FINITE $p$-GROUPS WHICH ARE NOT GENERATED BY THEIR NON-NORMAL SUBGROUPS

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Abstract. Here we classify finite non-Dedekindian $p$-groups which are not generated by their non-normal subgroups. (Theorem 1).

The purpose of this paper is to classify non-Dedekindian finite $p$-groups which are not generated by their non-normal subgroups. It is surprising that such $p$-groups must be of class 2 with a cyclic commutator subgroup.

We consider here only finite $p$-groups and our notation is standard (see [1]). We prove the following result.

**Theorem 1.** Let $G$ be a non-Dedekindian $p$-group and let $G_0$ be the subgroup generated by all non-normal subgroups of $G$, where we assume $G_0 < G$. Then $G$ is of class 2, $G/G_0$ is cyclic and for each $g \in G - G_0$, $\{1\} \neq \langle g \rangle \cap G_0 \leq G$ and $G/((\langle g \rangle \cap G_0)$ is abelian so that $G'$ is cyclic.

**Proof.** Since our group $G$ has at least $p$ (non-normal) conjugate cyclic subgroups, it follows that the subgroup $G_0$ is noncyclic. Let $x \in G-G_0$. Then $\langle x \rangle \not\leq G$, by hypothesis, and so $G'$ centralizes $\langle x \rangle$. It follows from $(G-G_0) = G$ that $G' \leq Z(G)$ and so $\text{cl}(G) = 2$.

Let $g \in G - G_0$. Then $Z = \langle g \rangle \triangleleft G$. Write $Z_0 = Z \cap G_0$; then $Z_0$, being the intersection of two $G$-invariant subgroups, is $G$-invariant. We claim that $G/Z_0$ is Dedekindian. Indeed, let $X/Z_0$ be any proper subgroup in $G/Z_0$. We have to show that $X \triangleleft G$. If $X \not\leq G_0$, then $X \not\leq Z_0$. Now assume that $X < G_0$ (the subgroup $G_0$ is $G$-invariant). Then $XZ = ZX$ is normal in $G$.

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We have and since \( a, b \) is a subgroup of order 2 in \( G \), we have proved that in case \( p > ab \) and so again follows that \( s \) where \( \langle \cdot \rangle \cap \langle \cdot \rangle = \langle \cdot \rangle \cap \langle \cdot \rangle \geq G \). Hence there is an element \( D \) cyclic. If \( G/Z \) is Dedekindian. In particular, \( Z_0 \neq \{1\} \) since \( G \) is non-Dedekindian, by hypothesis. If \( p > 2 \), then \( G/Z_0 \) is abelian and so \( G' \leq Z_0 \) and \( G' \) is cyclic. If \( p = 2 \), then \( G/Z_0 \) is either abelian or Hamiltonian (= nonabelian Dedekindian).

It follows from the above that \( \Omega_1(G) \leq G_0 \).

Now assume that \( p > 2 \). Amongst all elements in the set \( G - G_0 \), we choose an element \( a \) of the smallest possible order. Then \( a^p \in G_0 \) and \( G' \leq \langle \langle a^p \rangle \rangle \) (see the previous paragraph). We set \( |G'| = p^d, d \geq 1 \). Suppose that \( G/G_0 \) is not cyclic. Then there is \( b \in G \setminus \langle G_0(a) \rangle \) such that \( b^p \in G_0 \). We have \( \langle a \rangle \cap \langle b \rangle \geq G' \) and \( o(b) \geq o(a) \) by the minimality of \( o(a) \). Set
\[
|\langle a \rangle \cap \langle (a) \cap \langle b \rangle \rangle | = p^s, \quad \text{where} \quad s \geq 1 \quad \text{and} \quad o(a) \geq p^{d+s}.
\]
Hence there is \( b' \in \langle b \rangle - \langle a \rangle \) such that \( a^p = (b')^{-p^s} \). In that case, since \( cl(G) = 2 \), one obtains
\[
(ab')^p = a^{p^s} (b')^{p^s} [b', a]^{(p^s)} = [b', a]^{(p^s)},
\]
where \( s \geq 1, o(a) \geq p^{d+s} \) and \( \langle [b', a]^{(p^s)} \rangle < G' \) so that \( o([b', a]^{(p^s)}) < p^d \). It follows that
\[
o(ab') < p^{d+s} \quad \text{and so} \quad o(ab') < o(a).
\]

If \( b' \in \langle b^p \rangle \leq G_0 \), then \( ab' \in G - G_0 \). If \( \langle b' \rangle = \langle b \rangle \), then \( ab' \in G - \langle G_0(a) \rangle \) and so again \( ab' \in G - G_0 \). But this contradicts the minimality of \( o(a) \). We have proved that in case \( p > 2 \), \( G/G_0 \) is cyclic.

