THE LANGLANDS QUOTIENT THEOREM FOR FINITE CENTRAL EXTENSIONS OF $p$-ADIC GROUPS II: INTERTWINING OPERATORS AND DUALITY

Dubravka Ban and Chris Jantzen
Southern Illinois University and East Carolina University, USA

Abstract. In this paper, we prove that the Langlands quotient may be realized as the image of a standard intertwining operator in the context of finite central extensions of connected, reductive $p$-adic groups. We also verify that the duality of Aubert and Schneider-Stuhler holds in this context.

1. Introduction

This paper is a continuation of [3].

The Langlands quotient theorem is proved in [3] for finite central extensions of connected reductive $p$-adic groups. However, the proof is algebraic in nature and does not bring standard intertwining operators into the picture. As the connection with standard intertwining operators is important in some contexts, we address that here. In particular, we show that the Langlands quotient may be realized as the image of a standard intertwining operator. To do so, we use the results of [10] on standard intertwining operators for finite central extensions and properties of the Langlands quotient for such groups established in [3].

Although not directly related to the Langlands classification, another useful tool in representation theory—and which is often used in conjunction with the Langlands classification—is the duality of Aubert and Schneider, Stuhler (see [1, 12]; also [4]). We include a sketch of the proof here, but note that it is essentially the same as that of [1]. We also take the opportunity to remove an
unneeded hypothesis from [3]—namely, the assumption that the characteristic of the underlying field is zero (see Remark 2.1).

2. Notation and preliminaries

In this section, we briefly review background material and introduce notation needed in the remainder of the paper. The reader is referred to [3] for more details.

Let $F$ be a nonarchimedean local field. Note that we make no assumption on the characteristic of $F$ (see Remark 2.1 below).

Let $G$ be the group of $F$-points of a connected reductive group defined over $F$. We call $(\tilde{G}, \rho)$ a finite central extension of $G$ if the following hold:

1. $\rho : \tilde{G} \rightarrow G$ is a surjective homomorphism of topological groups.
2. $C = \ker(\rho)$ is a finite subgroup of $Z(\tilde{G})$, where $Z(H)$ denotes the center of $H$.
3. $\rho$ is a topological covering (as described in [11]). In particular, there is an open neighborhood $O$ of the identity in $G$ and a homeomorphism $j : \rho^{-1}(O) \rightarrow O \times C$ such that $\text{pr}_1 \circ j = \rho$ on $\rho^{-1}(O)$.

We introduce some terminology for finite central extensions. If $(\tilde{G}, \rho)$ is a finite central extension of $G$, a section of $\rho$ is a continuous map $\mu : G \rightarrow \tilde{G}$ such that $\rho \circ \mu = \text{id}_G$. A lifting of a subgroup $H$ of $G$ is a continuous homomorphism $s : H \rightarrow \tilde{G}$ such that $\rho \circ s = \text{id}_H$. Obviously, if $G$ lifts to $\tilde{G}$—in other words, if the sequence $1 \rightarrow C \rightarrow \tilde{G} \rightarrow G \rightarrow 1$ splits—then $\tilde{G} \cong G \times C$. As in [3], we retain the following convention: if $H$ be a subgroup of $G$, the preimage of $H$ in $\tilde{G}$ will be denoted by $\tilde{H}$ and a lifting of $H$ (if it exists) will be denoted by $\hat{H}$. Hence, $\hat{H} = \rho^{-1}(H)$ and $\tilde{H} \cong H$.

The group $\tilde{G}$ has a neighborhood basis of the identity consisting of compact open subgroups, and we can define smooth and admissible representations in the standard way. We denote by $R(\tilde{G})$ the Grothendieck group of the category of smooth finite length representations of $\tilde{G}$. Let $\pi_1$ and $\pi_2$ be smooth finite length representations of $\tilde{G}$. If $\pi_1$ and $\pi_2$ have the same irreducible components appearing with the same multiplicities, we write $\pi_1 = \pi_2$. If $\pi_1$ and $\pi_2$ are actually equivalent as representations, we write $\pi_1 \cong \pi_2$.

