Induced representations of Hilbert modules over locally C*-algebras and the imprimitivity theorem

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Abstract. We study induced representations of Hilbert modules over locally C*-algebras and their non-degeneracy. We show that if \( V \) and \( W \) are Morita equivalent Hilbert modules over locally C*-algebras \( A \) and \( B \), respectively, then there exists a bijective correspondence between equivalence classes of non-degenerate representations of \( V \) and \( W \).

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1. Introduction

Morita equivalence and induced representations of C*-algebras were first introduced by Rieffel [16, 17]. Two C*-algebras \( A \) and \( B \) are Morita equivalent if there exists a full Hilbert \( A \)-module \( E \) such that \( B \) is isomorphic to the C*-algebra \( K_A(E) \) of all compact operators on \( E \). Some properties of C*-algebras that are preserved under Morita equivalence were investigated in [2, 4, 15, 21]. Indeed, Rieffel defined induced representations of C*-algebras, that are now known as Rieffel induced representations, by using tensor products of Hilbert modules and established an equivalence between the categories of non-degenerate representations of Morita equivalent C*-algebras. Joita [10, 11] defined the notions of Morita equivalence and induced representations in the category of locally C*-algebras. Joita and Moslehian [12] have recently introduced a notion of Morita equivalence in the category of Hilbert C*-modules considered to obtain induced representations of Hilbert modules over locally C*-algebras. This enables us to prove the imprimitivity theorem for induced representations of Hilbert modules over locally C*-algebras.

Let us quickly recall the definition of locally C*-algebras and Hilbert modules over them. A locally C*-algebra is a complete Hausdorff complex topological \( \ast \)-algebra \( A \) whose topology is determined by its continuous C*-seminorms in the sense that the net \( \{a_i\}_{i \in I} \) converges to 0 if and only if the net \( \{p(a_i)\}_{i \in I} \) converges to 0 for every continuous C*-seminorm \( p \) on \( A \). Such algebras appear in the study of

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certain aspects of C*-algebras such as tangent algebras of C*-algebras, a domain of closed \( \ast \)-derivations on C*-algebras, multipliers of Pedersen’s ideal, noncommutative analogues of classical Lie groups, and K-theory. These algebras were first introduced by Inoue [6] as a generalization of C*-algebras and studied more in [5, 14] with different names. A (right) pre-Hilbert module over a locally C*-algebra \( A \) is a right \( A \)-module \( E \) compatible with the complex algebra structure and equipped with an \( A \)-valued inner product \( \langle \cdot, \cdot \rangle : E \times E \to A \), \( (x, y) \mapsto \langle x, y \rangle \), which is \( A \)-linear in the second variable \( y \) and has the properties:

\[
\langle x, y \rangle = \langle y, x \rangle^*, \quad \text{and} \quad \langle x, x \rangle \geq 0 \quad \text{with equality if and only if} \quad x = 0.
\]

A pre-Hilbert \( A \)-module \( E \) is a Hilbert \( A \)-module if \( E \) is complete with respect to the topology determined by the family of seminorms \( \{ p_E \}_{p \in S(A)} \), where \( p_E(\xi) = \sqrt{p(\langle \xi, \xi \rangle)} \), \( \xi \in E \). Hilbert modules over locally C*-algebras have been studied systematically in the book [8] and the papers [7, 14, 20].

Joita and Moslehian [12], and Skeide [18] defined Morita equivalence for Hilbert C*-modules with two different methods. In the recent sense of Joita and Moslehian, two Hilbert modules \( V \) and \( W \) over C*-algebras \( A \) and \( B \), respectively, are called Morita equivalent if \( K_A(V) \) and \( K_B(W) \) are strong Morita equivalent as C*-algebras. We consider this definition, which is weaker than Skeide’s definition and also fitted to our paper.

In this paper, we first present some definitions and basic facts about locally C*-algebras and Hilbert modules over them. In [19], Skeide proved that if \( E \) is a Hilbert module over a C*-algebra \( A \), then every representation of \( A \) induces a representation of \( E \). We use this fact to reformulate the induced representations of Hilbert C*-modules and some of their properties which have been studied in [1]. These enable us to obtain the notion of induced representations of Hilbert modules over locally C*-algebras. We finally define the concept of Morita equivalence for Hilbert modules over locally C*-algebras. We prove that two full Hilbert modules over locally C*-algebras are Morita equivalent if and only if their underlying locally C*-algebras are strong Morita equivalent and then we give a module version of the imprimitivity theorem. Indeed, we show that for Morita equivalent Hilbert modules \( V \) and \( W \) over locally C*-algebras \( A \) and \( B \), respectively, there is a bijective correspondence between equivalence classes of non-degenerate representations of \( V \) and \( W \).

2. Preliminaries

Let \( A \) be a locally C*-algebra, \( S(A) \) the set of all continuous C*-seminorms on \( A \) and \( p \in S(A) \). We set \( N_p = \{ a \in A : p(a) = 0 \} \), then \( A_p = A/N_p \) is a C*-algebra in the norm induced by \( p \). For \( p, q \in S(A) \) with \( p \geq q \), the surjective morphisms \( \pi_{pq} : A_p \to A_q \) defined by \( \pi_{pq}(a + N_p) = a + N_q \) induce the inverse system \( \{ A_p; \pi_{pq} \}_{p, q \in S(A), p \geq q} \) of C*-algebras and \( A = \varprojlim_{\leftarrow, p} A_p \), i.e., the locally C*-algebra \( A \) can be identified with \( \varprojlim_{\leftarrow, p} A_p \). The canonical map from \( A \) onto \( A_p \) is denoted by \( \pi_p \) and \( a_p \) is reserved to denote \( a + N_p \). A morphism of locally C*-algebras is a continuous morphism of \( \ast \)-algebras. An isomorphism of locally C*-algebras is
a morphism of locally C*-algebras which possesses an inverse morphism of locally C*-algebras.