Suppose \( p = 2 \) and \( G/G_0 \) is nonabelian. Then for each \( g \in G \setminus G_0 \), \( G/\langle (g) \cap \langle g \rangle \rangle \) is Hamiltonian (i.e., Dedekindian nonabelian). Let \( Q/G_0 \) be a subgroup of \( G/G_0 \), which is isomorphic to \( Q_8 \) and let \( R/G_0 \) be a unique subgroup of order 2 in \( Q/G_0 \). Then for each \( x \in Q \setminus R, x^2 \in R \setminus G_0 \). Let \( a, b \in Q \setminus R \) be such that \( \langle a, b \rangle \) covers \( Q/R \cong E_4 \). Note that \( \langle a \rangle \leq G, \langle b \rangle \leq G \) and since \( \langle a \rangle \cap G_0 \neq \{1\} \) and \( \langle b \rangle \cap G_0 \neq \{1\} \), we get \( o(a) = 2^s, s \geq 3 \), and \( o(b) \geq 2^3 \). Because
\[
[a, b] \in R - G_0 \quad \text{and} \quad [a, b] \in \langle a \rangle \cap \langle b \rangle.
\]
we have
\[
\langle a \rangle \cap \langle b \rangle = \langle a^2 \rangle = \langle b^2 \rangle = \langle [a, b] \rangle.
\]
But then \( C = \langle a, b \rangle \) is a 2-group of maximal class and order \( 2^{s+1} \), \( s \geq 3 \), and in this case \( \langle a \rangle \) is a unique cyclic subgroup of order \( 2^s \) in \( C \), contrary to the fact that \( o(b) = 2^s \). We have proved that in case \( p = 2 \), \( G/G_0 \) must be abelian and so \( G' \leq G_0 \).

Suppose that \( G' \) is noncyclic. By the above, \( p = 2 \) and for each \( g \in G - G_0 \), \( \{1\} \neq \langle g \rangle \cap G_0 \leq G \), where \( G/(\langle g \rangle \cap G_0) \) is Hamiltonian (=nonabelian Dedekindian). Set \( D = \langle g \rangle \cap G_0 \) and \( R/D = (G/D)' \cong C_2 \), where \( R = G'D \). We know that \( G' \leq G_0 \) (since \( G/G_0 \) is abelian) and so \( R \leq G_0 \) and \( G/R \) is elementary abelian. In particular, \( G/G_0 \neq \{1\} \) is elementary abelian and \( \langle g^2 \rangle = D \). Note that all quaternion subgroups in a Hamiltonian 2-group \( X \) generate \( X \). Hence there is a quaternion subgroup \( K/D \cong Q_8 \) in the Hamiltonian group \( G/D \) such that \( K \not\leq G_0 \). We have \( K > R \) and \( K/R \cong E_4 \) so that for each \( x \in K - R \), \( x^2 \in R - D \). We may choose some elements \( a, b \in K - G_0 \) such that \( Q = \langle a, b \rangle \) covers \( K/R \) and so \( Q \) also covers \( K/D \).