Fix a maximal split torus $A$ in $G$. We denote by $W = W(G, A)$ the Weyl group of $G$ with respect to $A$. Let $\Phi = \Phi(G, A)$ be the set of roots. Fix a minimal parabolic subgroup $P_0$ containing $A$. The choice of $P_0$ determines the set of simple roots $S$ and the set of positive roots $\Phi^+ \subset \Phi$. For $I \subset S$, we denote by $P_I$ the standard parabolic subgroup of $G$ determined by $I$ and by $L_I$ the standard Levi subgroup of $P_I$.

Let $P = MU$ be a parabolic subgroup of $G$. We call

$$\tilde{P} = \rho^{-1}(P)$$
a parabolic subgroup of $\tilde{G}$. Let $\tilde{M} = \rho^{-1}(M)$ and $\tilde{U}$ the canonical lifting of $U$ to $\tilde{G}$ described in the first appendix to [11]. Then

$$\tilde{P} = \tilde{M}\tilde{U}$$

serves as the Levi factorization. Set $\tilde{A}_M = \rho^{-1}(A_M)$, where $A_M$ is the split component of the center of $M$. The normalized parabolic induction $\text{Ind}_{\tilde{P}}^{\tilde{G}}$ and the Jacquet functor $r_{\tilde{P}}^\tilde{G}$ are defined as usual. In the case when $\tilde{P} = \tilde{M}\tilde{U}$ is a standard parabolic subgroup, we also denote these two functors by $i_{\tilde{M}}^\tilde{G}$ and $r_{\tilde{M}}^\tilde{G}$, respectively. We use the same notation for the corresponding homomorphisms $i_{\tilde{M}}^\tilde{G}$ and $r_{\tilde{M}}^\tilde{G}$ on the Grothendieck groups.

**Remark 2.1.** In [3], it was assumed that the characteristic of $F$ was zero. This is not needed, however, for reasons detailed below.

Recall that $G$ acts on $\tilde{G}$ by conjugation: $g \in G$ acts on $\tilde{x} \in \tilde{G}$ by

$$\tilde{x} \mapsto g\tilde{x}g^{-1},$$

where $\tilde{g}$ is any element of $\rho^{-1}(g)$. Let $P = MU$ be a parabolic subgroup of $G$. Let $s_U : U \to \tilde{G}$ be the canonical lifting described in [11, Appendix I]. Then $s_U$ is the unique $P$-equivariant homomorphism of $U$ into $\tilde{G}$. Take $a \in A_M$ and let $\tilde{a}$ be any element of $\rho^{-1}(a)$. Define $s_{\tilde{a}} : U \to \tilde{G}$ by

$$s_{\tilde{a}}(x) = \tilde{a}^{-1}s_U(axa^{-1})\tilde{a}.$$

Since $s_U$ is $P$-equivariant, we have $s_{\tilde{a}} = s_U$.

The assumption of characteristic zero was made to ensure the lifting from $U$ to $\tilde{G}$ was unique. This was used in several proofs to conclude $s_{\tilde{a}} = s_U$. However, as observed above, one can directly obtain $s_{\tilde{a}} = s_U$, which is what is really needed. Thus the assumption of characteristic zero may be removed.

### 3. Standard Intertwining Operators and the Langlands Classification

In this section, we show that the Langlands quotient may be realized as the image of a standard intertwining operator.

For $P = MU$ a standard parabolic subgroup of $G$, we let $X(M)$ denote the set of rational characters of $M$. Let $q$ be the number of elements in the residue field of $F$. There is a homomorphism (cf. [8]) $H_M : M \to a_M = \text{Hom}(X(M), \mathbb{R})$ such that $q^{\nu, H_M(m)} = |\nu(m)|$ for all $m \in M, \nu \in X(M)$.

