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THE EXTENDIBILITY OF $D(4)$-PAIRS $\{F_{2k}, F_{2k+6}\}$ AND $\{P_{2k}, P_{2k+4}\}$

Ljubica Baćić Ðuračković and Alan Filipin

Abstract. Let $k \geq 1$ be an integer and let F_k be the k-th Fibonacci number and P_k k-th Pell number. In this paper we prove that the pairs $\{F_{2k}, F_{2k+6}\}$ and $\{P_{2k}, P_{2k+4}\}$ cannot be extended to a $D(4)$-quintuple.

1. Introduction

If $n \neq 0$ is an integer, a set $\{a_1, a_2, \ldots, a_m\}$ of m distinct positive integers is called a $D(n)$-m-tuple if $a_i a_j + n$ is a perfect square for all i, j with $1 \leq i < j \leq m$.

In this paper we consider the case $n = 4$. There is a conjecture that there does not exist a $D(4)$-quintuple. Moreover, [7, Conjecture 1] states that if $\{a, b, c, d\}$ is a $D(4)$-quadruple such that $a < b < c < d$, then

$$d = d_+ = a + b + c + \frac{1}{2}(abc + rst),$$

where r, s and t are positive integers defined by $ab + 4 = r^2$, $ac + 4 = s^2$ and $bc + 4 = t^2$. The $D(4)$-quadruple $\{a, b, c, d\}$, where $d > \max\{a, b, c\}$ is called a regular quadruple if $d = d_+$. It is easy to see that a regular quadruple $\{a, b, c, d\}$ has the property $d > abc$. We also define $d_- = a + b + c + 1/2\,(abc - rst)$. The set $\{a, b, c, d_-\}$ is also a $D(4)$-quadruple with $< d_- < c$ provided $d_- \neq 0$. There are many results that support this Conjecture (see [1, 2, 7, 10–15]).

Here we will generalize the results from [11], where authors proved the uniqueness of the extensions of parametric $D(4)$-triples $\{F_{2k}, F_{2k+6}, 4F_{2k+2}\}$, $\{F_{2k}, F_{2k+6}, 4F_{2k+4}\}$, $\{P_{2k}, P_{2k+4}, 4P_{2k+2}\}$ and $\{P_{2k}, P_{2k+4}, 8P_{2k+2}\}$, where $k \geq 1$ is an integer and F_k and P_k denote the k-th Fibonacci and the k-th Pell number respectively. Remember that Pell numbers are given by $P_0 = 0$, $P_1 = 1$ and $P_{k+2} = 2P_{k+1} + P_k$ for $k \geq 0$.

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Let now $\{a,b\}$ be a fixed $D(4)$-pair, and let r be a positive integer given by $r^2 = ab+4$. For such pair we will define $c_\nu^\pm$ with

$$c_0^\pm = 0, c_1^\pm = a+b\pm 2r, c_{\nu+2}^\pm = (ab+2)c_{\nu+1}^\pm - c_\nu^\pm + 2(a+b), \nu \geq 2. \tag{1.1}$$

The set $\{a,b,c_\nu^\pm\}$ is a $D(4)$-triple. The main results of our paper is the following theorem.

THEOREM 1.1. *Let $\{a,b\} = \{F_{2k}, F_{2k+6}\}$ or $\{P_{2k}, P_{2k+4}\}$, $k \in \mathbb{N}$. If $\{a,b,c,d\}$ is a $D(4)$-quadruple, then there exist $\nu \in \mathbb{N}$ such that $c = c_\nu^\pm$ and $d = d_- = c_{\nu-1}^\pm$ or $d = d_+ = c_{\nu+1}^\pm$ for $\nu > 1$ and $d = d_+ = c_2^\pm$ for $\nu = 1$.*