Note that \( \langle a \rangle \leq G \), \( \langle b \rangle \leq G \) and \( [a, b] \in R - D \). Also,
\[
[a, b] \in \langle a \rangle \cap \langle b \rangle \text{ and so } ([a, b]) = \langle a^2 \rangle = \langle b^2 \rangle = \langle a \rangle \cap \langle b \rangle.
\]

This gives \( |Q : Q'| = 4 \) and so (by a well known result of O. Taussky) \( Q \) is a 2-group of maximal class with two distinct cyclic subgroups \( \langle a \rangle \) and \( \langle b \rangle \) of index 2. By inspection of 2-groups of maximal class (and noting that \( G \) is of class 2), we get \( o(a) = o(b) = 4 \) and \( Q \cong Q_8 \) with \( Q' = \langle a^2 \rangle = \langle b^2 \rangle \). Hence \( K = Q \times D \) since \( Q \leq G \) and \( Q \) covers \( K/D \cong Q_8 \). Also, \( \langle g \rangle \leq G \) and \( Q \cap \langle g \rangle = \{1\} \) and so \( Q \) centralizes \( \langle g \rangle \). The factor-group \( G/\langle a^2 \rangle \) is Hamiltonian and so
\[
o(g) = 4, \ D = \langle g^2 \rangle \cong C_2 \text{ and } G' = \langle a^2, g^2 \rangle \cong E_4
\]
since \( G' \) covers \( \langle a^2, g^2 \rangle / \langle a^2 \rangle \) and \( G' \) is noncyclic. For each \( x \in G \),
\[
x^4 \in \langle a^2 \rangle \cap \langle b^2 \rangle = \{1\} \text{ and so } \exp(G) = 4.
\]

Let \( K_1/\langle a^2 \rangle \cong Q_8 \) with \( K_1 \not\leq G_0 \). Then choose \( a_1, b_1 \in K_1 - G_0 \) such that \( \langle a_1, b_1 \rangle \) covers \( K_1/\langle a^2 \rangle \). We get
\[
Q_1 = \langle a_1, b_1 \rangle \cong Q_8 \text{ with } Q \cap Q_1 = \{1\} \text{ and } Q_1' = \langle a_1^2 \rangle = \langle b_1^2 \rangle,
\]
so \( (Q, Q_1) = Q \times Q_1 \).

Set \( a^2 = t, a_1^2 = t_1 \) and let \( x \in Q - \langle t \rangle, x_1 \in Q_1 - \langle t_1 \rangle \) so that \( xx_1 \) is one of 36 elements of order 4 with \( (xx_1)^2 = x^2 x_1^2 = tt_1 \). We claim that \( (xx_1) \) is not normal in \( Q \times Q_1 \) and so \( xx_1 \in G_0 \). Indeed, let \( y \in Q - \langle x \rangle \) so that
\[
(xx_1)^y = x^{-1} x_1 = (xx_1)t, \text{ where } (xx_1)t \not\in \langle xx_1 \rangle.
\]

But all these 36 elements of order 4 generate \( Q \times Q_1 \) (of order 64) and so \( Q \times Q_1 \leq G_0 \), a contradiction. We have proved that also in case \( p = 2 \), \( G' \) is cyclic.

In the following five paragraphs we assume that \( G/G_0 \) is noncyclic. By the above, \( p = 2 \) and \( G/G_0 \) is abelian.
Assume that there are \(a_1, a_2 \in G - G_0\) such that \(\langle a_1 \rangle \cap \langle a_2 \rangle = \{1\}\). We know that \(G/(\langle a_1 \rangle \cap G_0)\) and \(G/(\langle a_2 \rangle \cap G_0)\) are Dedekindian and \([a_1, a_2] \in \langle a_1 \rangle \cap \langle a_2 \rangle = \{1\}\) and so \(\langle a_1, a_2 \rangle\) is abelian. If both \(G/(\langle a_1 \rangle \cap G_0)\) and \(G/(\langle a_2 \rangle \cap G_0)\) are abelian, then
\[
G' \leq (\langle a_1 \rangle \cap G_0) \cap (\langle a_2 \rangle \cap G_0) = \{1\},
\]
a contradiction. Assume for a moment that both \(G/(\langle a_1 \rangle \cap G_0)\) and \(G/(\langle a_2 \rangle \cap G_0)\) are Hamiltonian. Then for each \(x \in G\),
\[
x^4 \in ((\langle a_1 \rangle \cap G_0) \cap (\langle a_2 \rangle \cap G_0) = \{1\}\) and so \(\exp(G) = 4\).
In particular,
\[
o(a_1) = o(a_2) = 4, \; \langle a_1^2, a_2^2 \rangle \cong E_4\) with \(\langle a_1^2, a_2^2 \rangle \leq Z(G)\).
We have
\[
G' \leq \langle a_1^2, a_2^2 \rangle \ continents \\langle a_1^2 \rangle / \langle a_1^2 \rangle \) and \(\langle a_1^2, a_2^2 \rangle / \langle a_2^2 \rangle\) and \(G'\) is cyclic and so \(G' = \langle a_1^2, a_2^2 \rangle\). For each
\[
x \in G, \; [a_2, x] \in \langle a_2 \rangle \cap G' = \{1\}\) and so \(a_2 \leq Z(G)\).
But then in the Hamiltonian 2-group \(G/(\langle a_1^2 \rangle)\) the element \(\langle (a_2)\rangle / \langle a_1^2 \rangle \cong C_4\) of order 4 lies in its center, a contradiction. We have proved that if \(a_1, a_2 \in G - G_0\) are such that \(\langle a_1 \rangle \cap \langle a_2 \rangle = \{1\}\), then one of \(G/(\langle a_1 \rangle \cap G_0)\) and \(G/(\langle a_2 \rangle \cap G_0)\) is abelian and the other one is Hamiltonian.