Given $\nu \in a_M$, let us write

$$\exp \nu = q^{\nu, H_M(\cdot)}$$

for the corresponding character of $M$. As in [3, Note 2.4], there is then an associated unramified character of $\tilde{M}$; for clarity, we denote this character $\exp \nu$. 

We now recall the Langlands classification for $\tilde{G}$ in the quotient setting (see [3, Remark 4.2]). A set of Langlands data for $\tilde{G}$ is a triple $(\tilde{P}, \nu, \tau)$ with the following properties:

1. $\tilde{P} = \tilde{M}\hat{U}$ is a standard parabolic subgroup of $\tilde{G}$,
2. $\nu \in (a_{\tilde{M}})_+ = \{x \in a_{\tilde{M}} | \langle x, \alpha \rangle > 0, \text{ for all } \alpha \in S(P, A_M)\}$, and
3. $\tau$ is (the equivalence class of) an irreducible tempered representation of $\tilde{M}$,

where $S(P, A_M)$ denotes the set of simple roots for the pair $(P, A_M)$.

We now state the Langlands classification in the quotient setting.

**Theorem 3.1 (The Langlands classification).** Suppose $(\tilde{P}, \nu, \tau)$ is a set of Langlands data for $\tilde{G}$. Then the induced representation $i^\tilde{G}_M(\tilde{\exp} \nu \otimes \tau)$ has a unique irreducible quotient, which we denote by $L(\tilde{P}, \nu, \tau)$. Conversely, if $\pi$ is an irreducible admissible representation of $\tilde{G}$, then there exists a unique $(\tilde{P}, \nu, \tau)$ as above such that $\pi \cong L(\tilde{P}, \nu, \tau)$.

Let $\tilde{P} = \tilde{M}\hat{U}$ be a standard parabolic subgroup of $\tilde{G}$. Let $\tau$ be an irreducible tempered representation of $\tilde{M}$ and $\nu \in (a_{\tilde{M}})_+$. We consider the representation

$$\Pi = \text{Ind}_{\tilde{P}}^\tilde{G}(\tilde{\exp} \nu \otimes \tau).$$

Then the contragredient $\bar{\Pi}$ satisfies

$$\bar{\Pi} \cong \text{Ind}_{\tilde{P}}^\tilde{G}(\tilde{\exp} (-\nu) \otimes \bar{\tau}).$$

The triple $(\tilde{P}, -\nu, \bar{\tau})$ is a set of Langlands data (in the subrepresentation version of Langlands’ classification). It follows that $\Pi$ has a unique irreducible subrepresentation, and consequently, $\Pi$ has a unique irreducible quotient. Denote this quotient by $\pi$.

Recall that two parabolic subgroups of $G$ are called opposite if their intersection is a Levi subgroup of each of them. Let $P^-$ denote the opposite
parabolic subgroup of $P$ and $\tilde{P}$ be the unique standard parabolic subgroup conjugate to $P^-$. The integral defining

$$J_{\tilde{P}^-}(\exp \nu \otimes \tau)$$

is absolutely convergent ([10, Théorème 2.4.1]), so it defines an intertwining operator

$$\Pi \to \Pi' = \text{Ind}_{\tilde{P}^-}^{\tilde{G}} (\exp \nu \otimes \tau).$$

We claim that the image of $J_{\tilde{P}^-}(\exp \nu \otimes \tau)$ is irreducible. Let $w_\ell$ denote the longest element in the Weyl group of $G$ (or a representative thereof). Since $w_\ell(\Pi') \cong \Pi'$, we get an intertwining operator

$$\mathcal{J} : \Pi \to \Pi'' = \text{Ind}_{\tilde{P}}^{\tilde{G}} (\exp w_\ell(\nu) \otimes w_\ell(\tau)).$$

Again, we have a set of Langlands data $(\tilde{P}, w_\ell(\nu), w_\ell(\tau))$. It follows that $\Pi''$ has a unique irreducible subrepresentation; denote it by $\pi''$. Let $\pi' \cong \pi''$ be the corresponding subquotient in $\Pi'$. We claim $\pi \cong \pi'$.