Notice that the case when $\nu = 1$ is completely solved in [11]. Also in [3, Lemma 3] the authors proved the Theorem 1.1 for $k \leq 7$ and $k \leq 3$ respectively. More precisely, they proved the uniqueness of the extension of $D(4)$-triple $\{a,b,c\}$ with $a < b < c$ if $b \leq 10^4$. From this point onwards we will assume $k \geq 8$ when considering the extension of the pair $\{F_{2k}, F_{2k+6}\}$ and $k \geq 4$ when considering the $D(4)$-pair $\{P_{2k}, P_{2k+4}\}$. In the proof of the main Theorem we will use standard methods in solving those kind of problems. The main purpose of the paper is to furthermore illustrate the use of [3, Theorem 1] which reduces our proof to some special cases and so we have to consider the extendibility of $D(4)$-triples $\{F_{2k}, F_{2k+6}, c\}$ and $\{P_{2k}, P_{2k+4}, c\}$ only for few values of c. The other aim is to generalize the results from [11]. We use and apply more results from [3].

First, we transform our problem of extendibility of $D(4)$-triple to solving the system of simultaneous Pellian equations. It leads to finding the intersection of binary recurrent sequences which we solve combining the congruence method, linear forms in logarithms and applying Baker-Davenport reduction.

2. EXTENSION OF $D(4)$-PAIR $\{a,b\}$

Through this paper we solve the both parametric families simultaneously. So we fix $a = F_{2k}$, $b = F_{2k+6}$ or $a = P_{2k}$, $b = P_{2k+4}$. We use $b > 10^4$ (see [3, Lemma 3]) and also $17.9a < b < 34a$. The main goal of this section is to find all possible elements c that can extend $D(4)$-pair $\{a,b\}$.

PROPOSITION 2.1. *Let $\{a,b\} = \{F_{2k}, F_{2k+6}\}$ or $\{P_{2k}, P_{2k+4}\}$, $k \in \mathbb{N}$. If $\{a,b,c\}$ is $D(4)$-triple, then exists $\nu \in \mathbb{N}$ such that $c = c_\nu^+$ or $c = c_\nu^-$.*

PROOF. To prove this we can use the proof of [3, Lemma 1] (see page 449, last 4 rows). There, we proved that if $c \neq c_\nu^\pm$, then there exist $\{a, b', c', b\}$ a regular $D(4)$-quadruple such that $b'c' < b/a$. In the case of $D(4)$-pair $\{F_{2k}, F_{2k+6}\}$ and $k \geq 8$, we see that $17.9a < b < 18a$, so we conclude $b'c' < 18$. But because $b'c'+4$ should be a square, we have $b'c' \leq 12$. Now from

$$b = a + b' + c' + \frac{1}{2}(ab'c' + \sqrt{(ab'+4)(ac'+4)(b'c'+4)})$$

we get

$$b < 1.1a + \frac{1}{2}(12a + 15a) = 14.6a$$

which is a contradiction with $b > 17.9a$. In the last inequality we have used $a > 10^4/18$ and $b'c' \leq 12$. For $D(4)$-pair $\{P_{2k}, P_{2k+4}\}$ and $k \geq 4$, we have $33.9P_{2k} < P_{2k+4} < 34P_{2k}$. Again, $\{a, b', c', b\}$ has to be a regular quadruple. Then, we conclude $b'c' < 34$, but because $b'c' + 4$ should be a square, we have $b'c' \leq 32$. If $b'c' < 32$, then $b'c' \leq 21$ and from

$$b = a + b' + c' + \frac{1}{2}(ab'c' + \sqrt{(ab' + 4)(ac' + 4)(b'c' + 4)})$$

we get

$$b < 1.1a + \frac{1}{2}(21a + 24a) = 23.6a$$

which is a contradiction with $b > 33.9a$. In the last inequality we have used $a > 10^4/34$ and $b'c' \leq 21$. However, if $b'c' = 32$, we do not get a contradiction right away. In that case we consider $(b', c') = (1, 32), (2, 16), (4, 8)$. Those cases can be solved using linear forms in logarithms. It is known method for finding the pure powers in binary recurrence sequences. We give the sketch of the proof for the case $(b', c') = (1, 32)$ and other cases can be proved in the same fashion. In that case $P_{2k} + 4 = X^2$ for some positive integer X. We know that