Assume in addition that \((G_0(a_1, a_2))/G_0)\) is noncyclic. Set \(\Omega_1(\langle a_1 \rangle) = \langle t_1 \rangle\) and \(\Omega_1(\langle a_2 \rangle) = \langle t_2 \rangle\) so that \(\langle t_1, t_2 \rangle \cong E_4\) and \(\langle t_1, t_2 \rangle \leq Z(G)\). Without loss of generality we may suppose that \(G/(\langle a_1 \rangle \cap G_0)\) is abelian and \(G/(\langle a_2 \rangle \cap G_0)\) is Hamiltonian. Since \(G/G_0\) is elementary abelian, we get
\[
o(a_1) = 4, \; G' = \langle a_1^2 \rangle \cong C_2\) and \(1 \neq a_2^2 \in G_0\).
It follows that \((G_0(a_1, a_2))/G_0 \cong E_4\). Let \(a_2^2\) be an element of order 4 in \(\langle a_2 \rangle\) so that
\[
(a_1^2 a_2^2)^2 = a_1^2 (a_2^2)^2 = t_1 t_2\) and \(a_1 a_2^2 \in G - G_0\).
But then \(\langle a_1 \rangle, \; \langle a_2 \rangle, \; \langle a_1 a_2^2 \rangle\) are three cyclic subgroups in \(G\) which are not contained in \(G_0\) and they have pairwise a trivial intersection. By the previous paragraph, this is not possible. We have proved that whenever \(a_1, a_2 \in G - G_0\) are such that \((\langle a_1, a_2 \rangle)/G_0)\) is noncyclic, then \(\langle a_1 \rangle \cap \langle a_2 \rangle \neq \{1\}\).

Let \(E/G_0\) be a four-subgroup in the noncyclic abelian group \(G/G_0\). Amongst all elements in \(E - G_0\) choose an element \(a\) of the smallest possible order \(2^s\). We have \(s \geq 2\) since \(a^2 \neq 1\). Set \(F = G_0(a)\) and let \(b\) be any element in \(E - F\) so that \(o(b) \geq 2^s\). By the above, \(D = \langle a \rangle \cap \langle b \rangle \neq \{1\}\). Let \(b'\) be an element of order \(2^s\) in \(\langle b \rangle\) such that
\[
a^{2^n} = (b')^{-2^n}, \; \text{where} \; |\langle a \rangle : D| = |\langle b' \rangle : D| = 2^n, \; n \geq 1,
and \(D = \langle a^{2^n} \rangle = \langle (b')^{2^n} \rangle\).
We compute
\[(ab')^2 = a^{2^2} (b')^2 \in \langle b', a \rangle^2 = [b', a]^{2^{n-1}(2^n-1)}, \]
where \(ab' \in E - G_0\) and \([b', a] \in D\).

