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A note on the products $((m+1)^2+1)((m+2)^2+1)\dots(n^2+1)$ and $((m+1)^3+1)((m+2)^3+1)\dots(n^3+1)$

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Abstract. We prove that for any positive integer m there exists a positive real number N_m such that whenever the integer $n \geq N_m$, neither the product $P_m^n = ((m+1)^2+1)((m+2)^2+1)\dots(n^2+1)$ nor the product $Q_m^n = ((m+1)^3+1)((m+2)^3+1)\dots(n^3+1)$ is a square.

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1. Introduction

In 2008, J. Cilleruelo proved that $\prod_{k=1}^{n}(k^2+1)$ is a square only for $n = 3$ [3]. His technique was applied to the products of consecutive values of other polynomials like $4x^2+1$ and $2x^2-2x+1$ by Fang [5], and to x^3+1 by Gürel and Kişisel [7]. Later, an idea due to W. Zudilin was applied to x^p+1 by Zhang and Wang [8] and to $x^{p^k}+1$ by Chen et al. [2] for odd prime p. In the very recent article [6], the author proved that while the product $\prod_{k=1}^{n}(4k^4+1)$ becomes a square infinitely often as the integer n changes, the product $\prod_{k=1}^{n}(k^4+4)$ becomes a square only for $n = 2$ using techniques different from the articles mentioned above. The problem is still open for polynomials like $x^{2^k}+1$. In this paper, our purpose is to extend results of [3] and [7] for products of sufficiently many consecutive values of the polynomial x^2+1 and x^3+1 starting from any positive integer $x = m+1$ up to $x = n$ by using techniques similar to the ones used in the above mentioned articles. The main result of this paper is the following theorem.

Theorem 1. *For any positive integer m, there exists a positive real number N_m such that whenever the integer $n \geq N_m$, neither the product*

$$P_m^n = ((m+1)^2+1)((m+2)^2+1)\dots(n^2+1)$$

nor the product

$$Q_m^n = ((m+1)^3+1)((m+2)^3+1)\dots(n^3+1)$$

is a square.

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In order to prove the main result, we need following lemmas.

Lemma 1. *If p is a prime such that $p^2|P_m^n$ or $p^2|Q_m^n$, then $p<2n$.*

Proof. See the proof of Theorem 1 in [3] and the proof of Lemma 1 in [7]. □

Now, we can write

$$P_m^n=\prod_{p<2n}p^{\alpha_p},\ Q_m^n=\prod_{p<2n}p^{\bar{\alpha}_p}\ \text{and}\ \frac{n!}{m!}=\prod_{p\le n}p^{\beta_p}.$$

Comparing the products term by term, we can easily deduce that $(\frac{n!}{m!})^2<P_m^n$ and $(\frac{n!}{m!})^3<Q_m^n$. After taking the natural logarithms of both sides, we obtain the following inequalities under the condition that both P_m^n and Q_m^n are square-full.

$$\sum_{p\le n}\beta_p\ln p<\frac{1}{2}\sum_{p<2n}\alpha_p\ln p. \tag{1}$$

$$\sum_{p\le n}\beta_p\ln p<\frac{1}{3}\sum_{p<2n}\bar{\alpha}_p\ln p. \tag{2}$$

Lemma 2.

If $p\equiv 1 \pmod 4$ and $p\le n$, then $\frac{\alpha_p}{2}-\beta_p\le\frac{\ln(n^2+1)}{\ln p}$.

If $p\equiv 1 \pmod 3$ and $p\le n$, then $\frac{\bar{\alpha}_p}{3}-\beta_p\le\frac{\ln(n^3+1)}{\ln p}$.

If $p\equiv 2 \pmod 3$ and $p\le n$, then $\bar{\alpha}_p-\beta_p\le\frac{\ln(n^3+1)}{\ln p}$.

