x=(x_k)\in c$ such that $[A(r,s,t)-Ir]x=y$. Solving this equation, we get

$$x_k = \frac{1}{t}\sum_{i=0}^{k-2}\left(\frac{-s}{t}\right)^{k-2-i} y_i + \left(\frac{-s}{t}\right)^{k} x_1 \tag{17}$$

for $k \geq 2$. By (17), x_k satisfies

$$x_k = \frac{-s}{t}x_{k-1} + \frac{y_{k-2}}{t}$$

for $k \geq 2$. Since$|\frac{-s}{t}| < 1$ and $(y_{k-2}/t) \in c$, by Lemma 2.1 [3], $x=(x_k)\in c$. The operator $A(r,s,t)-Ir$ is onto. Hence, $r \in \sigma[A(r,s,t),c]A_3$. □

Theorem 8. *The following statements hold:*

(i) $\sigma_{ap}[A(r,s,t),c] = D_1$,

(ii) $\sigma_\delta[A(r,s,t),c] = D_1\backslash\{r\}$,

(iii) $\sigma_{co}[A(r,s,t),c] = \{r+s+t\}$.

Proof. (i) From Table 1, $\sigma_{ap}[A(r,s,t),c] = \sigma\,[A(r,s,t),c]\,\backslash\sigma\,[A(r,s,t),c]\,C_1$. By Theorem 4, we get $\sigma_r[A(r,s,t),c] = \sigma\,[A(r,s,t),c]\,C_1\cup\sigma\,[A(r,s,t),c]\,C_2=\emptyset$. Again by Table 1, one can see that $\sigma\,[A(r,s,t),c]\,C_1 = \sigma\,[A(r,s,t),c]\,C_2 = \emptyset$. Hence, $\sigma_{ap}[A(r,s,t),c] = D_1$.

(ii) The following equality

$$\sigma_\delta[A(r,s,t),c] = \sigma[A(r,s,t),c]\backslash\sigma\,[A(r,s,t),c]\,A_3$$

can be deduced from Table 1. By using Theorems 5 and 7 we conclude that $\sigma_\delta[A(r,s,t),c] = D_1\backslash\{r\}$.

(iii) From Table 1, we have

$$\sigma_{co}[A(r,s,t),c] = \sigma\,[A(r,s,t),c]\,C_1\cup\sigma\,[A(r,s,t),c]\,C_2\cup\sigma\,[A(r,s,t),c]\,C_3.$$

Thus, $\sigma_{co}[A(r,s,t),c] = \{r+s+t\}$ by Theorems 1 and 4. □

4. Fine spectra of triangular triple-band matrices over the space of ℓ_p, $(0<p<1)$

In this section, we determine the fine spectra of a lower triangular triple-band matrix over the sequence spaces ℓ_p, where $(0<p<1)$.

A lower triple-band infinite matrix is of the following form:

$$B(r,s,t) = \begin{bmatrix} r & 0 & 0 & 0 & \dots \\ s & r & 0 & 0 & \dots \\ t & s & r & 0 & \dots \\ 0 & t & s & r & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Let us begin with a theorem concerning the bounded linearity of the operator $B(r,s,t)$ acting on the sequence space ℓ_p, $(0<p<1)$.

Theorem 9. *The operator $B(r,s,t):\ell_p\to\ell_p$ is a bounded linear operator and*

$$\|B(r,s,t)\|_{(\ell_p:\ell_p)}=|r|^p+|s|^p+|t|^p. \tag{18}$$

Proof. The linearity of the operator $B(r,s,t)$ is clear. Now, we prove that (18) holds on the space ℓ_p. Let us take $e^{(0)}=(1,0,0,\cdots)\in\ell_p$. Then $B(r,s,t)e^{(0)}=(r,s,t,0,\ldots)$ and observe that

$$\|B(r,s,t)\|_{(\ell_p:\ell_p)}\geq\frac{\|B(r,s,t)e^{(0)}\|_p}{\|e^{(0)}\|_p}=|r|^p+|s|^p+|t|^p,$$