$$P_n = \frac{1}{2\sqrt{2}}\left(\alpha^n - \beta^n\right),$$

where $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. If we define linear form in logarithms

$$\Lambda = 2\log\left(\frac{\alpha^k}{X}\right) - \log(2\sqrt{2}),$$

then $P_{2k} + 4 = X^2$, for $k > 1000$, implies $\log|\Lambda| < \log(4.003) - 2\log X$. On the other hand we can get the lower bound on $|\Lambda|$ using the Baker's theory on linear forms in logarithms. We have used the well known Baker-Wüstholz theorem from [5]. Precisely, we get

$$|\Lambda| > -2.315 \cdot 10^7 \log X.$$

Combining those bounds, we have $X \leq 30955$, which cannot be satisfied for $k > 1000$. It is left to check what is happening for $k \leq 1000$. The only possible k that will give $P_{2k} + 4$ to be a square is $k = 2$. Then, $P_{2k} + 4 = 16$. However, in that case $12 \cdot 32 + 4 = 388$ is not a square. So we get a contradiction with the fact that $\{1, 32, P_{2k}\}$ is a $D(4)$-triple. Also the case $k = 2$ was already solved as we mentioned above. $\square$

3. Problem of the extension of $D(4)$-triples $\{a,b,c_\nu^\pm\}$

Let us mention that Theorem 1.1 implies the following corollary.

Corollary 3.1. *Let $\{a,b\} = \{F_{2k}, F_{2k+6}\}$ or $\{P_{2k}, P_{2k+4}\}$, $k \in \mathbb{N}$. The pair $\{a,b\}$ cannot be extended to $D(4)$-quintuple.*

From now on we assume that $c = c_\nu^\pm$ is minimal in some sense.

Assumption 3.2. *Assume that $c = c_\nu^\pm$ is minimal such that $\{a,b,c,d\}$ is a $D(4)$-quadruple with $d > d_+$ and that $\{a,b,c',c\}$ is not a $D(4)$-quadruple for any c' with $0 < c' < c_{\nu-1}^\pm$.*

Remark 3.3. Notice that this assumption is not restrictive in any sense because we know all possible values of c (and we know how they are ordered). Otherwise, there would exist some $c' < c_{\nu-1}^\pm$ and $\{a,b,c',c\}$ would be an irregular $D(4)$-quadruple with $c' < d_+(a,b,c') < c$ which contradicts the minimality of c.

Lemma 3.4. *([3, Theorem 1]) Let $\{a,b,c\}$ be a $D(4)$-triple with $a < b$. Suppose that $\{a,b,c,d\}$ is a $D(4)$-quadruple with $d > d_+$ and that $\{a,b,c',c\}$ is not a $D(4)$-quadruple for any c' with $0 < c' < d_-$.*
(1) *If $b < 1.5a$, then $c < b^6$.*
(2) *If $1.5a \le b < 5a$, then $c < b^5$.*
(3) *If $b \ge 5a$, then $c < 6b^5$.*

The previous Lemma implies that we have to consider the extensions of our triples $\{a,b,c\}$ only with $c = c_2^\pm, c_3^\pm$ and that is what we will do now. Because we are in the case (3) of the previous Lemma, for $\nu \ge 4$ we have

$$c \ge c_4^- > a^3b^3(a+b-2r) > a^3b^3 \cdot 10a > 10b^7/(34^4) > 6b^5$$

for $b > 10^4$. As we mentioned, the case $c = c_1^\pm$ is completely solved in [11].