From [3, Remark 4.5], we know

$$\exp (-\nu) \otimes \tilde{\tau} \leq r_{\tilde{P}}(\tilde{\pi})$$

and $\tilde{\pi}$ is the only irreducible subquotient of $\tilde{\Pi}$ having $\exp (-\nu) \otimes \tilde{\tau}$ in its Jacquet module with respect to $\tilde{P}$. Then, using [7] and [3],

$$\exp \nu \otimes \tau \leq r_{\tilde{P}}(\pi).$$

On the other hand,

$$\exp w_\ell(\nu) \otimes w_\ell(\tau) \leq r_{\tilde{P}}(\pi'')$$

and again we have uniqueness. Conjugation with $w_\ell$ gives

$$\exp \nu \otimes \tau \leq r_{\tilde{P}}(\pi').$$

Now, uniqueness implies $\pi \cong \pi'$. Finally, simple properties of intertwining operators tell us that $\pi'$ is the image of $J_{\tilde{P}^-}(\exp \nu \otimes \tau)$.

We summarize:

**Theorem 3.2.** The Langlands subrepresentation $L(P, \nu, \tau)$ may be realized as the image of the standard intertwining operator $J_{\tilde{P}^-}(\exp \nu \otimes \tau)$.

**Remark 3.3.** As in [9, Lemma 1.1], the above discussion may be used to relate the data for the subrepresentation version of Langlands’ classification with that for the quotient version, as well as relating the data for $\tilde{\pi}$ with that for $\pi$. 
4. Duality

In this section, we define the duality operator \( D_{\hat{G}} \) and study its properties. Following [1], for \( I, J \subset S \), define

\[
D(I, J) = \{ w \in W \mid w^{-1} \cdot I \subset \Phi^+, w \cdot J \subset \Phi^+ \}.
\]

We denote by \( w_J \) the longest element of \( D(J, \emptyset) \).

The following covers the basic properties (1.1)–(1.4) of [1], which are the key properties needed to prove the basic properties of duality:

**Proposition 4.1.** Let \( \hat{M}, \hat{L} \) be the standard Levi factors of \( \hat{G} \) corresponding to \( I, J \subset S \).

1. If \( \hat{L} < \hat{M} \), then \( i_{\hat{M}} \circ i_{\hat{M}}^L = i_{\hat{L}} \circ i_{\hat{L}}^{\hat{M}} = r_{\hat{L}} \).
2. We have

\[
r_{\hat{M}} \circ i_{\hat{L}} = \sum_{w \in D(I, J)} i_{\hat{M}'}^M \circ w \circ r_{\hat{L}'}^L,
\]

where \( \hat{L}' = \hat{L} \cap w^{-1}(\hat{M}) \) and \( \hat{M}' = \hat{M} \cap w(\hat{L}) \).
3. If \( \hat{M} = wLw^{-1} \) for an element \( w \in W \), then

\[
i_{\hat{M}}^L \circ w = i_{\hat{L}}^L.
\]
4. If \( \hat{M} = w_J^{-1}\hat{L}w_J \), then

\[
\sim \circ r_{\hat{L}}^L = w_J \circ r_{\hat{M}}^L \circ \sim,
\]

where \( \sim \) denotes contragredient.

**Proof.** The first of these is simply [6, Proposition 1.9], which holds in the generality needed here. The second is done in [3, Proposition 3.3] and is essentially a corollary of [6, Theorem 5.2].

The proof of (3) in [5] relies on three results: the linear independence of characters, the Langlands classification, and the geometric lemma of [6] (or [7, Theorem 6.5]). The linear independence of characters is general, and holds for the groups we are considering (cf. [13, Lemma 1.13.1]); the Langlands classification for the groups under consideration is done in [3]. The geometric lemma is just (2) above. With these observations, the proof from [5] extends to cover the groups under consideration.