Since \(a\) was an element of the smallest possible order in the set of all elements in \(E - G_0\), we get
\[
n = 1, \quad a^2 \in D, \quad \text{and } \langle [b', a] \rangle = D \neq \{1\}.
\]

On the other hand,
\[
[b, a]^2 = [b, a^2] = [b, a] = 1.
\]
Hence, if \(b' \in \langle b^2 \rangle\) (in case \(\exp(b) > \exp(a) = 2^s\)), we get \([b', a] = 1\) and so \(D = \{1\}\), a contradiction. Hence, if \(b' \in \langle b^2 \rangle\), we get \(\langle [b', a] \rangle = D \cong C_2\), where \(D = \langle a^2 \rangle = \langle b^2 \rangle\).

Hence
\[
s = 2, \quad \exp(a) = \exp(b) = 2^s, \quad \text{and } Q = \langle a, b \rangle \cong Q_8.
\]

We have proved that all elements in \(E - F\) are of order 4 and each such element has the same square \(a^2\). We know that \(G'\) is cyclic, \(G' \leq G_0\), \(G' \leq Z(G)\) and \(G' \geq \langle a^2 \rangle = \langle a, b' \rangle\). Suppose that \(G' > \langle a^2 \rangle\) and let \(x \in G' - \langle a^2 \rangle\) be such that \(x^2 = a^2\), where \([x, a] = 1\). But then \(xa\) is an involution in \(E - G_0\), a contradiction. Hence \(G' = \langle a^2 \rangle \cong C_2\). Since all elements in \(E - F\) are of order 4 and they generate \(E\) and \(E' = \langle a^2 \rangle \cong C_2\), we get \(\exp(G) = 4\). In particular, all elements in \(F - G_0\) are of order 4 and let \(y \in F - G_0\). Then \(y\) is also of the smallest possible order 4 in \(E - G_0\). By repeating the above argument with the element \(y\) (instead of \(a\)), we get that for each \(b \in E - F\), \(b^2 = y^2\) and so \(y^2 = a^2\). We have proved that for each \(x \in E - G_0\), \(x^2 = a^2\).

For any \(x, y \in G\),
\[
[x^2, y] = [x, y]^2 = 1 \quad \text{since } G' = \langle a^2 \rangle \cong C_2.
\]

Hence \(O_1(G) \leq Z(G)\).

Let \(c\) be an element of order 4 in \(G_0\). Then
\[
ac \in E - G_0 \quad \text{and so } a^2 = (ac)^2 = a^2c^2 = [c, a]
\]
implies \(c^2 = [a, c] \in \langle a^2 \rangle\) and \(c^2 = a^2\).

But then \(\langle c \rangle \leq G\) and so there is \(b \in E - G_0\), which centralizes \(\langle c \rangle\). It follows that \(bc\) is an involution in \(E - G_0\), a contradiction. We have proved that \(G_0\) is elementary abelian. If \(G_0 \not\leq Z(E)\), then there are \(t \in G_0 - \langle a^2 \rangle\) and \(x \in E - G_0\) such that \([t, x] = a^x = x^a\). But then \(\langle t, x \rangle \cong D_8\) and so there are involutions in \(\langle t, x \rangle - G_0\), a contradiction. We have proved that \(E\) is Hamiltonian and so \(E \neq G\) because \(G\) is not Dedekindian.

Let \(v \in G - E\) be such that \(v^2 \in E\). Since \(O_1(G) \leq Z(G)\), we get
\[
1 \neq v^2 \in Z(E) = G_0.
\]

Then, by the above, \(\langle v \rangle \cap \langle a \rangle \neq \{1\}\) and so \(v^2 = a^2\).
Let \( a, b \in E - G_0 \) be such that \( \langle a, b \rangle \) covers \( E/G_0 \). Because there are no
involutions in \( G - G_0 \), we have
\[
[v, a] = [v, b] = [a, b] = a^2 \quad \text{and} \quad [v, ab] = [v, a] = [v, b] = a^2 a^2 = a^4 = 1.
\]
But then \((ab)^2 = v^2 = a^2\) implies that \((ab)v\) is an involution in \( G - G_0 \), a final
contradiction. We have proved that also in case \( p = 2 \), \( G/G_0 \) is cyclic.

If \( g \) is nonabelian. We have
\[
[a] = 1, \quad \text{then} \quad \langle ab \rangle \leq \langle a \rangle \subseteq \langle a^2 \rangle = \langle ab \rangle.
\]
It follows that in any case \( G \) is abelian. Our theorem is proved.

