

Spectral Analysis of Generalized Triangular and Welch Window Functions using Fractional Fourier Transform

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Original scientific paper

The paper presents a new closed-form expression for the fractional Fourier transform of generalized Triangular and Welch window functions. Fractional Fourier Transform (FrFT) is a parameterized transform having an adjustable transform parameter which makes it more flexible and superior over ordinary Fourier transform in several applications. It is an important tool used in signal processing for spectral analysis. The analysis of generalized Triangular and Welch window functions in fractional Fourier domain establishes a direct relationship between their FrFTs and fractional angle. Based on the mathematical model obtained, it is observed that adjustable spectral parameters of these functions can be obtained by modifying the fractional angle. The various values of spectral parameters such as half main-lobe width, side lobe fall-off rate and maximum side-lobe level with change in order of fractional Fourier transform are also obtained for these functions.

Key words: Fractional Fourier transform, Generalized triangular function, Spectral analysis, Welch window, Window function

Spektralna analiza poopćene trokutaste i Welchove prozorske funkcije korištenjem frakcijske Fourierove transformacije. U radu je prikazan novi izraz za zatvoreni oblik frakcijske Fourierove transformacije poopćene trokutaste i Welchove prozorske funkcije. Frakcijska Fourierova transformacija (FrFT) parametrizirana je transformacija s podesivim parametrom transformacije koja je u određenim primjenama fleksibilnija i superiornija u odnosu na uobičajenu Fourierovu transformaciju. Ističe se kao važan alat u obradi signala i spektralnoj analizi. Analiza poopćene trokutaste i Welchove prozorske funkcije u području frakcijske Fourierove transformacije uspostavlja izravni odnos između FrFT-a i frakcijskog kuta. Koristeći dobiveni matematički model, uočeno je da se podesivi spektralni parametri ovih funkcija mogu izvesti mijenjanjem frakcijskog kuta. Različite vrijednosti spektralnih parametara, kao što su polovica širine spektralnog vrha, stopa snižavanja amplituda viših harmonika ili najveća amplituda viših harmonika, odnosno njihova ovisnost u odnosu na red frakcijske Fourierove transformacije, također se mogu odrediti upotrebom ovih funkcija.

Ključne riječi: Frakcijska Fourierova transformacija, poopćena trokutasta funkcija, spektralna analiza, Welchov prozor, prozorska funkcija

1 INTRODUCTION

Windows are weighting functions that attenuate signals at their discontinuities. The window functions are applied to the time-domain signals and the process of multiplying the signal with the smoothly ending window function is called windowing technique. Windowing is done to make an infinitely long function finite in length so that frequency content of a signal of interest can be measured. As a result of this, the truncated signal exhibits different spectral characteristics from the original continuous-time signal. Frequency domain characteristics of several window functions are analyzed to determine their suitability for a specific application and to reduce the spectral leakage that results because of selecting a finite time interval signal.

Window functions are real, non-zero and time limited functions. They peak in the middle frequencies and decrease to zero at the edges in order to reduce the effects of the discontinuities that results because of finite duration. They are frequently used in various areas of signal processing and communications such as speech processing, digital filter design, and spectrum estimation. No existing window function is best in all aspects [16].

Thus, it is needed to select an appropriate window function according to the requirement of a particular application on the basis of the performance features [4], such as the attenuation at the maximum height of a side lobe, generally the first side-lobe (the side-lobe level), the rate at which peak of the side-lobes decrease in magnitude (side-

lobe fall-off rate) and the main-lobe width of (width of main-lobe at -3dB below main-lobe peak). The narrower the main-lobe width, the better will be the frequency resolution; and the lower the side lobe level, the better will be the amplitude accuracy or noise suppression. The narrow main-lobe width and reduced side-lobe level are conflicting requirements. When the main lobe width decreases, the remaining energy spreads out to side-lobes thereby increasing spectral leakage. Thus, the problem lies in deciding which window function is the best to apply on the signal being studied in order to estimate the spectral characteristics of a finite duration signal.

There are numerous standard window functions that can be chosen for the prevention of spectral leakage in the signal and to provide the specified side-lobe level [7, 10]. But the reduction of side-lobe leakage due to the applied window function introduces leakage from the expansion of main-lobe in ordinary frequency domain. This reduces spectral resolution and also some gain is lost because of main-lobe spreading. Thus, we need to increase the length of the selected window function to improve spectral resolution. Applying fractional Fourier transform to the window function, one can improve spectral resolution with no need of changing the length of the window function [9, 15]. Therefore, computational time and design time can be saved.

This paper presents a new mathematical model for obtaining the FrFT of generalized Triangular and Welch window functions. Based on which, it is found that these functions can be used as an adjustable windows in fractional Fourier domain for doing the signal spectral analysis. By changing the value of fractional order parameter, the spectral parameters such as Half Main-Lobe Width (HMLW), Maximum Side-Lobe Level (MSLL) and Side-Lobe Fall-Off Rate (SLFOR) of the resulting windows can be controlled. Therefore, trade-off problem between the frequency resolution and spectral leakage can be easily solved. The variations in spectral parameters of these window functions are studied for different values of fractional order parameter. A plot of spectrum of Welch window function is shown below in Fig. 1 to define these spectral parameters.