(4) In [3], we explained how parts of Casselman’s work [7] for \( G \) can be applied to \( \hat{G} \). More specifically, in [3, Section 2] we proved the structure results for \( \hat{G} \) which are a basis for Casselman’s proof in sections 4.1 and 4.2 of [7]. Then [7, Corollary 4.2.5] holds for \( \hat{G} \). Conjugation by \( w_J \) gives (4).

We define the duality operator \( D_{\hat{G}} \) on the Grothendieck group \( R(\hat{G}) \) as in [1, 12]:

\[
D_{\hat{G}} = \sum_{I \subset S} (-1)^{|I|} i_{\hat{M}'}^L \circ r_{\hat{L}'}^L.
\]
The following is [1, Théorème 1.7]:

**Theorem 4.2.** The duality operator $D_{\tilde{G}}$ has the following properties:

1. $D_{\tilde{G}} \circ \cdot = \cdot \circ D_{\tilde{G}}$.
2. For $J \subset S$, one has
   \[ D_{\tilde{G}} \circ i_{L,J} = i_{L,J} \circ D_{\tilde{G}}, \]
   \[ r_{L,J} \circ D_{\tilde{G}} = Ad(w_J) \circ D_{\tilde{L,J}} \circ r_{L,J}', \]
   where $w_J$ is as above and $J' = w_J^{-1}(J)$.
3. $D_{\tilde{G}}^2 = id$ (i.e., $D_{\tilde{G}}$ is an involution).
4. If $\pi$ is supercuspidal, $D_{\tilde{G}}(\pi) = (-1)^{|S|} \pi$.

**Proof.** (1) follows immediately from Proposition 4.1 (4) and the definition of $D_{\tilde{G}}$. Both parts of (2) follows from Proposition 4.1 (1)–(3) via the same calculations as in [1]. Note that the result of Solomon used for the first part of (2) is essentially a combinatorial identity on the Weyl group, so applies equally well to finite central extensions. (3) follows from (2) and induction/restriction in stages (Proposition 4.1 (1)) via the same calculation as in [1]. (4) is immediate from the definition.

It remains to verify that irreducibility is preserved (up to $\pm$) by duality. The proof is essentially a sketch of that from [1]. We start by reviewing the notation from [1].

Let $E$ be a $\tilde{G}$-module. For $I \subset S$, we denote by $E(U_I)$ the subspace of $E$ spanned by the elements $ux - x$, where $x \in E$ and $u \in U_I$. Set $E_{U_I} = E/E(U_I) = r_{L,I}(E)$. Define

\[ E_I = (i_{L,I} \circ r_{L,I})(E). \]

Let $I \subset J \subset S$. The natural projection from $E_{U_J}$ to $E_{U_I}$ induces a map $\phi_I^J : E_J \to E_I$. If $J = I \cup \{\alpha\}$, we define

\[ E_J = E_J \otimes \Lambda^{S-J}(C^{S-J}) \quad \text{and} \quad \phi_J^I = \phi_I^J \otimes \epsilon_J^I, \]

where $\epsilon_J^I : \Lambda^{S-J}(C^{S-J}) \to \Lambda^{S-I}(C^{S-I})$ is the map given by $\omega \mapsto \omega \wedge \alpha$. Define

\[ E_{I,J} = r_{L,I}(E_J). \]

**Theorem 4.3.** The duality operator $D_{\tilde{G}}$ takes irreducible representations to irreducible representations, up to sign.

**Proof.** The proof follows that of [1]. We outline the argument, indicating any changes needed for the case of finite central extensions. We remark that the principal changes are (1) the use of [6] in place of [7] (as the results of [6] are done in the generality of groups of totally disconnected type, rather than just reductive $p$-adic groups), and (2) a minor correction to Aubert’s proof.
Let $I \subset S$. Let $E$ be an irreducible representation of $\tilde{G}$ such that $E$ has supercuspidal support on (the associate class of) $\tilde{L}_I$.