4. Extension of $D(4)$-triples $\{a,b,c_2^\pm\}$ and $\{a,b,c_3^\pm\}$

Let $\{a,b,c,d\}$ be a $D(4)$-quadruple with $c = c_\nu^\pm$ with $\nu \ge 2$, which is given by (1.1). Notice that in the proof of the Corollary 3.1 we showed that all c's which extend the pair $\{a,b\}$ are given by (1.1). Moreover, let r, s and t be positive integers defined by $ab+4 = r^2$, $ac+4 = s^2$, $bc+4 = t^2$. Furthermore, there exist integers x, y and z such that

$$ad+4 = x^2,\ bd+4 = y^2,\ cd+4 = z^2. \tag{4.1}$$

Eliminating d, we obtain the following system of simultaneous Pellian equations

$$az^2 - cx^2 = 4(a-c), \tag{4.2}$$
$$bz^2 - cy^2 = 4(b-c). \tag{4.3}$$

From [7, Lemma 2] we know that if (z, x) and (z, y) are positive solutions of (4.2) and (4.3) respectively, then there exist indices i and m such that $z = v_m^{(i)}$, where

$$v_0^{(i)} = z_0^{(i)}\ , \quad v_1^{(i)} = \frac{1}{2}\Big(sz_0^{(i)} + cx_0^{(i)}\Big)\ , \quad v_{m+2}^{(i)} = sv_{m+1}^{(i)} - v_m^{(i)}, \tag{4.4}$$

and there exist indices j and n such that $z = w_n^{(j)}$, where

$$w_0^{(j)} = z_1^{(j)}\ , \quad w_1^{(j)} = \frac{1}{2}\Big(tz_1^{(j)} + cy_1^{(j)}\Big)\ , \quad w_{n+2}^{(j)} = tw_{n+1}^{(j)} - w_n^{(j)}. \tag{4.5}$$

Here $(z_0^{(i)}, x_0^{(i)})$ and $(z_1^{(j)}, y_1^{(j)})$ are fundamental solutions of (4.2) and (4.3) respectively. So now we have transformed the problem of solving the system of simultaneous Pellian equations to solving finitely many Diophantine equations of the form $z = v_m^{(i)} = w_n^{(j)}$. For the simplicity's sake, from now on, we will omit the superscripts (i) and (j). Initial terms of the sequences (v_m) and (w_n) are almost completely determined in the following lemma.

LEMMA 4.1. *([9, Lemma 9]) Let $a < b < c$.*

(i) *If the equation $v_{2m} = w_{2n}$ has a solution, then $z_0 = z_1$. Moreover, $|z_0| = 2$ or $|z_0| = \frac{1}{2}(cr - st)$ or $|z_0| < 1.608a^{\frac{-5}{14}}c^{\frac{9}{14}}$.*

(ii) *If the equation $v_{2m+1} = w_{2n}$ has a solution, then $|z_0| = t$, $|z_1| = \frac{1}{2}(cr - st)$, $z_0z_1 < 0$.*

(iii) *If the equation $v_{2m} = w_{2n+1}$ has a solution, then $|z_1| = s$, $|z_0| = \frac{1}{2}(cr - st)$, $z_0z_1 < 0$.*

(iv) *If the equation $v_{2m+1} = w_{2n+1}$ has a solution, then $|z_0| = t$, $|z_1| = s$, $z_0z_1 > 0$.*

LEMMA 4.2. *Let $c = c_2^{\pm}, c_3^{\pm}$. Then, solving the equations $v_m = w_n$ it is enough to consider*