which gives the following

$$\|B(r,s,t)\|_{(\ell_p:\ell_p)}\geq|r|^p+|s|^p+|t|^p. \tag{19}$$

Let $x=(x_k)\in\ell_p$, where $0<p<1$. Then, by using the triangle inequality and taking $x_{-1}=0$, we have

$$\begin{aligned}\|B(r,s,t)x\|_p&=\sum_{k=0}^{\infty}|rx_k+sx_{k-1}+tx_{k-2}|^p\\&\leq\sum_{k=0}^{\infty}|rx_k|^p+\sum_{k=0}^{\infty}|sx_{k-1}|^p+\sum_{k=0}^{\infty}|tx_{k-2}|^p\\&=|r|^p\sum_{k=0}^{\infty}|x_k|^p+|s|^p\sum_{k=0}^{\infty}|x_{k-1}|^p+|t|^p\sum_{k=0}^{\infty}|x_{k-2}|^p\\&=|s|^p\|x\|_p+|r|^p\|x\|_p+|t|^p\|x\|_p\\&=(|r|^p+|s|^p+|t|^p)\|x\|_p,\end{aligned}$$

which gives

$$\|B(r,s,t)\|_{(\ell_p:\ell_p)}\leq|r|^p+|s|^p+|t|^p. \tag{20}$$

Therefore, by combining inequalities (19) and (20) we complete the proof. □

If $T:\ell_p\longrightarrow\ell_p$ is a bounded matrix operator with the matrix A, then it is known that the adjoint operator $T^*:\ell_p^*\longrightarrow\ell_p^*$ is defined by the transpose of the matrix A and the dual space ℓ_p^* of ℓ_p is isomorphic to ℓ_∞, where $0<p<1$.

Theorem 10. $\sigma[B(r,s,t),\ell_p]=D_1$.

Proof. It is known from Cartlidge [9] that if a matrix operator A is bounded on c, then $\sigma(A,c)=\sigma(A,\ell_\infty)$. So we have $\sigma[A(r,s,t),\ell_\infty]=D_1$. By Proposition 1, $\sigma[B(r,s,t),\ell_p]=\sigma[B^*(r,s,t),\ell_p^*]=\sigma[A(r,s,t),\ell_\infty]=D_1$, which completes the proof. □

Theorem 11. $\sigma_p[B(r,s,t),\ell_p]=\emptyset$.

Proof. Consider $B(r,s,t)x = \alpha x$ with $x \neq \theta = (0,0,0,\ldots)$ in ℓ_p. Then, by solving the system of linear equations

$$\begin{aligned} rx_0 &= \alpha x_0 \\ sx_0 + rx_1 &= \alpha x_1 \\ tx_0 + sx_1 + rx_2 &= \alpha x_2 \\ tx_1 + sx_2 + rx_3 &= \alpha x_3 \\ &\vdots \end{aligned}$$

Let x_k be the first nonzero entry of x. Then the system of equations reduces to

$$\begin{aligned} rx_k &= \alpha x_k \\ sx_k + rx_{k+1} &= \alpha x_{k+1} \\ tx_k + sx_{k+1} + rx_{k+2} &= \alpha x_{k+2} \\ tx_{k+1} + sx_{k+2} + rx_{k+3} &= \alpha x_{k+3} \\ &\vdots \end{aligned}$$

From the first equation we get $\alpha = r$ and using the other equations in the given order we get $s,t = 0$, which contradicts the fact that $s,t \neq 0$. □

Theorem 12. $\sigma_p[B^*(r,s,t), \ell_p^*] = \sigma_p[A(r,s,t), \ell_\infty] = D_1$.