As in Aubert, $d_k$ is defined as follows: If $|J| = k$, we let $d_k$ be defined on $\tilde{E}_J$ by

$$
\bigoplus_{I \subset J} \tilde{\phi}_I^J.
$$

It is a straightforward matter to show that

$$
0 \longrightarrow \tilde{E}_J \longrightarrow \bigoplus_{[J]=[S]-1} \tilde{E}_J \longrightarrow \bigoplus_{[J]=[S]-2} \tilde{E}_J \longrightarrow \cdots \longrightarrow \tilde{E}_\emptyset \longrightarrow 0
$$

is a complex. As in [2], the key is to show that it is in fact an exact sequence.

Suppose $w \in D(J, I)$. It is a straightforward argument to check that the map

$$
\left( \tilde{P}_j w \tilde{P}_i \circ \tilde{r}_L^G \right)(E) \longrightarrow \left( \tilde{P}_t \circ \tilde{r}_L^G \circ \text{Ad}(w^{-1}) \right)(E)
$$

with $\Phi_f$ defined by $\Phi_f(p) = f(wp)$ for all $p \in \tilde{P}_I$, is an isomorphism of $\tilde{P}_I$-modules. Taking the Jacquet modules, we may then obtain

$$
\left[ \left( \tilde{P}_j w \tilde{P}_i \circ \tilde{r}_L^G \right)(E) \right]_{U_I} \cong \left[ \left( \tilde{P}_t \circ \tilde{r}_L^G \circ \text{Ad}(w^{-1}) \right)(E) \right]_{U_I} \cong \left( \tilde{L}_I \circ \tilde{r}_L^G \circ \text{Ad}(w^{-1}) \right)(E).
$$

If this is nonzero, we must have $w\tilde{L}_I w^{-1} \cap \tilde{L}_J$ conjugate to $\tilde{L}_I$ (since $E \in \text{Alg}(\{I\})$), hence $w\tilde{L}_I w^{-1} \subset \tilde{L}_J$. In this case, we may conclude that (4.1) reduces to the following:

$$
\left[ \left( \tilde{P}_j w \tilde{P}_i \circ \tilde{r}_L^G \right)(E) \right]_{U_I} \cong \left( \tilde{L}_I \circ \tilde{r}_L^G \circ \text{Ad}(w^{-1}) \right)(E)
$$

as $\tilde{L}_I$-modules.

Choose $W = \theta_1 \supset \theta_2 \supset \cdots \supset \theta_{k+1} = \emptyset$ with $\theta_i - \theta_{i+1} = w_i$ so that $Y_j = \tilde{P}_j \theta_{j+1}, \tilde{P}_0$ satisfies the hypotheses of Theorem 5.2 [6] (for $\tilde{P}_0$ double-cosets). We note that the $\Phi^J_k$—and hence $d_k, U_I$—respect the filtration by $\theta_i$. As in Aubert’s proof, if

$$
0 \longrightarrow \tilde{E}_{U_I}^{\theta_1}/\tilde{E}_{U_I}^{\theta_1+1} \longrightarrow \bigoplus_{[J]=[S]-1} \tilde{E}_{J,I}^{\theta_1}/\tilde{E}_{J,I}^{\theta_1+1} \longrightarrow \bigoplus_{[J]=[S]-2} \tilde{E}_{J,I}^{\theta_1}/\tilde{E}_{J,I}^{\theta_1+1} \longrightarrow \cdots \longrightarrow \bigoplus_{[J]=[I]} \tilde{E}_{J,I}^{\theta_1}/\tilde{E}_{J,I}^{\theta_1+1}
$$

is exact for all $i = 1, \ldots, t$ (where $\tilde{E}_{J,I}^{\theta_i}$ denotes the image of the elements of $\tilde{E}_{J,I}$ supported on $\tilde{P}_0 \theta_i \tilde{P}_0$, then

$$0 \to E_{U_J} \xrightarrow{d_{|S|-1}} \bigoplus \tilde{E}_{J,I} \xrightarrow{d_{|S|-2}} \bigoplus \tilde{E}_{J,I} \cdots \xrightarrow{d_{|J|-1}} \bigoplus \tilde{E}_{J,I}$$

is also exact.