1. Half Main-Lobe Width (**HMLW**): It is the frequency at which the Main lobe drops to the peak ripple value of the side lobes.
2. Maximum Side-Lobe Level (**MSLL**): It is the largest side lobe level in decibels relative to the main lobe peak gain.
3. Side-Lobe Fall-Off Rate (**SLFOR**): It is the asymptotic decay rate of side-lobe level in decibels per

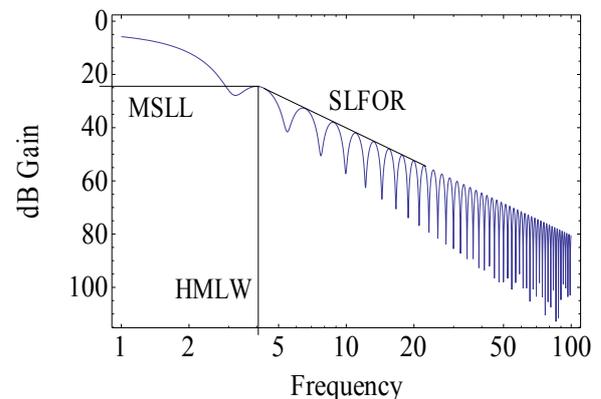


Fig. 1. Frequency Response (Log Magnitude Plot) of Welch window to define spectral parameters

decade/octave of frequency of the peaks of the side lobes.

It is proposed in the work that by adjusting fractional order parameter to different values, main-lobe width can be minimized and side-lobe fall-off rate can be raised to maximum. Thus, a choice can be made between detection and resolution. Detection means detecting a desired signal in the presence of noise. Resolution refers to the ability of distinguishing narrowband spectral components.

2 FRACTIONAL FOURIER TRANSFORM

FrFT is the generalization of ordinary Fourier transform (FT) that depends on a parameter α and can be interpreted as a rotation by an angle α in the time-frequency plane [2, 11, 12]. It is represented by R^α and $\alpha = a\pi/2$, where α is the angle of rotation and a is the fractional order parameter. The time domain and frequency domain are the special cases of the FrFT domain with α being $2n\pi$ and $2n\pi + \pi/2$, respectively, where n is an integer. It can also be viewed as a fractional power of the Fourier operator. It is more affluent in theory, more stretchy in applications, and implementation cost is same as that of computing FT [13, 14].

The continuous fractional Fourier transform (CFrFT) [2, 12] of a signal $x(t)$ represented along time axis denoted by t , with rotation angle α is computed as:

$$F^\alpha[x(t)] = X_\alpha(u) = \int_{-\infty}^{\infty} x(t)K_\alpha(t, u)dt, \quad (1)$$

where the transform kernel $K_\alpha(t, u)$ of the FrFT is given

by:

$$K_\alpha(t, u) = \begin{cases} \sqrt{\frac{1-j \cot \alpha}{2\pi}} \exp \left\{ j \frac{t^2+u^2}{2} \cot \alpha - jut \csc \alpha \right\}, & \text{if } \alpha \text{ is not a multiple of } \pi, \\ \delta(t-u), & \text{if } \alpha \text{ is a multiple of } 2\pi, \\ \delta(t+u), & \text{if } \alpha + \pi \text{ is a multiple of } 2\pi, \end{cases} \quad (2)$$

with

$$C_\alpha = \sqrt{\frac{1-j \cot \alpha}{2\pi}}, \quad p = \frac{1}{2} \cot \alpha \text{ and } q = \csc \alpha. \quad (3)$$

Here $\delta(t)$ denotes Dirac-delta distribution. The Fourier transform is a special case of FrFT for $\alpha = \frac{\pi}{2}$.

The FrFT is a unified time-frequency transform, which reveals the characteristics of a signal gradually changing from time domain to frequency domain with the fractional order α changing from 0 to 1. Fractional Fourier Transform (FrFT) is a parameterized transform with an adjustable transform parameter. It has an extra degree of freedom and can achieve better performance over ordinary Fourier transform. FrFT has become a very efficient mathematical tool in the various applications like harmonic analysis, signal synthesis, time-frequency analysis, digital watermarking, image encryption, modulation and multiplexing in communications, etc. FrFT holds all the properties of Fourier transform and therefore, has tremendous potential for improvement in areas where FT has been used [3, 5, 8]. The development of fast algorithms for computation of FrFT has made application of FrFT functional in real-time digital signal processing.

3 DERIVATION OF FRFT OF GENERALIZED TRIANGULAR FUNCTION

The generalized Triangular function denoted by $x(t)$ for any scaling parameter $L \neq 0$, is defined as:

$$x(t) = \begin{cases} 1 - \frac{t}{L} & 0 \leq t \leq L \\ 1 + \frac{t}{L} & -L \leq t \leq 0 \\ 0 & \text{otherwise} \end{cases}. \quad (4)$$

For parameter value, $L = 1$, $x(t)$ is equivalent to Triangular window.