Proof. When $p\equiv 1$ (mod 4), the equation $x^2+1\equiv 0 \pmod{p^j}$ has two solutions in an interval of length p^j. That can be used to bound α_p as follows:

$$\alpha_p=\sum_{j\le\frac{\ln(n^2+1)}{\ln p}}\#\{m<k\le n,p^j|(k^2+1)\}\le\sum_{j\le\frac{\ln(n^2+1)}{\ln p}}2\left\lceil\frac{n}{p^j}\right\rceil-\sum_{j\le\frac{\ln(m^2+1)}{\ln p}}2\left\lfloor\frac{m}{p^j}\right\rfloor.$$

When $p\equiv 1$ (mod 3), the equation $x^3+1\equiv 0 \pmod{p^j}$ has at most three solutions in an interval of length p^j. That can be used to bound $\bar{\alpha}_p$ as follows:

$$\bar{\alpha}_p=\sum_{j\le\frac{\ln(n^3+1)}{\ln p}}\#\{m<k\le n,p^j|(k^3+1)\}\le\sum_{j\le\frac{\ln(n^3+1)}{\ln p}}3\left\lceil\frac{n}{p^j}\right\rceil-\sum_{j\le\frac{\ln(m^3+1)}{\ln p}}3\left\lfloor\frac{m}{p^j}\right\rfloor.$$

When $p\equiv 2$ (mod 3), the equation $x^3+1\equiv 0 \pmod{p^j}$ has exactly one solution in an interval of length p^j (see Lemma 2 in [7]). That can be used to bound $\bar{\alpha}_p$ as follows:

$$\bar{\alpha}_p=\sum_{j\le\frac{\ln(n^3+1)}{\ln p}}\#\{m<k\le n,p^j|(k^3+1)\}\le\sum_{j\le\frac{\ln(n^3+1)}{\ln p}}\left\lceil\frac{n}{p^j}\right\rceil-\sum_{j\le\frac{\ln(m^3+1)}{\ln p}}\left\lfloor\frac{m}{p^j}\right\rfloor.$$

We also know that

$$\beta_p = \sum_{j \le \frac{\ln(n)}{\ln p}} \#\{m < k \le n, p^j | k\} = \sum_{j \le \frac{\ln(n)}{\ln p}} \left\lfloor \frac{n}{p^j} \right\rfloor - \sum_{j \le \frac{\ln(m)}{\ln p}} \left\lfloor \frac{m}{p^j} \right\rfloor.$$

Therefore, when $p \equiv 1 \pmod 4$,

$$\begin{aligned} \frac{\alpha_p}{2} - \beta_p \le & \sum_{j \le \frac{\ln(n)}{\ln p}} \left(\left\lceil \frac{n}{p^j} \right\rceil - \left\lfloor \frac{n}{p^j} \right\rfloor \right) + \sum_{\frac{\ln(n)}{\ln p} < j \le \frac{\ln(n^2+1)}{\ln p}} \left\lceil \frac{n}{p^j} \right\rceil \\ & - \sum_{\frac{\ln(m)}{\ln p} < j \le \frac{\ln(m^2+1)}{\ln p}} \left\lfloor \frac{m}{p^j} \right\rfloor \le \frac{\ln(n^2+1)}{\ln p}. \end{aligned} \tag{3}$$

When $p \equiv 1 \pmod 3$,

$$\begin{aligned} \frac{\bar{\alpha}_p}{3} - \beta_p \le & \sum_{j \le \frac{\ln(n)}{\ln p}} \left(\left\lceil \frac{n}{p^j} \right\rceil - \left\lfloor \frac{n}{p^j} \right\rfloor \right) + \sum_{\frac{\ln(n)}{\ln p} < j \le \frac{\ln(n^3+1)}{\ln p}} \left\lceil \frac{n}{p^j} \right\rceil \\ & - \sum_{\frac{\ln(m)}{\ln p} < j \le \frac{\ln(m^3+1)}{\ln p}} \left\lfloor \frac{m}{p^j} \right\rfloor \le \frac{\ln(n^3+1)}{\ln p}. \end{aligned} \tag{4}$$