A representation of a locally C*-algebra $A$ is a continuous *-morphism $\varphi : A \to B(H)$, where $B(H)$ is the $C^*$-algebra of all bounded linear maps on a Hilbert space $H$. If $(\varphi, H)$ is a representation of $A$, then there is $p \in S(A)$ such that $\|\varphi(a)\| \leq p(a)$, for all $a \in A$. The representation $(\varphi_p, H)$ of $A_p$, where $\varphi_p \circ \pi_p = \varphi$ is called a representation of $A_p$, associated to $(\varphi, H)$. We refer to [5, 11] for basic facts and definitions about the representation of locally C*-algebras.

Suppose $E$ is a Hilbert $A$-module and $(E, E)$ is the closure of linear span of $\{ \langle x, y \rangle : x, y \in E \}$. The Hilbert $A$-module $E$ is called full if $(E, E) = A$. One can always consider any Hilbert $A$-module as a full Hilbert module over locally C*-algebra $\langle E, E \rangle$. For each $p \in S(A)$, $N_p^E = \{ \xi \in E : \bar{p}_E(\xi) = 0 \}$ is a closed submodule of $E$ and $E_p = E/N_p^E$ is a Hilbert $A_p$-module with the action $(\xi + N_p^E)\pi_p(a) = \xi a + N_p^E$ and the inner product $(\xi + N_p^E, \eta + N_p^E) = \pi_p((\xi, \eta))$. The canonical map from $E$ onto $E_p$ is denoted by $\sigma_p^E$ and $\xi_p$ is reserved to denote $\sigma_p^E(\xi)$. For $p, q \in S(A)$ with $p \geq q$, the surjective morphisms $\sigma_{pq}^E : E_p \to E_q$ defined by $\sigma_{pq}^E(\sigma_p^E(\xi)) = \sigma_q^E(\xi)$ induce the inverse system $\{E_p; \ A_p; \ \sigma_{pq}^E; \ \pi_{pq}\}_{p, q \in S(A), p \geq q}$ of Hilbert C*-modules in the following sense:

- $\sigma_{pq}^E(\xi_p a_p) = \sigma_{pq}^E(\xi_p) \pi_{pq}(a_p)$, $\xi_p \in E_p$, $a_p \in A_p$, $p, q \in S(A)$, $p \geq q$,
- $(\sigma_{pq}^E(\xi_p), \sigma_{pq}^E(\eta_p)) = \pi_{pq}(\xi_p, \eta_p)$, $\xi_p, \eta_p \in E_p$, $p, q \in S(A)$, $p \geq q$,
- $\sigma_{pq}^E \circ \sigma_{pr}^E = \sigma_{pr}^E$ if $p, q, r \in S(A)$ and $p \geq q \geq r$,
- $\sigma_{pq}^E(\xi_p) = \xi_p$, $\xi \in E$, $p \in S(A)$.

In this case, $\varprojlim_{E_p}$ is a Hilbert $A$-module which can be identified with $E$. Let $E$ and $F$ be Hilbert $A$-modules and $T : E \to F$ an $A$-module map. The module map $T$ is called bounded if for each $p \in S(A)$ there is $k_p > 0$ such that $\bar{p}_F(Tx) \leq k_p \bar{p}_E(x)$ for all $x \in E$. The module map $T$ is called adjointable if there exists an $A$-module map $T^* : F \to E$ with the property $(Tx, y) = (x, T^*y)$ for all $x \in E, y \in F$. It is well-known that every adjointable map is bounded. The set $L_A(E, F)$ of all bounded adjointable $A$-module maps from $E$ into $F$ becomes a locally convex space with the topology defined by the family of seminorms $\{\bar{p}(T) = \|(\pi_p)_*\xi\|_{L_A_p(E_p, F_p)} : (\pi_p)_* : L_A(E, F) \to L_A_p(E_p, F_p)\}$ for all $p \in S(A)$ and $p \geq q$ defined by $(\pi_p)_*(T)\xi + N_p^F = T\xi + N_p^E$ for all $T \in L_A(E, F)$, $\xi \in E$. For $p, q \in S(A)$ with $p \geq q$, the morphisms $(\pi_{pq})_* : L_{A_p}(E_p, F_p) \to L_{A_q}(E_q, F_q)$ defined by $(\pi_{pq})_*(T_p(\sigma_{pq}^E(\xi))) = \sigma_{pq}^F(T_p(\sigma_{pq}^E(\xi)))$ induce the inverse system