(i) *$v_{2m} = w_{2n}$ if $z_0 = z_1 = \pm 2$ and*

(ii) *$v_{2m+1} = w_{2n+1}$ if $z_0 = \pm t, z_1 = \pm s$ and $z_0z_1 > 0$.*

PROOF. The proof for this follows immediately from the proof for [9, Lemma 9] and the assumption 3.2. First, we can remove the case $|z_0| = (cr - st)/2$ from Lemma 4.1 (i) because we get exactly the same intersections as in the case $v_{2m+1} = w_{2n+1}$ with $|z_0| = t$, $|z_1| = s$ and $z_0z_1 > 0$. The same is true for the cases (ii) and (iii) in that Lemma, i.e. $v_{2m+1} = w_{2n}$ with $|z_0| = t$, $|z_1| = \frac{1}{2}(cr - st)$, $z_0z_1 < 0$ and $v_{2m} = w_{2n+1}$ with $|z_1| = s$, $|z_0| = \frac{1}{2}(cr - st)$, $z_0z_1 < 0$ gives the exactly same intersections as in the case $v_{2m+1} = w_{2n+1}$ with $|z_0| = t$, $|z_1| = s$ and $z_0z_1 > 0$. Finally, the assumption 3.2 helps us to remove the third case of (i) from Lemma 4.1 because in that case we must have an irregular $D(4)$-quadruple $\{a, b, d_0, c\}$ with $0 < d_0 < c$ which contradicts the assumption. ∎

Now we will give the lower bounds of the indices m and n in the equation $v_m = w_n$ for $2 < n < m < 2n$ (where the relationship between m and n follows from [9, Lemma 5] if m and n have the same parity). It is not difficult to check that all solutions of $v_m = w_n$ with smaller indices will give the extension of $D(4)$-triple $\{a, b, c\}$ to a quadruple with $d = d_- = c^{\pm}_{\nu-1}$ or $d = d_+ = c^{\pm}_{\nu+1}$. So to prove there are no other extensions, we have to show that $v_m = w_n$ for $m > n > 2$ does not have a solution for $c = c^{\pm}_2, c^{\pm}_3$. In the proof we use $b > 10^4$ and $17.9a < b < 34a$. Also notice that in all those cases we have $c > a^2 b$ which implies that bounds we get in the next lemma are not trivial.

LEMMA 4.3. (i) *If $v_{2m} = w_{2n}$ has a solution for $n > 1$, then $m > 0.495b^{-0.5}c^{0.5}$.*

(ii) *If $v_{2m+1} = w_{2n+1}$ has a solution for $n \geq 1$, then $m^2 > 0.125b^{-1}c^{0.5}$.*

PROOF. (i) The statement follows from the proof of [4, Proposition 2.3]. We only have to use that $b > 10^4$.

(ii) In the case of odd indices, from [7, Lemma 3], inserting $z_0 = \pm t$, $z_1 = \pm s$ and $x_0 = y_1 = r$, we have

$$\pm\frac{1}{2}astm(m+1) + r(2m+1) \equiv \pm\frac{1}{2}bstn(n+1) + r(2n+1) \pmod{c}. \tag{4.6}$$

Using that $(st)^2 \equiv 16 \pmod{c}$, we conclude that $st \equiv \pm 4 \pmod{c'}$ for some c' which is a divisor of c, and $c' \geq \sqrt{c}$. Here the $\pm$ sign means that one of the congruences is true. Then, we have

$$\pm 2am(m+1) + r(2m+1) \equiv \pm 2bn(n+1) + r(2n+1) \pmod{c'}. \tag{4.7}$$

Let us now assume the opposite, i.e. $m^2 \leq 0.125b^{-1}c^{0.5}$. Then, it is easy to see that both sides of the congruence relation (4.7) are less that c' and they have the same sign. Precisely, we have

$$2am(m+1),\ r(2m+1),\ 2bn(n+1),\ r(2n+1) < 2bm(m+1)$$

and

$$2bm(m+1) \leq 4bm^2 \leq \frac{c'}{2}.$$

So we actually have an equation

$$\pm 2am(m+1) + r(2m+1) = \pm 2bn(n+1) + r(2n+1)$$

and

$$bn(n+1) - am(m+1) = r(m-n).$$

This leads to a contradiction, because

$$bn(n+1) - am(m+1) > 17.9an(n+1) - 2an(2n+1) > 11.9an^2$$

and

$$r(m-n) < rn < 6an \leq 6an^2.$$

□

5. Application of Baker's theory

Here we will combine the lower bounds for indices m and n together with the result obtained using the Baker's theory of linear forms in logarithms to prove the main Theorem for large values of k.