The FrFT $X_\alpha(u)$ of generalized Triangular function $x(t)$ is computed as follows: Substituting $x(t)$ in (1) results:

$$X_\alpha(u) = C_\alpha \exp(jpu^2) \times \left(\int_{-L}^0 \left(1 + \frac{t}{L}\right) \exp(jpt^2 - jqut) dt + \int_0^L \left(1 - \frac{t}{L}\right) \exp(jpt^2 - jqut) dt \right). \quad (5)$$

Changing t by $-t$ in the first integral, gives:

$$X_\alpha(u) = C_\alpha \exp(jpu^2) \times \left(\int_0^L \left(1 - \frac{t}{L}\right) \exp(jpt^2 + jqut) dt + \int_0^L \left(1 - \frac{t}{L}\right) \exp(jpt^2 - jqut) dt \right). \quad (6)$$

Equation (6) can be rewritten as:

$$X_\alpha(u) = C_\alpha \exp(jpu^2) (I_1 + I_2), \quad (7)$$

where

$$I_1 = \int_0^L \left(1 - \frac{t}{L}\right) \exp(jpt^2 + jqut) dt, \quad (8)$$

$$I_2 = \int_0^L \left(1 - \frac{t}{L}\right) \exp(jpt^2 - jqut) dt. \quad (9)$$

Solving for I_1 separately, the integral can be written as:

$$I_1 = \underbrace{\int_0^L 1 \exp(jpt^2 + jqut) dt}_{I_{11}} - \frac{1}{L} \underbrace{\int_0^L t \exp(jpt^2 + jqut) dt}_{I_{12}} \quad (10)$$

Now, first solving for $I_{12} = \int_0^L t \exp(jpt^2 + jqut) dt$, the integral can be rewritten as:

$$I_{12} = \frac{1}{2jp} \int_0^L (2jpt + jqu - jqu) \exp(jpt^2 + jqut) dt, \quad (11a)$$

or

$$I_{12} = \frac{1}{2jp} \int_0^L (2jpt + jqu) \exp(jpt^2 + jqut) dt - \frac{1}{2jp} \int_0^L (jqu) \exp(jpt^2 + jqut) dt = \frac{1}{2jp} \int_0^L \frac{d}{dt} \{ \exp(jpt^2 + jqut) \} - \frac{qu}{2p} \int_0^L \exp(jpt^2 + jqut) dt. \quad (11b)$$

Simplifying,

$$I_{12} = \frac{1}{2jp} \left[\exp(jpt^2 + jqut) \Big|_0^L - \frac{qu}{2p} \underbrace{\int_0^L \exp(jpt^2 + jqut) dt}_{I_{11}} \right]$$

$$= \frac{1}{2jp} (\exp(jpL^2 + jqL) - 1) - \frac{qu}{2p} I_{11}. \quad (11c)$$

Now, solving the integral $I_{11} = \int_0^L \exp(jpt^2 + jqut) dt$, the following expression results [1]:

$$I_{11} = \frac{\sqrt{\pi}}{2\sqrt{jp}} \exp\left(\frac{-jq^2u^2}{4p}\right) \times \left(\operatorname{Erfi}\left[\sqrt{j}\frac{(qu + 2pL)}{2\sqrt{p}}\right] - \operatorname{Erfi}\left[\sqrt{j}\frac{qu}{2\sqrt{p}}\right] \right), \quad (12)$$

where $\operatorname{Erfi}(z)$ is imaginary error function of z , which is defined in the whole complex z -plane.

By using (11c) and (12), solving for (10), one gets:

$$I_1 = \frac{\sqrt{\pi}}{2\sqrt{jp}} \left(1 + \frac{qu}{2pL}\right) \exp\left(\frac{-jq^2u^2}{4p}\right) \times \left(\operatorname{Erfi}\left[\sqrt{j}\frac{(qu + 2pL)}{2\sqrt{p}}\right] - \operatorname{Erfi}\left[\sqrt{j}\frac{qu}{2\sqrt{p}}\right] \right) - \frac{1}{2jpL} (\exp(jpL^2 + jqL) - 1). \quad (13)$$

By using $\operatorname{Erfi}(z) = -j\operatorname{Erf}(jz)$ and simplifying (13), one gets:

$$I_1 = \frac{\sqrt{j}}{2Lp^{3/2}} \exp\left(\frac{-jq^2u^2}{4p}\right) \left\{ \sqrt{\pi} (2pL + qu) \times \left(-\operatorname{Erf}\left[j^{3/2}\frac{(qu + 2pL)}{2\sqrt{p}}\right] + \operatorname{Erf}\left[j^{3/2}\frac{qu}{2\sqrt{p}}\right] \right) + \sqrt{jp} \left(\exp\left(\frac{jq^2u^2}{4p}\right) - \exp\left(\frac{j(qu + 2pL)^2}{4p}\right) \right) \right\}. \quad (14)$$