We now make a change--a minor correction, actually--to Aubert’s setup. We keep $\Theta$ the same as for Aubert: $\Theta$ consists of subsets $\theta \subset W$ having the property that if $w \in \theta$, then $w' \in \theta$ for any $w'$ having $\ell(w') > \ell(w)$. Suppose $I, J$ fixed as above. We define $\theta' \subset \theta$ to be the largest subset which is left-invariant under multiplication by $W_J$ and right-invariant under multiplication by $W_I$. In particular, a function in $E_{J,I}^{\theta'}$ is actually supported on $J, I$ and  $E_{J,I}^{\theta'}$ consists of minimal length double-coset representatives, the claim follows. Further, as claimed.

As in [1], it now suffices to show the exactness of $E_{J,I}^{\theta'/\theta}$ and $E_{J,I}^{\theta'/\theta}$ for all $i$. Let $\tilde{E}_{J,I}^{\theta'/\theta} \cong \text{Ad}(w^{-1}) \circ r_{\tilde{W}_{J,I}w^{-1}}(E)$ as $\tilde{E}_{J,I}$-modules (conversely, if $w \in D(J, I)$ and $w W_J w^{-1} \subset W_J$, then $E_{J,I}^{\theta'/\theta} \cong E_{J,I}^{\theta'/\theta}$). In the notation of section 5 [6], $E_{J,I}^{\theta'/\theta}$ plays the role of $F_Y$ (more precisely, $F_Y$ applied to $r_{\tilde{W}_{J,I}w^{-1}}(E)$). Then $E_{J,I}^{\theta'/\theta}$ appears in the role of $F_Z$ (again, applied to $r_{\tilde{W}_{J,I}w^{-1}}(E)$). Let $v$ be a representative of $w^{-1}$ in $\tilde{G}$ (note that (3) on p. 460 in [6] wants $\tilde{w} \in \tilde{G}$ such that $P_J \tilde{w}^{-1} \subset Z$; our $v$ plays this role). By Theorem 5.2 [6],

$$E_{J,I}^{\theta'/\theta} \cong \Phi_Z(r_{\tilde{M}_J}(E)) \cong \tilde{W}_{J,I}v w^{-1}(M_J) \circ r_{\tilde{W}_{J,I}w^{-1}}(E).$$

Since $E \in \text{Alg}(I)$, we must have $\tilde{M}_I \subset v(M_J)$, so $v^{-1} \tilde{M}_I v \subset \tilde{M}_J \Rightarrow w W_J w^{-1} \subset W_J$, as claimed. Therefore, we have

$$E_{J,I}^{\theta'/\theta} \cong \tilde{W}_{J,I}v w^{-1}(M_J) \circ r_{\tilde{W}_{J,I}w^{-1}}(E) \cong w^{-1} \circ r_{\tilde{W}_{J,I}w^{-1}}(E),$$

also as claimed.

As in [1], it now suffices to show the exactness of

$$\cdots \to \bigoplus_{|J| = k} w_i^{-1} \circ r_{\tilde{W}_{J,I}w_i^{-1}}(E) \to \cdots.$$
Note that $w_i \in \mathcal{D}(J, I)$ with $w_i W_1 w_i^{-1} \subset W_J$ if and only if $w_i \in \mathcal{D}(J, \emptyset)$ with $w_i(I) \subset J$. If we let $S^{w_i} = \{ \alpha \mid w_i^{-1} \cdot \alpha > 0 \}$, we see that $w_i \in \mathcal{D}(J, \emptyset)$ is equivalent to $J \subset S^{w_i}$. Thus, we are reduced to showing the exactness of

$$\cdots \rightarrow \bigoplus_{J \cap I \subset J \subset S^{w_i}} w^{-1} \circ r_{w \overline{M}_J w^{-1}}(E) \xrightarrow{\delta} \cdots$$