$\{L_{A_p}(E_p, F_p); (\pi_{pq})_*\}_{p, q \in S(A), p \geq q}$

of Banach spaces such that $\varprojlim_{A_p} L_{A_p}(E_p, F_p)$ can be identified to $L_A(E, F)$. In particular, topologizing, $L_A(E, E)$ becomes a locally C*-algebra which is abbreviated by $L_A(E)$. The set of all compact operators $K_A(E)$ on $E$ is defined as the closed linear subspace of $L_A(E)$ spanned by $\{\theta_{x,y} : \theta_{x,y}(\xi) = x(y, \xi)\}$ for all $x, y, \xi \in E$. This is a locally C*-subalgebra and a two-sided ideal of $L_A(E)$; moreover, $K_A(E)$ can be identified to $\varprojlim_{A_p} K_A_p(E_p)$. Induced representations of Hilbert modules. 87
Let $V$ and $W$ be Hilbert modules over locally $C^*$-algebras $A$ and $B$, respectively, and $\Psi : A \to \mathcal{L}_B(W)$ a continuous $*$-morphism. We can regard $W$ as a left $A$-module by $(a, y) \mapsto \Psi(a)y$, $a \in A$, $y \in W$. The right $B$-module $V \otimes_A W$ is a pre-Hilbert module with the inner product given by $\langle x \otimes y, z \otimes t \rangle = \langle y, \Psi(\langle x, z \rangle)t \rangle$. We denote by $V \otimes_W W$ the completion of $V \otimes_A W$, cf. [9] for more detailed information.

3. Induced representations of Hilbert modules

In this section, we first study induced representations of Hilbert $C^*$-modules and then we reformulate them in the context of Hilbert modules over locally $C^*$-algebras. Let $H$ and $K$ be Hilbert spaces. Then the space $B(H, K)$ of all bounded operators from $H$ into $K$ can be considered as a Hilbert $B(H)$-module with the module action $(T, S) \mapsto TS$, $T \in B(H, K)$ and $S \in B(H)$ and the inner product defined by $\langle T, S \rangle = T^*S$, $T, S \in B(H, K)$. Murphy [13] showed that any Hilbert $C^*$-module can be represented as a submodule of the concrete Hilbert module $B(H, K)$ for some Hilbert spaces $H$ and $K$. This allows us to extend the notion of a representation from the context of $C^*$-algebras to the context of Hilbert $C^*$-modules. Let $V$ and $W$ be two Hilbert modules over locally $C^*$-algebras $A$ and $B$, respectively, and $\varphi : A \to B$ be a morphism of $C^*$-algebras. A map $\Phi : V \to W$ is said to a $\varphi$-morphism if $\langle \Phi(x), \Phi(y) \rangle = \varphi(\langle x, y \rangle)$ for all $x, y \in V$. A $\varphi$-morphism $\Phi : V \to B(H, K)$, where $\varphi : A \to B(H)$ is a representation of $A$ is called a representation of $V$. When $\Phi$ is a representation of $V$, we assume that an associated representation of $A$ is denoted by the same lowercase letter $\varphi$, so we will not explicitly mention $\varphi$. Let $\Phi : V \to B(H, K)$ be a representation of a Hilbert $A$-module $V$. We say $\Phi$ is a non-degenerate representation if $\overline{\Phi(V)}(H) = K$ and $\overline{\Phi(V)^*}(K) = H$. Two representations $\Phi_1 : V \to B(H_1, K_1)$ of $V$, $i = 1, 2$ are said to be unitarily equivalent if there are unitary operators $U_1 : H_1 \to H_2$ and $U_2 : K_1 \to K_2$, such that $U_2\Phi_1(v) = \Phi_2(v)U_1$ for all $v \in V$. Representations of Hilbert modules have been investigated in [1, 3, 19].

Lemma 1. Let $V$ be a full Hilbert $A$-module and $\Phi_1 : V \to B(H_1, K_1)$ and $\Phi_2 : V \to B(H_2, K_2)$ two non-degenerate representations of $V$. If $\Phi_1$ and $\Phi_2$ are unitarily equivalent, then $\varphi_1$ and $\varphi_2$ are unitarily equivalent.

Proof. Let $U_1 : H_1 \to H_2$ and $U_2 : K_1 \to K_2$ be unitary operators and $U_2\Phi_1(x) = \Phi_2(x)U_1$ for all $x \in V$. Then we have

$$U_1\varphi_1(\langle x, y \rangle)h = U_1\Phi_1(x)^*\Phi_1(y)h = \Phi_2(x)^*\Phi_2(y)U_1h = \varphi_2(\langle x, y \rangle)U_1h,$$

for every $x, y \in V$ and $h \in H_1$. Since $V$ is full, we conclude that $U_1\varphi_1(a)h = \varphi_2(a)U_1h$ for every $a \in A$ and $h \in H_1$, and consequently, $\varphi_1$ and $\varphi_2$ are unitarily equivalent.

Skeide [19] recovered the result of Murphy by embedding every Hilbert $A$-module $E$ into a matrix $C^*$-algebra as a lower submodule. He proved that every representation of $B$ induces a representation of $E$. We describe his induced representation as follows.
Construction 1. Let $B$ be a $C^*$-algebra and $E$ a Hilbert $B$-module and $\varphi : B \to B(H)$ a $*$-representation of $B$. Define a sesquilinear form $(\langle \cdot \rangle, \langle \cdot \rangle)$ on the vector space $E \otimes_{alg} H$ by $\langle x \otimes h, y \otimes k \rangle = \langle h, \varphi((x,y))k \rangle_H$, where $\langle \cdot, \cdot \rangle_H$ denotes the inner product on the Hilbert space $H$. By [19, Proposition 3.8], the sesquilinear form is positive and $E \otimes_{alg} H$ is a semi-Hilbert space. Then $(E \otimes_{alg} H)/N_\varphi$ is a pre-Hilbert space with the inner product defined by