Similarly I_2 can be computed by replacing qu by $-qu$ in equation (14), i.e.,

$$I_2 = \frac{\sqrt{j}}{2Lp^{3/2}} \exp\left(\frac{-jq^2u^2}{4p}\right) \left\{ \sqrt{\pi} (2pL - qu) \times \left(-\operatorname{Erf}\left[j^{3/2}\frac{(-qu + 2pL)}{2\sqrt{p}}\right] + \operatorname{Erf}\left[-j^{3/2}\frac{qu}{2\sqrt{p}}\right] \right) + \sqrt{jp} \left(\exp\left(\frac{jq^2u^2}{4p}\right) - \exp\left(\frac{j(-qu + 2pL)^2}{4p}\right) \right) \right\}. \quad (15)$$

Solving for equation (7) by using (3), (14) and (15),

one gets:

$$X_\alpha(u) = A \left(\begin{matrix} B \left(\begin{matrix} 2 \exp(ju^2 \csc 2\alpha) - \\ \exp\left(\frac{j(\cot \alpha L + \csc \alpha u)^2}{2 \cot \alpha}\right) - \\ \exp\left(\frac{j(\cot \alpha L - \csc \alpha u)^2}{2 \cot \alpha}\right) \end{matrix} \right) + \\ C \left(\begin{matrix} \operatorname{Erf}\left[(-1)^{3/4} \sqrt{\csc \alpha} u\right] - \\ \operatorname{Erf}\left[(-1)^{3/4} \frac{\cot \alpha L + \csc \alpha u}{\sqrt{2 \cot \alpha}}\right] \end{matrix} \right) + \\ D \left(\begin{matrix} \operatorname{Erf}\left[(-1)^{3/4} \sqrt{\csc \alpha} u\right] - \\ \operatorname{Erf}\left[(-1)^{3/4} \frac{\cot \alpha L - \csc \alpha u}{\sqrt{2 \cot \alpha}}\right] \end{matrix} \right) \end{matrix} \right), \quad (16)$$

where

$$A = \sqrt{\frac{1 - j \cot \alpha}{\pi \cot^3 \alpha}} \frac{1}{L} \left(\frac{1}{4} + \frac{j}{4}\right) \exp\left(\frac{ju^2 \tan \alpha}{2}\right), \quad (17)$$

$$B = (-\sqrt{2} - \sqrt{2j}) \sqrt{\cot \alpha}, \quad (18)$$

$$C = (\cot \alpha L + \csc \alpha u) \sqrt{2\pi}, \quad (19)$$

$$D = (\cot \alpha L - \csc \alpha u) \sqrt{2\pi}. \quad (20)$$

From equation (16) to (20), it can be seen that FrFT of generalized Triangular function is directly dependent on the fractional angle α .

4 DERIVATION OF FRFT OF WELCH WINDOW

The Welch window function denoted by $x(t)$, is defined as:

$$x(t) = \begin{cases} 1 - t^2 & -1 \leq t \leq 1 \\ 0 & \text{elsewhere} \end{cases} \quad (21)$$

The FrFT $X_\alpha(u)$ of Welch window is computed as follows: Substituting $x(t)$ in (1) results:

$$X_\alpha(u) = C_\alpha \exp(jpu^2) \times \left(\int_{-1}^1 (1 - t^2) \exp(jpt^2 - jqut) dt \right). \quad (22)$$

Equation (22) can be rewritten as:

$$X_\alpha(u) = C_\alpha \exp(jpu^2) \left(\underbrace{\int_{-1}^1 \exp(jpt^2 - jqut) dt}_{I_3} - \underbrace{\int_{-1}^1 t^2 \exp(jpt^2 - jqut) dt}_{I_4} \right). \quad (23)$$

First solving for I_4 separately, the integral can be written as:

$$I_4 = \int_{-1}^1 \underbrace{t \exp(-jqut)}_u \cdot \underbrace{t \exp(jpt^2)}_{dv} dt. \quad (24)$$

Let $u = t \exp(-jqut)$ and $v = t \exp(jpt^2) dt$, which gives: $du = -jqut \exp(-jqut) dt + \exp(-jqut) dt$ and $v = \frac{1}{2jp} \exp(jpt^2)$. Now, solving equation (24) via integrating by parts, one gets:

$$I_4 = \left| \frac{t}{2jp} \exp(jpt^2 - jqut) \right|_{-1}^1 + \frac{1}{2jp} \underbrace{\int_{-1}^1 (jqut - 1) \exp(jpt^2 - jqut) dt}_{I_5} = \frac{1}{2jp} (\exp(jp - jq) + \exp(jp + jq)) + \frac{1}{2jp} I_5. \tag{25}$$

Similarly solving for I_5 , the integral can be written as:

$$I_5 = jq \underbrace{\int_{-1}^1 t \exp(jpt^2 - jqut) dt}_{I_6} - \underbrace{\int_{-1}^1 \exp(jpt^2 - jqut) dt}_{I_3}. \tag{26}$$