Proof. Assume that $\alpha \in D_1$. By using the methodology used in the proof of Theorem 3, and it is easy to see that $\alpha \in D_1$ implies $|x_1| \leq 1$, we can see that $(x_n) = (x_1)^n$, as in equation (12). Thus $x_n \in \ell_\infty$. Moreover, assume that $\alpha \notin D_2$, which implies $|\lambda| < 1$. Using the same reasoning given in Case 1 and Case 2 in the proof of Theorem 3, $x_n \notin \ell_\infty$. Therefore $\sigma_p[A(r,s,t), \ell_\infty] = D_1$. □

Theorem 13. $\sigma_r[B(r,s,t), \ell_p] = D_1$.

Proof. We show that the operator $B(r,s,t) - \alpha I$ has an inverse and $\overline{R[B(r,s,t) - \alpha I]} \neq \ell_p$ for α satisfying $2\big|r-\alpha\big| \leq \big|-s+\sqrt{s^2-4t(r-\alpha)}\big|$. For $\alpha \neq r$, $B(r,s,t) - \alpha I$ is a triangle so it has an inverse. For $\alpha = r$, the operator $B(r,s,t) - \alpha I$ is one to one by Theorem 11. So it has an inverse. By Theorem 12, the operator $[B(r,s,t) - \alpha I)]^* = B(r,s,t)^* - \alpha I$ is not one to one for $\alpha \in \mathbb{C}$ such that $2\big|r-\alpha\big| \leq \big|-s+\sqrt{s^2-4t(r-\alpha)}\big|$. Hence the range of the operator $B(r,s,t) - \alpha I$ is not dense in ℓ_p by Lemma 2. So, $\sigma_r[B(r,s,t), \ell_p] = D_1$. □

Theorem 14. $\sigma_c[B(r,s,t), \ell_p] = \emptyset$.

Proof. Since the parts $\sigma_c[B(r,s,t), \ell_p]$, $\sigma_r[B(r,s,t), \ell_p]$ and $\sigma_p[B(r,s,t), \ell_p]$ are pairwise disjoint and their union is $\sigma[B(r,s,t), \ell_p]$, the proof is immediate, from Theorems 10, 11 and 13. □

Theorem 15. *If* $|t| > |-s|$, $r \in \sigma[B(r,s,t), \ell_p]C_1$.

Proof. From Theorem 13, $r \in \sigma_r[B(r,s,t),\ell_p]$. It is sufficient to show that the operator $[B(r,s,t)-Ir]^{-1}$ is continuous. By Lemma 3, it is enough to show that $[B(r,s,t)-Ir]^*$ is onto and for given $y=(y_k)\in \ell_p^*=\ell_\infty$, we have to find $x=(x_k)\in \ell_\infty$ such that $[B(r,s,t)-Ir]^*x=y$. Solving the system of linear equations

$$\begin{aligned} sx_1+tx_2 &= y_0\\ sx_2+tx_3 &= y_1\\ sx_3+tx_4 &= y_2\\ &\vdots\\ sx_k+tx_{k+1} &= y_{k-1}\\ &\vdots \end{aligned}$$

one can easily observe that

$$x_k=\frac{1}{t}\sum_{i=0}^{k-2}\left(\frac{-s}{t}\right)^{k-2-i}y_i+\left(\frac{-s}{t}\right)^k x_1.$$

We can easily see that $x=(x_k)\in\ell_\infty$ since $|t|>|-s|$. This shows that $[B(r,s,t)-Ir]^*$ is onto. Hence, $r\in\sigma[B(r,s,t),\ell_p]C_1$. □

Theorem 16. *The following statements hold:*

(i) $\sigma_{ap}[B(r,s,t),\ell_p]=D_1\backslash\{r\}$ *for* $|t|>|-s|$,

(ii) $\sigma_\delta[B(r,s,t),\ell_p]=D_1$,

(iii) $\sigma_{co}[B(r,s,t),\ell_p]=D_1$.

Proof. (i) From Table 1, we get $\sigma_{ap}[B(r,s,t),\ell_p]=\sigma\left[B(r,s,t),\ell_p\right]\backslash\sigma\left[B(r,s,t),\ell_p\right]C_1$. By Theorem 15, one can obtain $\sigma\left[B(r,s,t),\ell_p\right]C_1=\{r\}$. Hence, $\sigma_{ap}[B(r,s,t),\ell_p]=D_1\backslash\{r\}$.