$$\langle x \otimes h + N_\varphi, y \otimes k + N_\varphi \rangle = \langle x \otimes h, y \otimes k \rangle,$$

where $N_\varphi$ is the vector subspace of $E \otimes_{alg} H$ generated by $\{x \otimes h \in E \otimes_{alg} H : \langle x \otimes h, x \otimes h \rangle = 0\}$. The completion of $(E \otimes_{alg} H)/N_\varphi$ with respect to the above inner product is denoted by $E \otimes H$. We identify the elements $x \otimes h$ with the equivalence classes $x \otimes h + N_\varphi \in E \otimes H$. Suppose $x \in E$ and $L_x h = x \otimes h$ then $\|L_x h\|^2 = \langle h, \varphi((x,x))h \rangle \leq \|h\|^2 \|x\|^2$, i.e. $L_x \in B(H,E)$. We define $\eta_\varphi : E \to B(H,E)$ by $\eta_\varphi(x) = L_x$. Then for $x, x' \in E$, $h, h' \in H$ and $b \in B$ we have $\langle \eta_\varphi(x), \eta_\varphi(x') \rangle = \varphi((x,x'))$ and $\eta_\varphi(bx) = \eta_\varphi(x) \varphi(b)$, and so $\eta_\varphi$ is a representation of $E$.

Lemma 2. Let $\varphi_1 : B \to B(H_1)$ and $\varphi_2 : B \to B(H_2)$ be two non-degenerate representations of $B$. If $\varphi_1$ and $\varphi_2$ are unitarily equivalent, then $\eta_{\varphi_1}$ and $\eta_{\varphi_2}$ are unitarily equivalent.

**Proof.** Suppose $U : H_1 \to H_2$ is a unitary operator such that $U \varphi_1(b) = \varphi_2(b)U$ for all $b \in B$. Then $id_E \otimes U : E \otimes_{alg} H_1 \to E \otimes_{alg} H_2$ given by $x \otimes h_1 \mapsto x \otimes h_2$ can be extended to a unitary operator $V$ from $E H_1$ onto $E H_2$ and $V \eta_{\varphi_1}(x) = \eta_{\varphi_2}(x)U$ for all $x \in E$. Hence, $\eta_{\varphi_1}$ and $\eta_{\varphi_2}$ are unitarily equivalent.

The above argument enables us to extend the Rieffel induced representations from the case of $C^*$-algebras to the context of Hilbert $C^*$-modules. For this, let $V$ and $W$ be two full Hilbert modules over $C^*$-algebras $A$ and $B$, respectively. Let $E$ be a Hilbert $B$-module and $A$ acts as adjointable operators on the Hilbert $C^*$-module $E$, and $\Phi : W \to B(H,K)$ is a non-degenerate representation of $W$. Using [15, Proposition 2.66], the formula $A \eta_\varphi(x \otimes h) = (a,x) \otimes h$ extends to obtain a (Rieffel induced) representation of $A$ as bounded operators on Hilbert space $E H$. In view of Construction 1, the representation $A \eta_{\varphi} : A \to B(EH)$ of the $C^*$-algebra $A$ obtains the representation $A \eta_{\varphi}^\Phi : V \to B(\Phi E, \Phi H)$ of the Hilbert $A$-module $V$. The constructed representation $A \eta_{\varphi}^\Phi$ is called the Rieffel induced representation from $W$ to $V$ via $E$ and denoted by $\eta_{\varphi}^\Phi$. The following result can be found in [1, Proposition 3.3] that we derive from Lemmas 1 and 2. Our argument seems to be shorter.

**Lemma 3.** Let $W$ be a full Hilbert $B$-module and $\Phi_1 : W \to B(H_1, K_1)$ and $\Phi_2 : W \to B(H_2, K_2)$ two non-degenerate representations of $W$. If $\Phi_1$ and $\Phi_2$ are unitarily equivalent, then $\Phi_1^\Phi$ and $\Phi_2^\Phi$ are unitarily equivalent.

**Corollary 1.** If $\Phi : W \to B(H, K)$ and $\oplus_{i \in I} \Phi_i : W \to B(\oplus_{i \in I} H_i, \oplus_{i \in I} K_i)$ are unitarily equivalent, then $\Phi_1^\Phi$ and $\oplus_{i \in I} \Phi_i^\Phi$ are unitarily equivalent.

Now, we reformulate representations of the Hilbert module from the case of $C^*$-algebras to the case of locally $C^*$-algebras. Let $V$ and $W$ be two Hilbert modules over
locally C*-algebras $A$ and $B$, respectively, and $\varphi: A \rightarrow B$ a morphism of locally C*-algebras. A map $\Phi: V \rightarrow W$ is said to be a $\varphi$-morphism if $\langle \Phi(x), \Phi(y) \rangle = \varphi(\langle x, y \rangle)$, for all $x, y \in V$. A $\varphi$-morphism $\Phi: V \rightarrow B(H, K)$, where $\varphi: A \rightarrow B(H)$ is a representation of $A$, is called a representation of $V$. We can define non-degenerate representations and unitarily equivalent representations for Hilbert modules over locally C*-algebras like a Hilbert C*-modules case.