Solving (26) for I_6 in a similar manner as solving for I_{12} using (11a):

$$I_6 = \frac{1}{2jp} \left| \exp(jpt^2 - jqut) \right|_{-1}^1 + \frac{qu}{2p} \underbrace{\int_{-1}^1 \exp(jpt^2 - jqut) dt}_{I_3} = \frac{1}{2jp} (\exp(jp - jq) - \exp(jp + jq)) + \frac{qu}{2p} I_3. \tag{27}$$

By using (26) and (27), solving for I_5 , one gets:

$$I_5 = \frac{qu}{2p} (\exp(jp - jq) - \exp(jp + jq)) + \left(\frac{jq^2u^2}{2p} - 1 \right) I_3. \tag{28}$$

By using (25) and (28), solving for I_4 , one gets:

$$I_4 = \frac{1}{2jp} (\exp(jp - jq) + \exp(jp + jq)) + \frac{qu}{4jp^2} (\exp(jp - jq) - \exp(jp + jq)) + \frac{1}{2jp} \left(\frac{jq^2u^2}{2p} - 1 \right) I_3. \tag{29}$$

Now, solving the integral $\int_{-1}^1 \exp(jpt^2 - jqut) dt$ the following expression results [1]:

$$I_3 = \frac{\sqrt{\pi}}{2\sqrt{jp}} \exp\left(\frac{-jq^2u^2}{4p}\right) \left(Erfi \left[\sqrt{j} \frac{(-qu + 2p)}{2\sqrt{p}} \right] + Erfi \left[\sqrt{j} \frac{(qu + 2p)}{2\sqrt{p}} \right] \right). \tag{30}$$

By using (29) and (30), solving for (22) and rearranging, one gets:

$$X_\alpha(u) = C_\alpha \frac{\exp(jpu^2 - jq^2u^2/4p)}{8j^{3/2}p^{5/2}} \left\{ \exp[jp - jq] + (jq^2u^2/4p) \sqrt{j} p [-2qu + 2qu \exp(2jq) - 4p - 4p \exp(2jq)] + \sqrt{\pi} (4jp^2 - jq^2u^2 + 2p) \left(Erfi \left[\sqrt{j} \frac{(-qu + 2p)}{2\sqrt{p}} \right] + Erfi \left[\sqrt{j} \frac{(qu + 2p)}{2\sqrt{p}} \right] \right) \right\}. \tag{31}$$

Putting values of C_α , p and q using (3) in (31) and simplifying, one gets:

$$X_\alpha(u) = A_1 \left\{ \exp\left(\frac{j}{2} \cot \alpha (u - 1)^2\right) [-2u \csc \alpha + 2u \csc \alpha \exp(2ju \csc \alpha) - 2 \cot \alpha - 2 \cot \alpha \exp(2ju \csc \alpha)] + B_1 \left(Erf \left[(-1)^{3/4} \frac{(\cot \alpha - u \csc \alpha)}{\sqrt{2 \cot \alpha}} \right] + Erfi \left[(-1)^{3/4} \frac{(\cot \alpha + u \csc \alpha)}{\sqrt{2 \cot \alpha}} \right] \right) \right\}, \tag{32}$$

where

$$A_1 = \frac{1}{4j^{3/2}} \sqrt{\frac{1 - j \cot \alpha}{\pi \cot^3 \alpha}}, \tag{33}$$

$$B_1 = \sqrt{\pi} (2j \cot^2 \alpha + ju^2 \csc^2 \alpha - \cot \alpha). \tag{34}$$

From equation (32) to (34), it can be seen that FrFT of Welch window function is directly dependent on the fractional angle α .

5 RESULTS AND DISCUSSIONS

The plot of Triangular function for $-0.5 \leq t \leq 0.5$ is shown in Fig. 2. The plot of Welch window as a function of time is shown in Fig. 3. The magnitude of FrFT