The next step is to identify $\delta_k$. Consider the following part of a commutative diagram:

$$\tilde{E}_{J,I}^f / \tilde{E}_{I,J}^f \xrightarrow{\delta_j} \bigoplus_{\alpha \in J} \tilde{E}_{J-(\alpha),I}^f / \tilde{E}_{J-(\alpha),J}^f$$

$$\downarrow F_{j}^{J,I} \quad \downarrow \sum_{\alpha \in J} F_{j}^{J-(\alpha),I}$$

$$\text{Ad}(w_j)(r_{w_j \overline{M}_J w_j^{-1}}(E)) \xrightarrow{\delta} \bigoplus \text{Ad}(w_j)(r_{w_j \overline{M}_J w_j^{-1}}(E)).$$

Here, $F_{J,I}^f$ is inherited from $F_j = F_{\theta_j} / F_{\theta_{j+1}}$ (as in section 5 [6]) (note that up to sign, the image of $F_{J,I}^f$ is $\Phi_j^f(E_J) \cong \text{Ad}(w_j)(r_{w_j \overline{M}_J w_j^{-1}}(E)))$. We would like to show that up to sign, $\delta$ is just the identity.

To make the connection between our setup and that of [6] clear, we pause to note that for [6], $P = \tilde{P}_J$, $Q = \tilde{P}_I$, and $w = w_j^{-1}$. Recall that the equivalence $F_Z \cong \Phi_Z$ is defined by an intertwining operator $A$ constructed from an intertwining operator $\tilde{A}$ with certain properties. In our case,

$$\bar{A}f = \int_{U_I \cap w_j^{-1}(\tilde{P}_J) \setminus U_I} r_{w(\check{M}_I),\check{M}_J}(f(w_j \hat{u}_I \check{M}_J)) \, d\mu(\hat{u}_I).$$

Therefore, if $f + E_{J,I}^{\theta_j+1} \in E_{J,I}^f / E_{I,J}^f$,

$$F_{J,I}^f(f + E_{J,I}^{\theta_j+1}) = \int_{U_I \cap w_j^{-1}(\tilde{P}_J) \setminus U_I} r_{w(\check{M}_I),\check{M}_J}(f(w_j \hat{u}_I \check{M}_J)) \, d\mu(\hat{u}_I).$$

On the other hand,

$$F_{J-(\alpha),J}^f \circ \phi_{J-(\alpha)}^f(f + E_{J,I}^{\theta_j+1}) = F_{J-(\alpha),J}^f(r_{J-(\alpha)}^f(f) + E_{J-(\alpha),J}^{\theta_j+1})$$

$$= \int_{U_I \cap w_j^{-1}(\tilde{P}_J-(\alpha)) \setminus U_I} r_{w(\check{M}_I),\check{M}_J-(\alpha)}(r_{J-(\alpha)}^f(f)(w_j \hat{u}_I \check{M}_J)) \, d\mu(\hat{u}_I).$$

Observe that $r_{w(\check{M}_I),\check{M}_J-(\alpha)} \circ r_{J-(\alpha)}^f = r_{w(\check{M}_I),\check{M}_J}$. As

$$w \in \mathcal{D}(J, I), \alpha \in J \Rightarrow U_I \cap w_j^{-1}(\tilde{P}_J-(\alpha)) = U_I \cap w_j^{-1}(\tilde{P}_J),$$

these integrals are the same. Therefore, up to sign, $\delta$ is the identity.

The rest of the proof is now the same as in [1]. □
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D. Ban
Department of Mathematics
Southern Illinois University
Carbondale, IL 62901
USA
E-mail: dban@siu.edu

C. Jantzen
Department of Mathematics
East Carolina University
Greenville, NC 27858
USA
E-mail: jantzenc@ecu.edu

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