Suppose $A$ is a locally C*-algebra, $V$ is a Hilbert $A$-module and $\varphi: A \rightarrow B(H)$ is a representation of $A$ on some Hilbert space $H$. Suppose $p \in S(A)$ and $\varphi_p$ is a representation of $A_p$ associated to $\varphi$; then there exist a Hilbert space $K$ and a representation $\Phi_p: V_p \rightarrow B(H, K)$ which is a $\varphi_p$-morphism. For details we refer to the proof of [13, Theorem 3.1]. It is easy to see that the map $\Phi: V \rightarrow B(H, K)$, $\Phi(v) = \Phi_p(\sigma^V_p(v))$ is a $\varphi$-morphism, i.e., it is a representation of $V$.

**Lemma 4.** Let $V$ be a Hilbert module over locally C*-algebra $A$ and $\Phi: V \rightarrow B(H, K)$ a representation of $V$. If $p \in S(A)$ and $\varphi_p$ is a representation of $A_p$ associated to $\varphi$, then the map $\Phi_p: V_p \rightarrow B(H, K)$, $\Phi_p(\sigma^V_p(v)) = \Phi(v)$ is a $\varphi_p$-morphism. Specifically, $\Phi_p$ is a representation of $V_p$ and $\Phi$ is non-degenerate if and only if $\Phi_p$ is. In this case, we say that $\Phi_p$ is a representation of $V_p$ associated to $\Phi$.

**Proof.** Let $v, v' \in V$ and $\pi_v(v-v') = 0$. Since $\|\varphi(a)\| \leq p(a)$ for all $a \in A$, we have $\Phi(v - v'), \Phi(v - v') - \varphi(\langle v - v', v - v' \rangle) = 0$, which shows $\Phi_p$ is well-defined. We also have

$$
\langle \Phi_p(\sigma^V_p(v)), \Phi_p(\sigma^V_p(v')) \rangle = \langle \Phi(v), \Phi(v') \rangle = \varphi(\langle v, v' \rangle) = \varphi_p(\sigma^V_p(v), \sigma^V_p(v'))
$$

Then, by definition of $\Phi_p$, the representation $\Phi$ is non-degenerate if and only if $\Phi_p$ is non-degenerate.

Let $V$ and $W$ be two full Hilbert modules over locally C*-algebras $A$ and $B$, respectively. Let $E$ be a Hilbert $B$-module, $\Psi: A \rightarrow L_B(E)$ a non-degenerate continuous $*$-morphism and $\Phi: W \rightarrow B(H, K)$ a non-degenerate representation of $W$. We construct a non-degenerate representation from $W$ to $V$ via $E$ as follows.

**Construction 2.** We define a sesquilinear form $\langle \cdot, \cdot \rangle$ on the vector space $E \otimes_{alg} H$ by $\langle x \otimes h, y \otimes k \rangle = \langle h, \varphi((x, y)k) \rangle$ and make the Hilbert space $E H$ as in Construction 1. The map $A \rightarrow B(E H)$ defined by

$$
A \rightarrow \varphi(a)(x \otimes h) = A \varphi(a)x \otimes h, \quad a \in A, \quad x \in E, \quad h \in H,
$$

is a representation of $A$. The representation $(E H, A \varphi\varphi)$ is called the Rieffel induced representation from $B$ to $A$ via $E$, cf. [11]. Since $A$ acts as an adjointable operator on Hilbert $B$-module $E$, we can construct interior tensor product $V \otimes_E E$ as a Hilbert $B$-module. Hence, we find the Hilbert spaces $E H$ and $V \otimes_E E H$. Let $v \in V$; then the map $E \otimes H \rightarrow V \otimes E H$, $(x, h) \mapsto v \otimes x \otimes h$ is a bilinear form and so there is a unique linear transformation $E \Phi(v): E \otimes_{alg} H \rightarrow V \otimes E H$ which can be extended to a bounded linear operator $E \Phi(v)$ from $E H$ to $V \otimes E H$. To see this, suppose $q \in S(B)$, $x \in E$, $h \in H$ and $\langle \varphi_q, H \rangle$ is a representation of $B_q$ associated to $(\varphi, H)$. We have
We now show that $V = \Phi(V(x \otimes h)) = (v \otimes x \otimes h, v \otimes x \otimes h)$

$$\langle h, \varphi((v \otimes x, v \otimes x))h \rangle_H
\leq \langle h, \varphi((\Psi((v, v))^{1/2}x, \Psi((v, v))^{1/2}x))h \rangle_H$$

$$\langle h, \varphi((\sigma_q(x))^{1/2}(\sigma_q(x)))h \rangle_H
\leq \langle h, \varphi_q((\sigma_q(x), \sigma_q(x)))h \rangle_H$$

$$\langle h, \varphi((x, x'))h \rangle_H
= \langle h, \varphi((x, x'))h \rangle_H$$

The following equalities hold for every $v, v' \in V, x, x' \in E$ and $h, h' \in H$

$$\langle (x \otimes h) , \; \sum_i \Phi(x \otimes h) \rangle = \langle \Phi((x \otimes h) , \; \sum_i \Phi(x \otimes h) \rangle$$

which imply $\Phi(V, V) = \Phi(V, V) = \Phi(V, V) = \Phi(V, V)$. That is, the map $\Phi : V \rightarrow B(E^*, E)$ is a $A_{\Phi}$-morphism and so it is a representation of $V$.

We now show that $\Phi(V)$ is non-degenerate. To see this, recall that $\Phi(A)(E) = E$ and $\Phi(V, V) = A$, which imply $\Phi((V, V)) = E$. Suppose $x, x' \in E$ and $h \in H$, we have

$$\| (x - x') \otimes h \|^2 = \langle h, \varphi((x - x', x - x'))h \rangle_H$$

Given $\epsilon > 0$, there exist $v_i, v_i' \in V$ and $x_i \in E$ such that $\Phi(\sum_i \Psi((v_i, v_i'))x_i - x) < \epsilon$.