When $p \equiv 2 \pmod 3$,

$$\begin{aligned} \bar{\alpha}_p - \beta_p \le & \sum_{j \le \frac{\ln(n)}{\ln p}} \left(\left\lceil \frac{n}{p^j} \right\rceil - \left\lfloor \frac{n}{p^j} \right\rfloor \right) + \sum_{\frac{\ln(n)}{\ln p} < j \le \frac{\ln(n^3+1)}{\ln p}} \left\lceil \frac{n}{p^j} \right\rceil \\ & - \sum_{\frac{\ln(m)}{\ln p} < j \le \frac{\ln(m^3+1)}{\ln p}} \left\lfloor \frac{m}{p^j} \right\rfloor \le \frac{\ln(n^3+1)}{\ln p}. \end{aligned} \tag{5}$$

□

Using this Lemma 2, we can update inequalities (1) and (2) as follows:

$$\sum_{\substack{p \equiv 3 (mod 4) \\ p \le n}} \beta_p \ln p < (\frac{\alpha_2}{2} - \beta_2) \ln 2 + \sum_{\substack{p \equiv 1 (mod 4) \\ p \le n}} \ln(n^2+1) + \sum_{n<p<2n} \frac{\alpha_p \ln p}{2}. \tag{6}$$

$$\begin{aligned} \sum_{\substack{p \equiv 2 (mod 3) \\ p \le n}} \frac{2\beta_p \ln p}{3} < & (\frac{\bar{\alpha}_3}{3} - \beta_3) \ln 3 + \sum_{\substack{p \equiv 1 (mod 3) \\ p \le n}} \ln(n^3+1) + \sum_{\substack{p \equiv 2 (mod 3) \\ p \le n}} \frac{\ln(n^3+1)}{3} \\ & + \sum_{n<p<2n} \frac{\bar{\alpha}_p \ln p}{3}. \end{aligned} \tag{7}$$

Lemma 3. *For any prime $p \le n$, $\beta_p \ge \frac{n-m}{p-1} - \frac{\ln(n+1)}{\ln p}$.*

Proof. Using the lower and the upper bounds of the prime powers in a factorial as in Lemma 1 [1] for n and m, respectively, we can easily obtain the desired inequality. $\square$

When $p = 2$, if k is even $k^2 + 1 \equiv 1 \pmod 4$, otherwise $k^2 + 1 \equiv 2 \pmod 4$, so,

$$\alpha_2 = \left\lceil \frac{n}{2} \right\rceil - \left\lceil \frac{m}{2} \right\rceil \leq \frac{n - m + 1}{2}$$

and using the preceding lemma,

$$\beta_2 \geq n - m - \frac{\ln(n+1)}{\ln 2}.$$

When $p = 3$, $k^3 + 1 \equiv 0 \pmod{3^t}$ has only one solution for $t = 1$ and two solutions for $t > 1$. So,

$$\bar{\alpha}_3 \leq (\left\lceil \frac{n}{3} \right\rceil - \left\lfloor \frac{m}{3} \right\rfloor) + 2(\left\lceil \frac{n}{9} \right\rceil - \left\lfloor \frac{m}{9} \right\rfloor) + \cdots \leq \frac{5(n-m)}{6} + 3 + \frac{3\ln n}{\ln 3} - \frac{3\ln m}{\ln 3}$$

and using the preceding lemma again,

$$\beta_3 \geq \frac{n - m}{2} - \frac{\ln(n+1)}{\ln 3}.$$

Lemma 4. *If $p > n$, then $\alpha_p \leq 2$ and $\bar{\alpha}_p \leq 3$.*

Proof. If $p > n$, and $\alpha_p \geq 3$, then it is easy to see that there exist distinct j, k, l such that $p|j^2 + 1$, $p|k^2 + 1$, and $p|l^2 + 1$. Then $p|(j - k)(j + k)$, so $p|j + k$, and similarly, $p|j + l$. Then $p|k - l$, a contradiction. A similar argument can be applied to $\bar{\alpha}_p$, as in [7]. $\square$