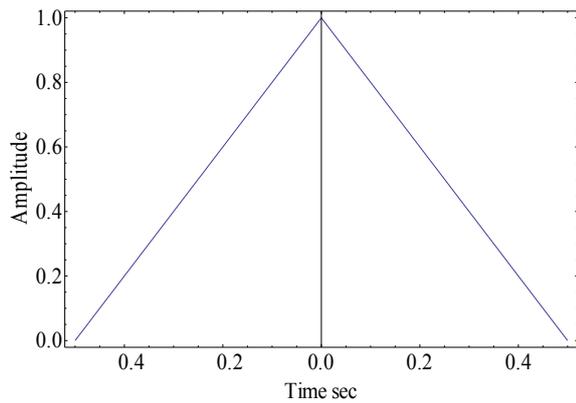


Fig. 2. The Triangular window function in time-domain

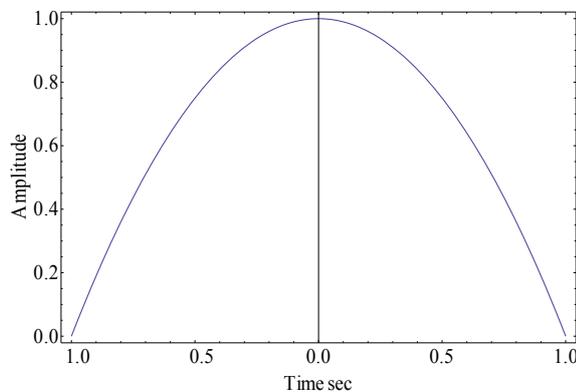


Fig. 3. The Welch window in time-domain

of these functions i.e. $|X_\alpha(u)|$ in dB is plotted versus frequency u (cycles/sec). Figure 4 shows the plot for calculating MSLL for generalized Triangular function by varying scaling parameter L to different values keeping fractional order $a = 0.5$. Figures 5 illustrate the plot for calculating MSLL for Triangular function by varying fractional order parameter to different values keeping scaling parameter $L = 0.5$. Figure 7 shows the MSLL plot for Welch window for different values of fractional order. The plots for calculating SLFOR for these functions are also shown in Fig. 6 and 8. The continuum of fractional Fourier transform of Triangular and Welch window functions to sinc as the fractional order is varied from 0 to 1 are also shown in Fig. 9 and Fig. 10 respectively.

The values of MSLL, HMLW and SLFOR for Triangular function (scaling parameter $L = 0.5$) and Welch window function are tabulated in Table 1 and Table 2 respectively for various values of fractional order parameter a . It is observed from the Figs. and Tables that spectral parameters of Triangular function and Welch window depend upon the value of fractional order parameter a in time-frequency plane. Main-lobe width of both the func-

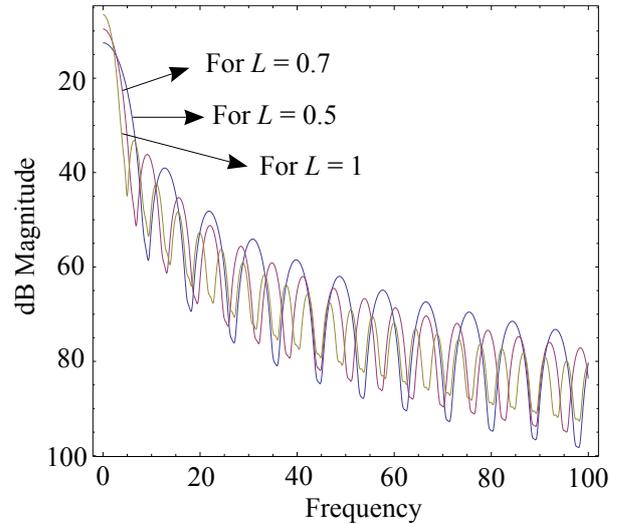


Fig. 4. MSLL plot for generalized Triangular function (fractional order $a = 0.5$) for values of scaling parameter $L = 0.5, 0.7$ and 1

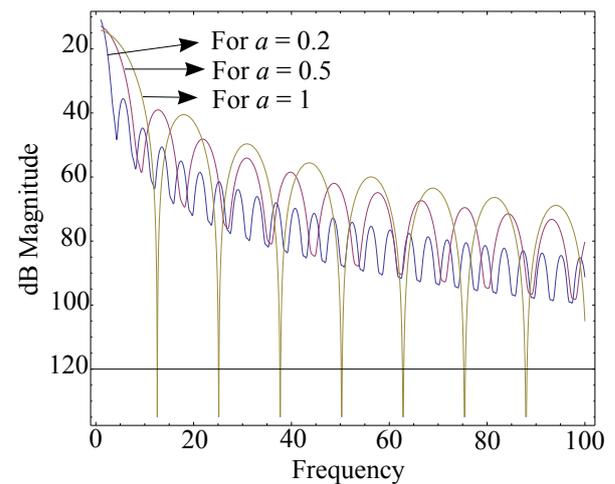


Fig. 5. MSLL plot for Triangular function (scaling parameter $L = 0.5$) for fractional orders $a = 0.2, 0.5$ and 1

tions increases regularly with increase in fractional order a . SLFOR for Triangular function shows variation between -8.9 dB/octave to -11 dB/octave with change in order parameter a . MSLL is also reduced with increase in fractional order a , e.g. MSLL for $a = 0.7$ is -25.8 db compared to is -27.8 db for $a = 1$. If the value of parameter L is taken to be equal to 1, Fig. 11 shows the spectral parameters of triangular window at fractional order $a = 1$.

From Table 2, the SLFOR for Welch window shows variation between -7.8 dB/octave to -9.52 dB/octave with change in order parameter a . MSLL is reduced with increase in fractional order a , e.g. MSLL for $a = 0.7$ is $-$

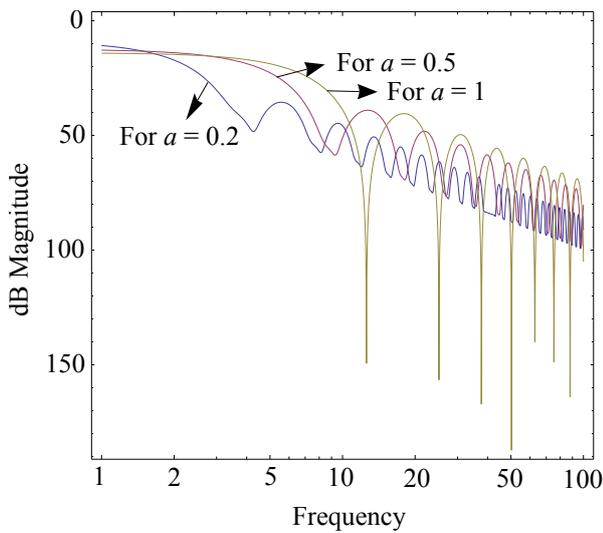


Fig. 6. SLFOR plot for Triangular function (scaling parameter $L = 0.5$) for fractional orders $a = 0.2, 0.5$ and 1