(ii) The following equality

$$\sigma_\delta[B(r,s,t),\ell_p]=\sigma[B(r,s,t),\ell_p]\backslash\sigma\left[B(r,s,t),\ell_p\right]A_3$$

is obtained from Table 1. By Theorems 10 and 11 $\sigma_\delta[B(r,s,t),\ell_p]=D_1$.

(iii) From Table 1, we have

$$\sigma_{co}[B(r,s,t),\ell_p]=\sigma\left[B(r,s,t),\ell_p\right]C_1\cup\sigma\left[B(r,s,t),\ell_p\right]C_2\cup\sigma\left[B(r,s,t),\ell_p\right]C_3=\sigma_p[B^*(r,s,t),\ell_p^*].$$

By Theorem 12, it is immediate that $\sigma_{co}[B(r,s,t),\ell_p]=D_1$. □

5. Graphical representation

In this section, we give the graphical representations of the spectrum of the triangular triple-band matrix over the sequence space c.

If we choose $r = t = 1$, $s = -2$, we get

$$\sigma[A(1,-2,1),c] = \left\{\alpha \in \mathbb{C} : \left|1-\sqrt{\alpha}\right| \le 1\right\}.$$

Then, in polar coordinates, the boundary of $\sigma[A(1,-2,1),c]$ is as follows:

Let $\alpha = \rho e^{i\theta}$. Then,

$$\begin{aligned} 1 &= \left|1-\sqrt{\rho}e^{\frac{i\theta}{2}}\right| \\ &= \left[1-\sqrt{\rho}\cos\left(\frac{\theta}{2}\right)\right]^2 + \left[-\sqrt{\rho}\sin\left(\frac{\theta}{2}\right)\right]^2 \\ &= -2\sqrt{\rho}\cos\left(\frac{\theta}{2}\right) + \rho + 1 \\ \rho &= 4\cos^2\left(\frac{\theta}{2}\right), \quad -\pi < \theta < \pi. \end{aligned}$$

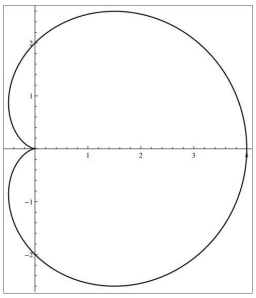

Figure 1: *The continuous spectrum of the operator $A(r,s,t)$ over the space c*

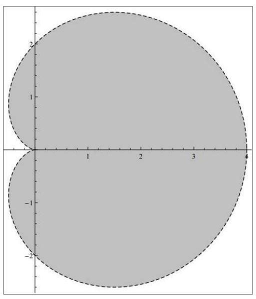

Figure 2: *The point spectrum of the operator $A(r,s,t)$ over the space c*

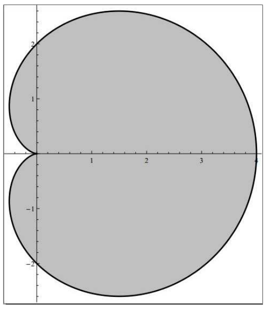

Figure 3: *The spectrum of the operator $B(r, s, t)$ and $A(r, s, t)$ over the spaces ℓ_p and c*

Conclusion

In this paper, we determine the spectrum, the continuous spectrum, thepoint spectrum and the residual spectrum of the operator triple-band matrix over the sequence spaces ℓ_p and c and give their graphical representations. We also obtain a new type of subspectral classes.

Finally, we should note that in the case $t = 0$, the operator $A(r, s, t)$ defined by an upper triangular triple-band matrix reduces to the operator $U(r, s)$ defined by an upper triangular double band matrix, and in the case $r = 1$, $s = -1$ and $t = 0$, the operator $A(r, s, t)$ is reduced to $\triangle^+$ defined by an upper triangular difference matrix. Our results are more general and more comprehensive than the corresponding results obtained by Karakaya and Altun [23], and Dündar and Başar [10], respectively.

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