In view of the above inequality, the term $\sum_i \Psi((v_i, v_i'))x_i \otimes h$ approximates $x \otimes h$ in $E^*H$. But we have

$$\sum_i \Psi((v_i, v_i'))x_i \otimes h = \sum_i \Phi((v_i, v_i'))(x_i \otimes h)$$

which implies $\Phi(V)^*(\Phi(V))(E^*H) = E^*H$. The equality $\Phi(V)(E^*H) = \Phi(V)(E^*H)$ follows from the definition of $\Phi$, i.e., $\Phi(V)$ is non-degenerate.
Definition 1. The representation $V_p^\Phi$ in Construction 2 is a called Rieffel induced representation from $W$ to $V$ via $E$.

Theorem 3. Let $V$ and $W$ be two full Hilbert modules over locally C*-algebras $A$ and $B$, respectively. Let $E$ be a Hilbert $B$-module, $\Psi : A \to L_B(E)$ a non-degenerate continuous $*$-morphism and $\Phi : W \to B(H,K)$ a non-degenerate representation. If $q \in S(B)$ and $(\varphi_q, H)$ is a non-degenerate representation of $B_q$ associated to $(\varphi, H)$, then there is $p \in S(A)$ such that $A_p$ acts non-degenerately on $E_q$ and the representations $V^\Phi_p \mid E_q$ and $V^\Psi_q \circ \sigma^V_p$ of $V$ are unitarily equivalent.

Proof. Continuity of $\Psi$ implies that there exists $p \in S(A)$ such that $q(\Psi(a)) \leq p(a)$ for each $a \in A$, which guarantees $\Psi_p : A_p \to L_B(E_q)$, $\Psi_p(\pi_p(a)) = (\pi_q)_*(\Psi(a))$ is a $*$-morphism of C*-algebras. Moreover, $\Psi_p$ is non-degenerate since

$$\Psi_p(A_p)(E_p) = \Psi_p(\pi_p(A))(\sigma^E_p(E)) = (\pi_q)_*(\Psi(A)\sigma^E_q(E))$$

$$= \sigma^E_q(\Psi(A)(E))$$

$$= \sigma^E_q(E) = E_q.$$

If $\Phi_q$ is a non-degenerate representation of $W_q$ associated to $\Phi$, then $V^\Phi_p \mid E_q : V_p \to B(E_q, H \otimes_{\varphi_q} E_q) H$ defined by $V^\Phi_p \mid E_q(\sigma^V_p(\tau))(\sigma^E_p(\pi \otimes h)) = \sigma^V_p(\tau) \otimes \sigma^E_p(\pi \otimes h)$ is a non-degenerate representation of $V_p$ which is also a $A_p^\Phi \otimes \varphi_q$-morphism. Indeed, $V^\Phi_p \mid E_q$ is the Rieffel induced representation from $W_q$ to $V_p$ via $E_q$. Hence, $V^\Phi_p \mid E_q \circ \sigma^V_p$ is a non-degenerate representation of $V$ and it is a $A_p^\Phi \otimes \varphi_q \circ \pi_p$-morphism. The representations $(A_p^\Phi \otimes \varphi_q \circ \pi_p \otimes H)$ of $A$ are unitarily equivalent by [11, proposition 3.4]. We define the linear map $U_1 : E \otimes_{alg} H \to E_q \otimes_{alg} H$, $U_1(x \otimes h) = \sigma^E_q(x) \otimes h$ which satisfies

$$\langle U_1(x \otimes h), U_1(x \otimes h) \rangle = \langle \sigma^E_q(x) \otimes h, \sigma^E_q(x) \otimes h \rangle$$

$$= \langle h, \varphi_q(\langle \sigma^E_q(x), \sigma^E_q(x) \rangle) \rangle_H$$

$$= \langle h, \varphi_q(\pi_q(x)) \rangle_H$$

$$= \langle h, \varphi_q(x) \rangle_H$$

$$= \langle x \otimes h, x \otimes h \rangle,$$

for all $x \in E$ and $h \in H$. Then $U_1$ can be extended to a bounded linear operator, which is again denoted by $U_1$ from $E$ onto $E_q H$. It is easy to see that $U_1$ is a unitary operator. We define the linear map $U_2 : V \otimes_{alg} E \otimes_{alg} H \to V_p \otimes_{alg} E_q \otimes_{alg} H$ by $U_2(v \otimes x \otimes h) = \sigma^V_p(v) \otimes \sigma^E_q(x) \otimes h$. For every $v \in V$, $x \in E$ and $h \in H$ we have

$$\langle U_2(v \otimes x \otimes h), U_2(v \otimes x \otimes h) \rangle = \langle \sigma^V_p(v) \otimes \sigma^E_q(x) \otimes h, \sigma^V_p(v) \otimes \sigma^E_q(x) \otimes h \rangle$$

$$= \langle h, \varphi_q \left( \langle \sigma^V_p(v) \otimes \sigma^E_q(x), \sigma^V_p(v) \otimes \sigma^E_q(x) \rangle \right) \rangle_H$$

$$= \langle h, \varphi_q(\langle \sigma^V_p(v), \sigma^V_p(v) \rangle) \rangle_H$$