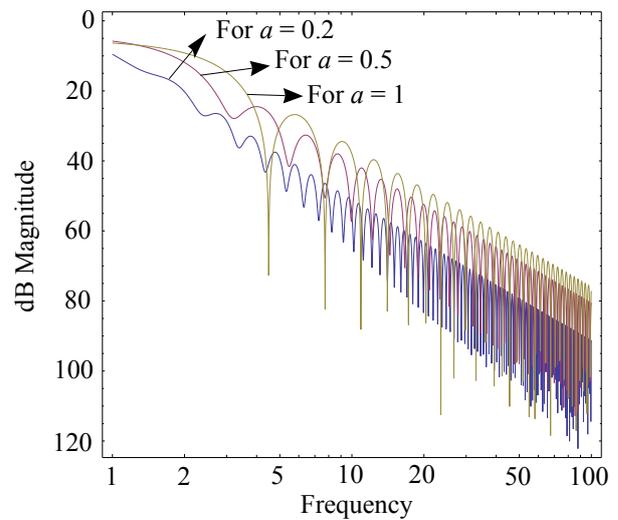


Fig. 8. SLFOR plot for Welch window for fractional orders $a = 0.2, 0.5$ and 1

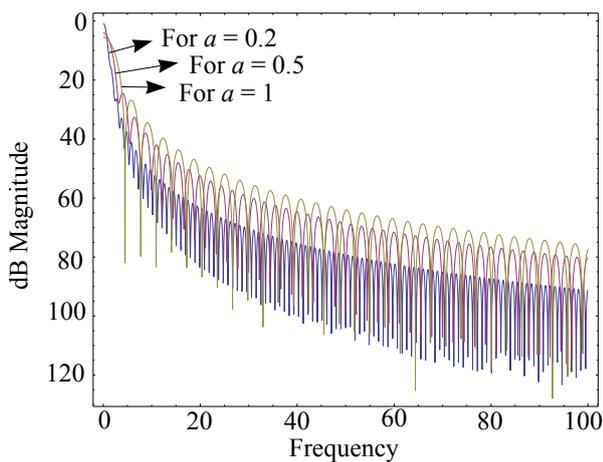


Fig. 7. MSLL plot for Welch window for fractional orders $a = 0.2, 0.5$ and 1

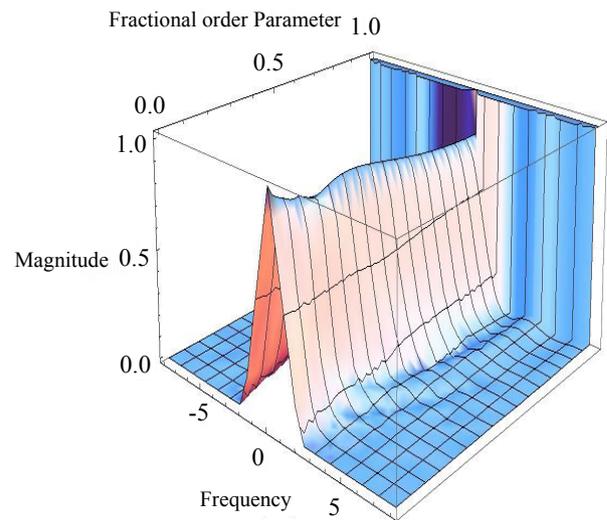


Fig. 9. The continuum of FrFT function for different values of fractional order a

21.19 db compared to is -22.5 db for $a = 1$.

Thus, an optimal domain can be selected in order to make a compromise between increase in main-lobe width and side-lobe level reduction.

6 APPLICATION OF FRFT IN TUNING WIDTH OF THE TRANSITION-BAND

The transition bandwidth of window-based FIR filters is proportional to the window main-lobe width, which in turn is inversely proportional to the length of the window function [15]. The transition width can be directly reduced by increasing the window length but at the cost of increased computations. This paper presents an alternate

methodology to tune the transition width using FrFT. A triangular window based low-pass FIR filter with cut-off frequency = 0.5π and length $N = 64$ is simulated.

The above Fig. 12 shows the variability in frequency response of the filter with change in value of fractional order parameter. As the fractional order a is reduced from 1 to 0, the transition width of window based FIR filter can be made narrow.