Using [5] the second author proved in [8] that $v_m = w_n$, for $n > 2$, implies

$$\frac{m}{\log(m+1)} < 6.543 \cdot 10^{15} \log^2 c.$$

So we combine this with Lemma 4.3. In the case of even indices from Lemma 4.3 (i) we get the inequality

$$\frac{2 \cdot 0.495 b^{-0.5} c^{0.5}}{\log(2 \cdot 0.495 b^{-0.5} c^{0.5} + 1)} < 6.543 \cdot 10^{15} \log^2 c.$$

If we put $a = F_{2k}$, $b = F_{2k+6}$ and $c = c_2^{\pm}, c_3^{\pm}$ by using the software package `Mathematica` 9, we get a contradiction for $k > 52$ in the worst case of $c = c_2^-$. In the case of odd indices from Lemma 4.3 (ii) we get the inequality

$$\frac{2 \cdot 0.125^{0.5} b^{-0.5} c^{0.25} + 1}{\log(2 \cdot 0.125^{0.5} b^{-0.5} c^{0.25} + 2)} < 6.543 \cdot 10^{15} \log^2 c.$$

For the same values of a, b and c we get a contradiction for $k > 225$ and $c = c_2^-$. In the same way when $a = P_{2k}$ and $b = P_{2k+4}$ we get a contradiction for $k > 122$ and $c = c_2^-$ in the case of odd indices. If $c \neq c_2^{\pm}$, we get the same or even better bound for k.

Now we are left to see what is happening for small values of k, i.e. when $a = F_{2k}$, $b = F_{2k+6}$ for $8 \leq k \leq 225$ and when $a = P_{2k}$, $b = P_{2k+4}$ for $4 \leq k \leq 122$. To solve this we use the well known Baker-Davenport reduction method (see [6, Lemma 5]). For this we also need the inequality which follows from $v_m = w_n$, $n > 2$ (that is [8, Lemma 9]),

$$\begin{aligned} 0 &< m \log\left(\frac{s+\sqrt{ac}}{2}\right) - n \log\left(\frac{t+\sqrt{bc}}{2}\right) + \log\frac{\sqrt{b}(x_0\sqrt{c}+z_0\sqrt{a})}{\sqrt{a}(y_1\sqrt{c}+z_1\sqrt{b})} \\ &< 2ac\left(\frac{s+\sqrt{ac}}{2}\right)^{-2m}. \end{aligned}$$

In the case of even indices we have $z_0 = z_1 = \pm 2$, $x_0 = y_1 = 2$ and in the case of odd indices we have $x_0 = y_1 = r$, $z_0 = \pm t$, $z_1 = \pm s$ and $z_0 z_1 > 0$. We have done the reduction using the software package `Mathematica` 9. In all cases, after at most 3 steps of reduction, we get that $w_m = w_n$ implies $n \leq m \leq 2$ which finishes the proof of Theorem 1.1. To run all programs and to finish our proof, it took us less than 2 hours on 2.80 GHz Intel Core 2 Duo 2.98GB.

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Proširenje $D(4)$-parova $\{F_{2k}, F_{2k+6}\}$ i $\{P_{2k}, P_{2k+4}\}$

Ljubica Baćić Đuračković i Alan Filipin

Sažetak. Neka je k prirodni broj, F_k k-ti Fibonaccijev broj i P_k k-ti Pellov broj. U ovom članku dokazali smo da se parovi $\{F_{2k}, F_{2k+6}\}$ i $\{P_{2k}, P_{2k+4}\}$ ne mogu proširiti do $D(4)$-petorke.

Ljubica Baćić Đuračković
Primary School Nikola Andrić
32000 Vukovar
Croatia
E-mail: ljubica.bacic@skole.hr

Alan Filipin
Faculty of Civil Engineering
University of Zagreb
10 000 Zagreb
Croatia
E-mail: filipin@master.grad.hr

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