$$= \langle h, \varphi_q(\langle \sigma^E_q(x), \pi_p(\langle v, v \rangle) \sigma^E_q(x) \rangle) \rangle_H.$$
\[ \langle U_2(v \otimes x \otimes h), U_2(v \otimes x \otimes h) \rangle = \langle h, \varphi_q \left( \left( \sigma_q^E(x), (\pi_q)_* (\Psi ((v, v))) \right) \right) \rangle_H \]

\[ = \langle h, \varphi_q \left( \left( \sigma_q^E(x), \sigma_q^E(\Psi ((v, v))) \right) \right) \rangle_H \]

\[ = \langle h, \varphi_q \left( \pi_q (\langle (x, \Psi ((v, v)x) \rangle ) \right) \rangle_H \]

\[ = \langle h, \varphi (\langle (v, \Psi ((v, v)x)h \rangle_H \]

\[ = \langle v \otimes x \otimes h, v \otimes x \otimes h \rangle, \]

and so \( U_2 \) can be extended to a bounded linear operator \( U_2 \) from \( V \odot q E \) onto \( V \odot q E \). It is easy to see that \( U_2 \) is unitary. Moreover, \( U_2 \odot q E \Phi (v) = \odot q E \Phi_1 \odot (\sigma_q^E) U_1 (v) \) for all \( v \in V \). Hence, the representations \( \odot q E \Phi \) and \( \odot q E \Phi_1 \odot (\sigma_q^E) \) are unitarily equivalent.

**Theorem 4.** Let \( \Phi_1 : W \to B(H_1, K_1) \) and \( \Phi_2 : W \to B(H_2, K_2) \) be two non-degenerate representations of \( W \). If \( \Phi_1 \) and \( \Phi_2 \) are unitarily equivalent, then \( \odot q E \Phi_1 \) and \( \odot q E \Phi_2 \) are unitarily equivalent, too.

**Proof.** Let \( q, q' \in \mathbb{S} (B) \), \( (\varphi_1, q_1, H_1) \) be a representation of \( B_q \) associated to \( \varphi_1 \) and let \( (\varphi_2, q_2, H_2) \) be a representation of \( B_{q'} \) associated to \( \varphi_2 \). Consider \( r \in \mathbb{S} (B) \) such that \( q, q' \leq r \). By Theorem 3, there exists \( p \in \mathbb{S} (A) \) such that \( A_p \) acts non-degenerately on \( E_r \) and the representation \( \odot q E \Phi_r \) is unitarily equivalent to \( \odot q E \Phi_i \odot (\sigma_q^E) \) for \( i = 1, 2 \).

Since \( \Phi_r \) and \( \Phi_i \) are unitarily equivalent representations of \( W_r \), Lemma 3 implies that the representations \( \odot q E \Phi_1 \) and \( \odot q E \Phi_2 \) are unitarily equivalent.

**Corollary 2.** If \( \Phi : W \to B(H, K) \) and \( \odot q E \Phi_i : W \to B(\oplus_{i \in I} H_i, \oplus_{i \in I} K_i) \) are unitarily equivalent, then \( \odot q E \Phi_1 \) and \( \odot q E \Phi_2 \) are unitarily equivalent.

**Proof.** Let \( q \in \mathbb{S} (B) \) and \( \Phi_q : W_q \to B(H, K) \) be a representation of \( W_q \) associated to \( \Phi \). For every \( i \in I \), define \( \Phi_{i q} : W_q \to B(H_i, K_i) \) by \( \Phi_{i q}(\sigma_q^{W_i}(w)) = \Phi_i(w) \).

If \( \sigma_q^{W_i}(w) = 0 \), then \( \Phi_q(\sigma_q^{W_i}(w)) = 0 \) and so \( \Phi(w) = 0 \). Since \( \Phi \) and \( \odot q E \Phi_i \) are unitarily equivalent, we conclude that \( \odot q E \Phi_i(w) = 0 \) and therefore, \( \Phi_i(w) = 0 \) for each \( i \in I \). It proves that \( \Phi_{i q} \) is well-defined for any \( i \in I \). It is easy to see that \( \Phi_q \) is unitarily equivalent to \( \odot q E \Phi_{i q} \). By Theorem 3, there exists \( p \in \mathbb{S} (A) \) such that \( A_p \) acts non-degenerately on \( E_q \) and the representations \( \odot q E \Phi_1 \) and \( \odot q E \Phi_{i q} \odot (\sigma_q^E) \) are unitarily equivalent. The representations \( \odot q E \Phi_1 \) and \( \odot q E \Phi_{i q} \odot (\sigma_q^E) \), \( i \in I \), are unitarily equivalent, too. On the other hand, Corollary 1 implies that the representations \( \odot q E \Phi_q \) and \( \odot q E \Phi_{i q} \) are unitarily equivalent. Consequently, the representations \( \odot q E \Phi_q \odot (\sigma_q^E) \) and \( \odot q E \Phi_{i q} \odot (\sigma_q^E) \) of \( V \) are unitarily equivalent.