7 CONCLUSION

The mathematical analysis for obtaining the fractional Fourier transform of Triangular function and Welch win-

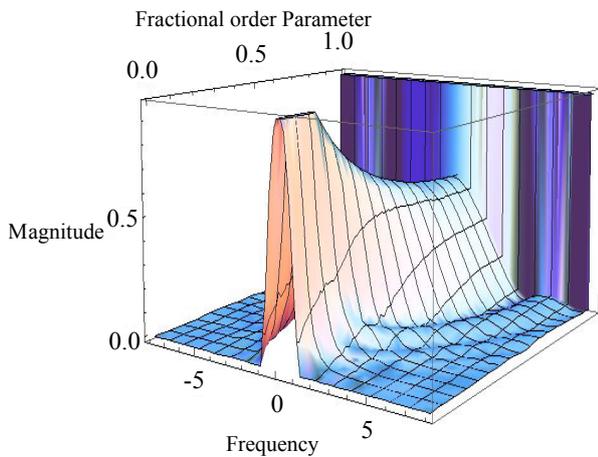


Fig. 10. The continuum of fractional Fourier transform of Welch window for different values of fractional order α

Table 1. Characteristics of Triangular function (scaling parameter $L = 0.5$) for different values of fractional order parameter α

Fractional Order α	MSLL (dB)	HMLW	SLFOR (dB/octave)
0.2	-24.1	3.4	-11
0.3	-24.4	5.21	-10.8
0.4	-24.9	6.11	-10.6
0.5	-25	7.02	-10.0
0.6	-25.5	7.92	-9.9
0.7	-25.8	8.83	-9.53
0.8	-26.3	9.73	-9.35
0.9	-26.8	10.2	-9.2
1	-27.8	10.6	-8.9

Table 2. Characteristics of Welch window for different values of fractional order parameter α

Fractional Order α	MSLL (dB)	HMLW	SLFOR (dB/octave)
0.2	-18.7	1.59	-9.52
0.3	-20.12	1.93	-9.47
0.4	-20.72	2.01	-9.08
0.5	-20.99	2.94	-8.99
0.6	-21.05	3.39	-8.57
0.7	-21.19	3.68	-8.43
0.8	-21.96	3.84	-8.13
0.9	-22.17	3.96	-7.92
1	-22.5	4.301	-7.8

down is presented in the paper. The different spectral parameters of these window functions are obtained by changing fractional order to different values. A good window can achieve low side-lobe levels with minimum increase in

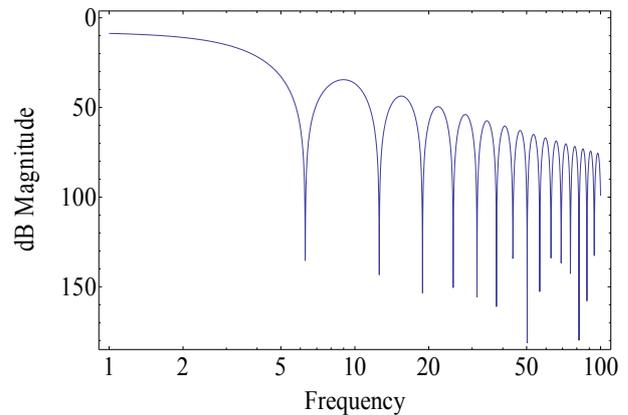


Fig. 11. SLFOR for Triangular window ($L = 1$) at fractional order $\alpha = 1$. The value of MSLL is -26.6 dB and SLFOR is -12 dB/octave

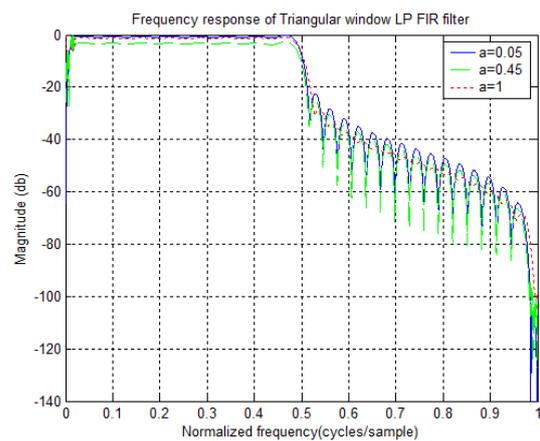


Fig. 12. The magnitude response of Triangular window based low-pass FIR filter for fractional order $\alpha = 0.05, 0.45$ & 1

main-lobe width. For Triangular and Welch window function, the side-lobe reduces but at the expense of HMLW. The analysis reveals that as the fractional order is reduced, main-lobe width can be minimized and SLFOR can be raised to maximum. Thus, it can be concluded that FrFT of Triangular function and Welch window varies directly with the change in fractional angle α . The new derived model for FrFT of these functions clearly shows that and a trade-off can be made between increase in main-lobe width and reduced side-lobe level which best suites the desired application. Triangular function and Welch window can be used as adjustable windows in the fractional Fourier domain for estimating the spectrum of a signal so that a choice can be made between amplitude accuracy and spectral resolution.

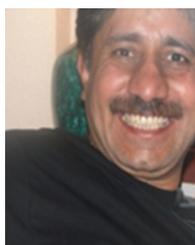
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