On the other hand, if \( \langle a \rangle \) is nonabelian, then \( G \) is abelian. We have \( [a, b] = [a, b] = a^2 \),
and so the fact that \((ab)v\) is an involution in \( G - G_0 \), a final
contradiction. We have proved that also in case \( p = 2 \), \( G/G_0 \) is cyclic.

Suppose that \( p = 2 \) and there is \( g \in G - G_0 \) such that \( G/(\langle g \rangle \cap G_0) \) is
Hamiltonian. We set \( D = \langle g \rangle \cap G_0 \neq \{1\} \) and note that \( G' \leq G_0 \) implies that
\( G/G_0 \) is elementary abelian. But \( G/G_0 \) is also cyclic and so \( |G : G_0| = 2 \). We
get \( g^2 \in G_0 \) and so \( D = \langle g^2 \rangle \neq \{1\} \). Since the Hamiltonian group \( G/D \) is
generated by its quaternion subgroups, there is a quaternion subgroup \( K/D \) in \( G/D \) such that \( K \not\leq G_0 \). Let \( a, b \in K - G_0 \) be such that \( Q = \langle a, b \rangle \) covers
\( K/D \), where \( ab \in G_0 \). Let \( R/D \) be a unique subgroup of order 2 in \( K/D \) so
that \( R \leq G_0 \) and \( G' \) covers \( R/D \). We have
\[
a^2 \in R - D, \quad b^2 \in R - D, \quad (ab)^2 \in R - D, \quad \text{and} \quad [a, b] \in R - D.
\]
On the other hand,
\[
[a, b] \in \langle a \rangle \cap \langle b \rangle \quad \text{and so} \quad \langle a \rangle \cap \langle b \rangle = \langle a^2 \rangle = \langle b^2 \rangle = \langle a, b \rangle.
\]
Since \( Q/Q' \cong \mathbb{Z}_2 \), \( Q \) is of maximal class (by O. Taussky) and since \( Q \) has two
distinct cyclic subgroups \( \langle a \rangle \) and \( \langle b \rangle \) of index 2, we get
\[
Q \cong \mathbb{Z}_2^3, \quad o(a) = o(b) = 2, \quad \langle [a, b] \rangle \cong \mathbb{Z}_2, \quad Q / \langle g^2 \rangle = \{1\}
\]
and so \( \langle Q, \langle g \rangle \rangle = Q \times \langle g \rangle \).

Also,
\[
G' \leq R \quad \text{and} \quad G' \geq \langle [a, b] \rangle \cong \mathbb{Z}_2,
\]
and so the fact that \( G' \) is cyclic implies \( G' \cap \langle g^2 \rangle = \{1\} \). It follows
\[
G' = \langle [a, b] \rangle = \langle a^2 \rangle \cong \mathbb{Z}_2
\]
and for any \( x, y \in G \),
\[
[x, y] = [x, y]^2 = 1 \quad \text{implying} \quad U_2(G) \leq Z(G).
\]
Since \( G' \cap \langle g \rangle = \{1\} \), we have \( \langle g \rangle \leq Z(G) \) and so \( G = G_0 * \langle g \rangle \) gives that \( G_0 \)
is nonabelian. We have \( ab \in G_0 \) and so \( abg \notin G_0 \) which implies \( \langle abg \rangle \leq G \).
We compute
\[
(abg)^2 = (ab)^2 g^2 = a^2 g^2.
\]
If \( g^4 \neq 1 \), then
\[
(abg)^4 = g^4 \neq 1 \quad \text{and so} \quad G' = \langle a^2 \rangle \nleq \langle abg \rangle.
\]
If \( g^4 = 1 \), then \( a^2 g^2 \) is an involution distinct from \( a^2 \) and so again \( G' \nleq \langle abg \rangle \).
It follows that in any case \( G' \nleq \langle abg \rangle \) and so \( \langle abg \rangle \leq Z(G) \). But then
\[
ab \in Z(G) \quad \text{giving} \quad C_4 \cong \langle ab \rangle \leq Z(Q),
\]
a contradiction. We have proved that for each \( g \in G - G_0 \), \( G/(\langle g \rangle \cap G_0) \) is
abelian. Our theorem is proved. □
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