Using Lemma 3 and Lemma 4, inequalities (6) and (7) become

$$\begin{aligned}\sum_{\substack{p\equiv 3(4)\\ p\leq n}} \frac{(n-m)\ln p}{p-1} &< (\frac{\alpha_2}{2} - \beta_2)\ln 2 + \sum_{\substack{p\equiv 1(mod 4)\\ p\leq n}} \ln(n^2+1) + \sum_{\substack{p\equiv 3(mod 4)\\ p\leq n}} \ln(n+1) \\ &+ \sum_{n<p<2n} \ln p.\end{aligned} \tag{8}$$

$$\begin{aligned}\sum_{\substack{p\equiv 2(mod 3)\\ p\leq n}} \frac{2(n-m)\ln p}{3(p-1)} &< (\frac{\bar{\alpha}_3}{3} - \beta_3)\ln 3 + \sum_{\substack{p\equiv 1(mod 3)\\ p\leq n}} \ln(n^3+1) \\ &+ \sum_{\substack{p\equiv 2(mod 3)\\ p\leq n}} \frac{\ln(n^3+1)}{3} + \sum_{\substack{p\equiv 2(mod 3)\\ p\leq n}} \frac{2\ln(n+1)}{3} \\ &+ \sum_{n<p<2n} \ln p.\end{aligned} \tag{9}$$

Employing the estimates for α_2, β_2, replacing $\ln(n+1)$ by $\ln(n^2+1)$ and replacing

$$\sum_{\substack{p\equiv 3(mod 4)\\ p\leq n}} \ln p + \sum_{n<p<2n} \ln p$$

by

$$\sum_{p<2n} \ln p,$$

the right-hand side of inequality (8) gets larger,

$$\sum_{\substack{p\equiv 3(4)\\ p\le n}} \frac{(n-m)\ln p}{(p-1)} < \frac{(1-3n+3m)\ln 2}{4} + \sum_{p\le n} \ln(n^2+1) + \sum_{p<2n} \ln p. \tag{10}$$

Dividing both sides by $n-m$ we obtain

$$\sum_{\substack{p\equiv 3(4)\\ p\le n}} \frac{\ln p}{(p-1)} < \left(\frac{1}{4(n-m)} - \frac{3}{4}\right)\ln 2 + \frac{\pi(n)\ln(n^2+1)}{n-m} + \frac{\sum_{p<2n}\ln p}{n-m}. \tag{11}$$

Likewise, employing the estimates for $\bar{\alpha}_3$, β_3, replacing $\ln(n+1)$ by $\ln(n^3+1)$ and replacing

$$\sum_{\substack{p\equiv 2(mod 3)\\ p\le n}} \frac{2\ln p}{3}$$

by

$$\sum_{p\le n} \ln p,$$

the right-hand side of inequality (9) gets larger,

$$\sum_{\substack{p\equiv 2(mod 3)\\ p\le n}} \frac{2(n-m)\ln p}{3(p-1)} < \frac{(9-n+m)\ln 3}{9} + \sum_{p\le n} \ln(n^3+1) + \sum_{p<2n} \ln p. \tag{12}$$

Dividing both sides by $\frac{2(n-m)}{3}$ we obtain

$$\sum_{\substack{p\equiv 2(mod 3)\\ p\le n}} \frac{\ln p}{(p-1)} < \left(\frac{3}{2(n-m)} - \frac{1}{6}\right)\ln 3 + \frac{3\pi(n)\ln(n^3+1)}{2(n-m)} + \frac{3\sum_{p<2n}\ln p}{2(n-m)}. \tag{13}$$

Since

$$\pi(n) \sim \frac{n}{\ln n} \text{ and } \sum_{p<2n} \ln p \sim 2n,$$

the limiting values of the right-hand sides in (12) and (13) are $\frac{16-3\ln 2}{4} \cong 3.48013$ and $\frac{45-\ln 3}{6} \cong 7.31689$, respectively. However, the left hand sides of these inequalities are divergent series exceeding the right-hand sides for some values of N_m for each inequality. That means any of these inequalities will no longer be satisfied and it will contradict P_m^n and/or Q_m^n is square-full, which proves our theorem.

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