4. The imprimitivity theorem for Hilbert modules

In this section, we introduce the concept of Morita equivalence between Hilbert modules over locally C*-algebras and give a module version of the imprimitivity theorem.
Let $A$ and $B$ be locally $C^*$-algebras. We say that $A$ and $B$ are strongly Morita equivalent, written $A \simeq_M B$, if there is a full Hilbert $A$ module $E$ such that locally $C^*$-algebras $B$ and $K_A(E)$ are isomorphic. Joita [10, Proposition 4.4] showed that strong Morita equivalence is an equivalence relation in the set of all locally $C^*$-algebras. The vector space $\bar{E} := K_A(E, A)$ is a full Hilbert $K_A(E)$-module with the following action and inner product

$$(T, S) \mapsto TS, \quad S \in K_A(E), \quad T \in K_A(E, A),$$

$$\langle T, S \rangle = T^*S, \quad T, S \in K_A(E, A).$$

Since locally $C^*$-algebras $B$ and $K_A(E)$ are isomorphic, $\bar{E}$ may be regarded as a Hilbert $B$-module. Moreover, the linear map $\alpha$ from $A$ to $K_B(\bar{E})$ defined by $\alpha(a)(\theta, \sigma) = \theta a \sigma x$ is an isomorphism of locally $C^*$-algebras by [10, Lemma 4.2 and Remark 4.3]. It is easy to see that for each $p \in S(A)$, the linear map $U_p : (\bar{E})_p \to \bar{E}_p$ defined by $U_p(T + N^E_p) = (\pi_p)_*(T)$ is unitary and so the Hilbert $K_A(E_p)$-modules $(\bar{E}_p)_p$ and $\bar{E}_p$ are the same.

**Definition 2.** Suppose $V$ and $W$ are Hilbert modules over locally $C^*$-algebras $A$ and $B$, respectively. The Hilbert modules $V$ and $W$ are called Morita equivalent if $K_A(V)$ and $K_B(W)$ are strong Morita equivalent as locally $C^*$-algebras. In this case, we write $V \simeq_M W$.

**Lemma 5.** Let $V$ be a full Hilbert module over locally $C^*$-algebra $A$. Then $K_A(V)$ is strong Morita equivalent to $(V, V)$.

**Proof.** The module $\bar{V} = K_A(V, A)$ is a full Hilbert $K_A(V)$-module by [10, Corollary 3.3]. Then locally $C^*$-algebras $K_K(V)(\bar{V})$ and $K_A(A)$ are isomorphic by Lemma 4.2 in [10]. Since $(V, V) = A \simeq K_A(A)$, locally $C^*$-algebras $K_A(V)$ and $(V, V)$ are strong Morita equivalent.

**Corollary 3.** Two full Hilbert modules over locally $C^*$-algebras are Morita equivalent if and only if their underlying locally $C^*$-algebras are strong Morita equivalent.

**Theorem 5.** Let $V$ and $W$ be two full Hilbert modules over locally $C^*$-algebras $A$ and $B$, respectively, such that $V \simeq_M W$. If $E$ is a Hilbert $A$-module which gives the strong Morita equivalence between $A$ and $B$, and $\Phi$ is a non-degenerate representation of $V$, then $\Phi$ is unitarily equivalent to $V^{(W)}_{E \backslash \bar{E}}(\Phi)$.

**Proof.** Let $p \in S(A)$ and $\Phi_p$ be a non-degenerate representation of $V_p$ associated to $\Phi$. Using [11, Lemma 4.1], there is $q \in S(B)$ such that $A_p \simeq_M B_q$ and $E_p$ gives the strong Morita equivalent between $A_p$ and $B_q$. The representations $\varphi_p$ and $\pi_{E_p} V^B \pi_{E_p} V^B$ of $A_p$ are unitarily equivalent by [15, Theorem 3.29]. Then the representations $\Phi_p$ and $\pi_{E_p} \pi_{E_p} (\Phi_p)$ of $V_p$ are unitarily equivalent by Lemma 2 and consequently, the representations $V^W_p (\Phi_p) \circ \sigma_p^V$ and $\Phi_p \circ \sigma_p^V = \Phi$ of $V$ are unitarily equivalent. In view of Theorems 3 and 4, we have

- the representations $E \Phi$ and $E_p \Phi_p \circ \sigma_q^W$ of $W$ are unitarily equivalent,
• the representations $V_{E}(W_{E} \Phi)$ and $V_{E}(W_{E} \Phi_{p} \circ \sigma_{q}^{W})$ of $V$ are unitarily equivalent, and

• the representations $V_{E}(W_{E} \Phi_{p} \circ \sigma_{q}^{W})$ and $V_{E}(W_{E} \Phi_{p} \circ \sigma_{q}^{W})$ of $V$ are unitarily equivalent.

The assertion now follows from the fact that $(W_{E} \Phi_{p} \circ \sigma_{q}^{W}) \approx W_{E} \Phi_{p}$. □

We now reformulate the imprimitivity theorem within the framework of Hilbert modules as follows.

**Theorem 6.** Let $V$ and $W$ be two Hilbert modules over locally $C^*$-algebras $A$ and $B$, respectively. If $V \sim_{M} W$, then there is a bijective correspondence between equivalence classes of non-degenerate representations of $V$ and $W$.

**Proof.** By replacing the underlying $C^*$-algebras $A$ and $B$, we may assume that $V$ and $W$ are full Hilbert modules over $A$ and $B$, respectively. Let $E$ be a Hilbert $A$-module which gives strong Morita equivalence between $A$ and $B$. Then, by Theorems 4 and 5, the map $\Phi \mapsto W_{E} \Phi$ from the set of all non-degenerate representations of $V$ to the set of all non-degenerate representations of $W$ induces a bijective correspondence between equivalence classes of non-degenerate representations of $V$ and $W$. □

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